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Local holomorphic dynamics of diffeomorphisms in dimension one

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Prologue

The study of the behavior of the sequence of iterates of a germ of holomorphic diffeomorphism f in $\mathbb{C}$ has been object of study since the time of Schröder and Fatou and Julia and it is still today a very flourishing sector of mathematics. Much of this theory has been used and improved by people interested in the dynamics of holomorphic foliations, relating local dynamics of germs with that of foliations by means of holonomy and Poincaré's time one map.

To understand dynamics of a germ, one tries first to simplify it by means of suitable changes of coordinates. In particular, the best situation one can hope to have is *linearization* of the germ. This means that suitably changing coordinates the map becomes a linear transformation. If the change of coordinates used to linearize the germ is holomorphic then the linear transformation obtained is the differential of the germ at the fixed point. However if the change of coordinates involved is only continuous then the linear transformation might not be the differential. Holomorphic linearization is the dream of people that study local holomorphic dynamics, for one can really think of the map as a linear transformation. Even topological linearization is useful (for instance it provides trajectories and behavior of orbits), quasiconformal conjugation (which might change the differential as well) and sometimes it may be useful also to have just formal linearization. Anyhow, the first derivative is the map which first approximates the dynamics of the map, and thus it is natural to classify and study dynamics according to it.

As we will see, a generic germ of holomorphic diffeomorphism is holomorphically linearizable. Unfortunately, the non-generic situation comes out often in celestial mechanics and physical problems. Thus one is forced to understand non-linearizable dynamical systems. These are not completely understood, even if from the pioneering work of Fatou, Dulac and Poincaré much has been done.

In these notes we provide a survey with detailed proofs about local dynamics of germs of holomorphic diffeomorphisms. The first part is related to formal classification, and we relate germs of diffeomorphisms with formal vector fields via the exponential map. Then we discuss holomorphic dynamics. The core part here is to provide a detailed proof of Yoccoz's wonderful qualitative result about holomorphic linearization for almost every elliptic germ. We also study the hyperbolic case and the parabolic case. Then we end up with few notes on the topological classification, especially Camacho's theorem for the parabolic case.

The survey is based on a PhD course I gave at Università di Roma "Tor Vergata" in 2007/08. The bibliography is not exhaustive at all, although I tried to give appropriated credits when possible. Proofs however are provided quite in details, trying to use a point of view suitable for further generalizations, especially in higher dimensions.

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1. Formal Normal Forms

1.1. Germs of formal diffeomorphisms. Let denote with $\widehat{\mathsf{Diff}}(\mathbb{C}, O)$ the set of formal power series of type

$$\widehat{f}(z) := \sum_{j\in\mathbb{N}} a_j z^j, \quad a_0 = 0, a_1 \neq 0.$$

Namely, the constant coefficient of $\widehat{f}$ is zero and the coefficient of the linear term is not zero. If $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ and the series is uniformly convergent on some open disc of positive radius, we write $\widehat{f} \in \mathsf{Diff}(\mathbb{C}, O)$. An element of $\widehat{\mathsf{Diff}}(\mathbb{C}, O)$ is called a *formal germ of diffeomorphism*, while an element of $\mathsf{Diff}(\mathbb{C}, O)$ is called a *germ of diffeomorphism*.

PROPOSITION 1.1. *The set $\widehat{\mathsf{Diff}}(\mathbb{C}, O)$ is a non-commutative group with respect to composition and $\mathsf{Diff}(\mathbb{C}, O)$ is a subgroup.*

PROOF. Let $\widehat{f}(z) := \sum a_j z^j$ and $\widehat{g}(z) := \sum b_j z^j$ be in $\widehat{\mathsf{Diff}}(\mathbb{C}, O)$. Define the composition

$$(\widehat{f} \circ \widehat{g})(z) := \sum_{j \geq 1} a_j \left(\sum_{k \geq 1} b_k z^k \right)^j$$
$$= a_1 b_1 z + (a_1 b_2 + a_2 b_1^2) z^2 + (a_1 b_3 + 2 a_2 b_1 b_2 + a_3 b_1^3) z^3 + \dots$$

Therefore, since $a_1 b_1 \neq 0$ then $\widehat{f} \circ \widehat{g} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ and the group sum is well defined. Clearly the germ defined by $\mathsf{id}(z) := z$ is the neutral element. Also the associative property is easy to be verified. It remains to prove that each $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ is invertible. Let $\widehat{f}(z) := \sum a_j z^j \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be given. We are looking for a germ $\widehat{g}(z) := \sum b_j z^j$ such that $\widehat{f} \circ \widehat{g} = \widehat{g} \circ \widehat{f} = \mathsf{id}$. From the very definition of composition we obtain the condition $b_0 = 0, b_1 = 1/a_1$. Then $a_1 b_2 + a_2 b_1^2 = 0$ which implies $b_2 = -a_2 b_1^2 / a_1$, and $a_1 b_3 + 2 a_2 b_1 b_2 + a_3 b_1^3 = 0$ which implies $b_3 = -(2 a_2 b_1 b_2 + a_3 b_1^3)/a_1$. More generally, since the coefficient of $\widehat{f} \circ \widehat{g}$ of position k is of the form $a_1 b_k +$[terms containing $b_1, \dots, b_{k-1}$] it follows that the equation $a_1 b_k +$[terms containing $b_1, \dots, b_{k-1}$] $= 0$ has a unique solutions in terms of $a_1, \dots, a_k, b_1, \dots, b_{k-1}$. Therefore $\widehat{g}$ is uniquely determined and $\widehat{f}$ is invertible. It is finally clear that $\mathsf{Diff}(\mathbb{C}, O)$ is a subgroup of $\widehat{\mathsf{Diff}}(\mathbb{C}, O)$ because the composition of two holomorphic functions is holomorphic. □

DEFINITION 1.2. We say that $\widehat{f}, \widehat{g} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ are *formally conjugated* if there exists $\widehat{h} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ such that $\widehat{h} \circ \widehat{f} = \widehat{g} \circ \widehat{h}$. In case $\widehat{f}, \widehat{g}, \widehat{h} \in \mathsf{Diff}(\mathbb{C}, O)$ we say that $\widehat{f}$ and $\widehat{g}$ are *holomorphically conjugated*.

A germ of (formal) diffeomorphism $\widehat{f}$ is *(formally) linearizable* if it is (formally) conjugated to a linear germ of the form $\widehat{g}(z) := \lambda z$.

PROPOSITION 1.3. *Let $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be given by $\widehat{f}(z) = \sum_{j \geq 1} a_j z^j$. Suppose that $\widehat{f}$ is formally conjugated to a formal germ $\widehat{g}(z) = \sum_{j \geq 1} b_j z^j$. Then $b_1 = a_1$.*

PROOF. Assume that $\widehat{h}(z) = \sum_{j \geq 1} c_j z^j$ conjugates $\widehat{f}$ to $\widehat{g}$, namely $\widehat{h} \circ \widehat{f}(z) = \widehat{g} \circ \widehat{h}(z)$. Expanding we find

$$\widehat{h} \circ \widehat{f}(z) = c_1 a_1 z + O(z^2), \quad \widehat{g} \circ \widehat{h}(z) = b_1 c_1 z + O(z^2),$$

and since $c_1 \neq 0$, it follows that $a_1 = b_1$. □

REMARK 1.4. If $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ is formally conjugated to the linear germ $\widehat{g}(z) = \lambda z$ then one can find a germ $\widehat{h} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ with $\widehat{h}(z) = z + \sum_{j \geq 2} b_j z^j$ which conjugates $\widehat{f}$ to $\widehat{g}$. Indeed, if $H(z) = \sum_{j \geq 1} c_j z^j$ solves $H \circ \widehat{f}(z) = \lambda H(z)$ then $\widehat{h}(z) = H(z)/c_1$ does the job.

From Proposition 1.3 it follows that the term a_1 is invariant under conjugation. Since dynamical properties of a germ are invariant under conjugation, the following definition is coherent:

DEFINITION 1.5. Let $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be given by $\widehat{f}(z) = \sum_{j\geq 1} a_j z^j$. Let $\lambda := a_1$. We say that $\widehat{f}$ is

(1) *hyperbolic* if $|\lambda| \neq 1$,
(2) *parabolic* if $\lambda^q = 1$ for some $q \in \mathbb{N} \setminus \{0\}$
(3) *elliptic* if $|\lambda| = 1$ and $\lambda^q \neq 1$ for all $q \in \mathbb{N} \setminus \{0\}$.

In the next subsections we examine formal linearization according to the previous classification.

1.2. Homological equation, resonances and the non-parabolic case. Let $\widehat{f}$ be in $\widehat{\mathsf{Diff}}(\mathbb{C}, O)$. In order to find a simpler conjugated form for $\widehat{f}$, or even to linearize $\widehat{f}$, one can try to dispose of one monomial after another, starting from the one of smallest degree. This operation is not always working due to resonances. This phenomenon is easily controlled in dimension one but plays an important role in higher dimension.

To enter into details, let

$$\widehat{f}(z) = \lambda z + a_j z^j + O(z^{j+1}),$$

where a_j for $j \geq 2$ denotes the first non-zero coefficient in the series of $\widehat{f}$. Let us try to use a (holomorphic) diffeomorphism of the form $\varphi(z) = z + \alpha z^k$ to dispose of the term of degree j, that is a_j, without introducing terms of degree less than j. Namely, we look for $\varphi(z)$ which solves the following functional equation:

$$\varphi \circ \widehat{f}(z) = \lambda\varphi(z) + O(z^{j+1}).$$

Expanding we obtain

$$\widehat{f}(z) + \alpha[\widehat{f}(z)]^k = \lambda z + \lambda\alpha z^k + O(z^{j+1}),$$

that is

$$\lambda z + a_j z^j + \alpha\lambda^k z^k + O(z^{j+1}, z^{k+1}) = \lambda z + \lambda\alpha z^k + O(z^{j+1}).$$

From this it follows that we have to choose $k = j$ and we come up with the following equation known as the *homological equation*:

$$a_j + \alpha\lambda(\lambda^{k-1} - 1) = 0. \tag{1.1}$$

Clearly, such an equation has a unique solution $\alpha = -a_j/\lambda(\lambda^{k-1} - 1)$ in case $\lambda^{k-1} \neq 1$.

This simple argument has a series of interesting consequences that we list.

THEOREM 1.6. *Let $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be a non-parabolic germ. Then $\widehat{f}$ is formally linearizable.*

PROOF. Let $\widehat{f}(z) = \lambda z + \sum_{j\geq 2} a_j z^j$. Let $T_2(z) = z + \alpha_2 z^2$ with $\alpha_2 = -a_2/\lambda(\lambda - 1)$. Since $\lambda^q \neq 1$ for every $q \in \mathbb{N}$ then α_2 is well defined. Notice that $T_2 = \mathrm{id}$ if $a_2 = 0$. Then let $\widehat{f}_2 := T_2 \circ \widehat{f} \circ T_2^{-1}$. Since $T_2^{-1}(z) = z - \alpha_2 z^2 + O(z^3)$ it follows that $\widehat{f}_2(z) = \lambda z + \sum_{j\geq 3} \tilde{a}_j z^j$.

More generally, we can define by induction T_k for $k \geq 2$ to be the (holomorphic) diffeomorphism of the form $T_k(z) = z + \alpha_k z^k$ which solves the homological equation (1.1) for the coefficient of degree k of $(T_{k-1} \circ \ldots \circ T_2) \circ \widehat{f} \circ (T_{k-1} \circ \ldots \circ T_2)^{-1}$. Then we let

$$T(z) = \lim_{k\to\infty} (T_k \circ \ldots \circ T_2)(z).$$

In other words, $T(z) = z + \sum_{j\geq 2} b_j z^j$ is the formal diffeomorphism whose coefficient b_j is the coefficient of degree j in $T_j \circ \ldots \circ T_2$. Notice that for $k > j$

$$T_k \circ (T_j \circ \ldots \circ T_2)(z) = (T_j \circ \ldots \circ T_2)(z) + O(z^k),$$

and therefore the coefficient b_j stabilizes in the limit $\lim_{k\to\infty}(T_k \circ \ldots \circ T_2)(z)$ and hence it is well-defined.

By construction $T \circ \widehat{f}(z) = \lambda T(z)$ and $\widehat{f}$ is linearizable. $\square$

It is worth to explicitly notice that, even if $\widehat{f}$ is a holomorphic non-parabolic germ, that is $\widehat{f} \in \mathsf{Diff}(\mathbb{C}, O)$, the formal diffeomorphism T in the proof of Theorem 1.6, given as infinite composition of holomorphic diffeomorphisms, may not be holomorphic. Namely, a germ $f \in \mathsf{Diff}(\mathbb{C}, O)$ may be formally linearizable but not holomorphically linearizable. Examples of such germs exist and we will see them later. The problem of convergence of the infinite composition is strongly related to how $\lambda^k - 1$ stays bounded away from zero. This problem is known as *small divisors problem*. We shall come back to this later.

For the moment, we notice that if one stops the process of linearization in the proof of Theorem 1.6 at degree k, then the germ $T_k \circ \ldots \circ T_2$ is holomorphic and conjugates $\widehat{f}$ to a germ of the form $\lambda z + O(z^{k+1})$. In particular if $\widehat{f}$ is holomorphic we have

COROLLARY 1.7. *Let $f \in \mathsf{Diff}(\mathbb{C}, O)$ be a non-parabolic germ. For any $k \in \mathbb{N}$ there exists $g \in \mathsf{Diff}(\mathbb{C}, O)$ such that g (holomorphically) conjugates f to a (holomorphic) germ of the type*

$$\lambda z + O(z^{k+1}).$$

Before moving to the parabolic case, we prove that the diffeomorphism which linearizes a non-parabolic germ is essentially unique:

PROPOSITION 1.8. *Let $\widehat{f}(z) = \lambda z + O(z^2) \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be a non-parabolic germ. Let $\widehat{g}_0, \widehat{g}_1 \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be such that $\widehat{g}_j \circ \widehat{f} = \lambda \widehat{g}_j$, $j = 0, 1$. Then there exists $a \in \mathbb{C} \setminus \{0\}$ such that $\widehat{g}_1 = a\widehat{g}_0$.*

PROOF. By hypothesis,

$$\widehat{g}_0^{-1}(\lambda \widehat{g}_0(z)) = \widehat{f}(z) = \widehat{g}_1^{-1}(\lambda \widehat{g}_1(z)),$$

which implies $\widehat{g}_1(\widehat{g}_0^{-1}(\lambda z)) = \lambda \widehat{g}_1(\widehat{g}_0^{-1}(z))$. Let $\widehat{h}(z) := \widehat{g}_1 \circ \widehat{g}_0^{-1}(z)$. Then

$$\widehat{h}(\lambda z) = \lambda \widehat{h}(z).$$

If $\widehat{h}(z) = \sum_{j\geq 1} a_j z^j$, expanding the previous expression we find

$$\sum_{j\geq 1} \lambda^j a_j z^j = \sum_{j\geq 1} \lambda a_j z^j.$$

Equating terms of the same degree we obtain

$$\lambda(\lambda^{j-1} - 1)a_j = 0.$$

Since $\widehat{f}$ is not parabolic, and then $\lambda^{j-1} \neq 1$ for all $j > 1$, it follows that $a_j = 0$ for $j > 1$. Therefore $\widehat{h}(z) = a_1 z$ and $\widehat{g}_1 \circ \widehat{g}_0^{-1}(z) = a_1 z$ which implies $\widehat{g}_1 = a_1 \widehat{g}_0$ as claimed. $\square$

The previous proposition allows to write quite explicitly the coefficients of the diffeomorphism which linearizes a non-parabolic germ:

PROPOSITION 1.9. *Let $f(z) = \lambda z + \sum_{j\geq 2} a_j z^j \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be a non-parabolic germ of formal diffeomorphism. Let $g(z) = z + \sum_{j=2}^{\infty} b_j z^j \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be such that $g^{-1} \circ f = \lambda g^{-1}$. Then*

$$b_n = \frac{1}{\lambda^n - \lambda} \sum_{j=2}^{n} (a_j \sum_{k_1+\ldots+k_j=n} b_{k_1} \cdots b_{k_j}). \tag{1.2}$$

PROOF. The functional equation $g^{-1} \circ f = \lambda g^{-1}$ is equivalent to $f(g(z)) = g(\lambda z)$. We compute

$$\begin{aligned} f(g(z)) &= \lambda(z + \sum_{j=2}^{\infty} b_j z^j) + \sum_{k\geq 2} a_k \left(z + \sum_{j=2}^{\infty} b_j z^j \right)^k \\ &= \lambda z + (\lambda b_2 + a_2) z^2 + (\lambda b_3 + a_2(b_1 b_2 + b_2 b_1) + a_3) z^3 + \ldots \\ &+ \left[\lambda b_n + a_2 (\sum_{j_1+j_2=n} b_{j_1} b_{j_2}) + \ldots \right. \\ &\left. + a_{n-1} (\sum_{j_1+\ldots+j_{n-1}=n} b_{j_1} \cdots b_{j_{n-1}}) + a_n \right] z^n + \ldots \end{aligned}$$

Also

$$g(\lambda z) = \lambda z + \lambda^2 b_2 z^2 + \ldots + \lambda^n b_n z^n + \ldots$$

Therefore equating the two expressions, we obtain

$$\lambda^n b_n = \lambda b_n + a_2 (\sum_{j_1+j_2=n} b_{j_1} b_{j_2}) + \ldots + a_{n-1} (\sum_{j_1+\ldots+j_{n-1}=n} b_{j_1} \cdots b_{j_{n-1}}) + a_n,$$

from which (1.2) follows. Finally, Proposition 1.8 assures that this is the only possible expression for g. □

The intertwining map, if convergent, is univalent in the elliptic case:

LEMMA 1.10. *Let $f(z) = \lambda z + \sum_{j\geq 2} a_j z^j \in \mathsf{Diff}(\mathbb{C}, O)$ with $\lambda = e^{2\pi i \theta}$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $h \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ with $h(0) = 0$, $h'(0) = 1$ be such that $f(h(z)) = h(\lambda z)$. Let $r > 0$ and assume that h is holomorphic on the disc $\mathbb{D}_r$. Then h is univalent on $\mathbb{D}_r$.*

PROOF. Assume $z_1, z_2 \in \mathbb{D}_r$, $z_1 \neq z_2$ and $h(z_1) = h(z_2)$. Since $f(h(z_1)) = f(h(z_2))$ then by the functional equation $f(h(z)) = h(\lambda z)$ it follows $h(\lambda^n z_1) = h(\lambda^n z_2)$ for all $n = 0, 1, \ldots$. But $\{\lambda^n\}$ is dense in $\partial\mathbb{D}$, hence $h(e^{\eta i} z_1) = h(e^{\eta i} z_2)$ for all $\eta \in \mathbb{R}$. Now $A : \zeta \mapsto h(\zeta z_1) - h(\zeta z_2)$ is a holomorphic function on a neighborhood of $\mathbb{D}$ such that $A|_{\partial\mathbb{D}} \equiv 0$. Thus $H \equiv 0$ on $\mathbb{D}$. Therefore $h(\zeta z_1) = h(\zeta z_2)$ for all $\zeta \in \mathbb{D}$. In particular h is not injective in any neighborhood of 0, against our assumption that $h'(0) = 1$. □

1.3. Formal normal forms in the parabolic case. In this subsection we examine the parabolic case. In this case the homological equation (1.1) has no solution if $\lambda^{k-1} = 1$. If this happens, we say that λ has a *resonance* in degree k. However, it might be possible that the coefficients which generate a resonance are already zero and then the linearization process works. This case is simply characterized:

PROPOSITION 1.11. *Let $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be a germ of parabolic type. Then $\widehat{f}$ is formally linearizable if and only if there exists $m \in \mathbb{N}$ such that $f^{\circ m} = \mathsf{id}$.*

PROOF. Assume first that $\widehat{f}(z) = \lambda z + O(z^2)$ is formally linearizable. Since $\widehat{f}$ is parabolic, there exists $m \in \mathbb{N}$ such that $\lambda^m = 1$. Thus there exists $\widehat{g} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ such that $\lambda z = \widehat{g} \circ \widehat{f} \circ \widehat{g}^{-1}(z)$. Hence

$$z = \lambda^m z = (\widehat{g} \circ \widehat{f} \circ \widehat{g}^{-1})^{\circ m}(z) = \widehat{g} \circ \widehat{f}^{\circ m} \circ \widehat{g}^{-1}(z).$$

This implies that $\mathsf{id} = \widehat{g} \circ \widehat{f}^{\circ m} \circ \widehat{g}^{-1}$. Composing on the left with $\widehat{g}^{-1}$ and on the right with $\widehat{g}$ this yields $\widehat{f}^{\circ m} = \mathsf{id}$.

Conversely, assume that there exists $m \in \mathbb{N}$ such that $\widehat{f}^{\circ m} = \mathsf{id}$. If $\widehat{f}(z) = \lambda z + O(z^2)$ then $\widehat{f}^{\circ m}(z) = \lambda^m z + O(z^2)$ and by hypothesis it follows that $\lambda^m = 1$. Now define

$$\widehat{g}(z) := \frac{1}{m} \sum_{j=0}^{m-1} \frac{\widehat{f}^{\circ j}(z)}{\lambda^j}.$$

Then, taking into account that $\widehat{f}^{\circ m} = \mathsf{id}$ and $\lambda^m = 1$, we have

$$\widehat{g} \circ \widehat{f}(z) = \frac{1}{m} \sum_{j=0}^{m-1} \frac{f^{\circ(j+1)}(z)}{\lambda^j} = \frac{1}{m} \sum_{j=1}^{m} \lambda \frac{f^{\circ j}(z)}{\lambda^j} = \frac{1}{m} \sum_{j=0}^{m-1} \lambda \frac{f^{\circ j}(z)}{\lambda^j} = \lambda \widehat{g}(z),$$

hence $\widehat{f}$ is linearizable. □

REMARK 1.12. Proposition 1.11 holds also in the holomorphic context, *i.e.*, $f \in \mathsf{Diff}(\mathbb{C}, O)$ parabolic is holomorphic linearizable if and only if there exists $m \in \mathbb{N}$ such that $f^{\circ m} = \mathsf{id}$. This follows from the same proof.

In the non-linearizable case it is however possible to obtain a simpler normal form.

THEOREM 1.13. *Let $\widehat{f}(z) := \lambda z + O(z^2) \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be a parabolic germ with $\lambda^m = 1$ and $\lambda^j \neq 1$ for $j = 1, \dots, m-1$. If $f^{\circ m} \neq \mathsf{id}$ then there exist $n \in \mathbb{N}$ and $a \in \mathbb{C}$ such that $\widehat{f}$ is formally conjugated to*

$$\lambda z + z^{mn+1} + a z^{2mn+1}.$$

Moreover, n and a are uniquely determined by the class of formal conjugation of $\widehat{f}$ in $\widehat{\mathsf{Diff}}(\mathbb{C}, O)$.

Moreover,

$$a = \frac{1}{2\pi i} \int_\gamma \frac{dz}{\lambda z - f(z)},$$

where γ is a positively oriented small loop around the origin.

PROOF. Let $\widehat{f}(z) = \lambda z + a_j z^j + O(z^{j+1})$ with $a_j \neq 0$ being the first non-zero coefficient. If λ has no resonance in degree j then the homological equation (1.1) can be solved and $\widehat{f}$ can be conjugated to a map of the type $\lambda z + O(z^k)$ for some $k > j$. Since by hypothesis $f^{\circ m} \neq \mathsf{id}$ and thus by Proposition 1.11 it is not linearizable, after a finite number of steps we have to encounter a resonant term. Notice that, being m the order of λ then such a resonant term must be of degree $nm + 1$ for some $n \geq 1$. We can thus assume that

$$\widehat{f}(z) = \lambda z + a_{mn+1} z^{mn+1} + O(z^{mn+2})$$

with $a_{nm+1} \neq 0$.

First of all, let conjugate $\widehat{f}$ with $D(z) = \alpha z$ for some $\alpha \in \mathbb{C} \setminus \{0\}$ to be chosen later. Then

$$\begin{aligned} D^{-1} \circ f \circ D(z) &= \frac{1}{\alpha}(\lambda \alpha z + a_{mn+1}\alpha^{mn+1}z^{mn+1} + O(z^{mn+2}) \\ &= \lambda z + a_{mn+1}\alpha^{mn+1}z^{mn+1} + O(z^{mn+2}). \end{aligned}$$

Choosing α such that $a_{mn+1}\alpha^{mn+1} = 1$, we have that $\widehat{f}$ is conjugated to

$$\widehat{f}_1(z) = \lambda z + z^{mn+1} + a_j z^j + O(z^{j+1}),$$

for some $j > mn + 1$ and $a_j \neq 0$. In order to dispose of the term of degree j we look for a germ of the form $T(z) = z + \alpha z^h$ which solves the functional equation

$$T \circ \widehat{f}_1(z) = \lambda T(z) + T(z)^{mn+1} + O(z^{j+1}). \tag{1.3}$$

Expanding we obtain

$$\begin{aligned} &\lambda z + z^{mn+1} + a_j z^j + \alpha(\lambda z + z^{mn+1} + a_j z^j)^h + O(z^{j+1}) \\ &= \lambda z + \lambda \alpha z^h + (z + \alpha z^h)^{mn+1} + O(z^{j+1}) \end{aligned}$$

that is,

$$\begin{aligned} &\lambda z + z^{mn+1} + a_j z^j + \alpha \sum_{k=0}^{h} \binom{h}{k} \lambda^k z^{(mn+1)h-kmn} + O(z^{j+1}) \\ &= \lambda z + \lambda \alpha z^h + \sum_{k=0}^{mn+1} \binom{mn+1}{k} \alpha^{mn+1-k} z^{(mn+1)h-k(h-1)} + O(z^{j+1}) \end{aligned}$$

namely,

$$\begin{aligned} &\lambda z + z^{mn+1} + a_j z^j + \alpha \lambda^h z^h + \alpha h \lambda^{h-1} z^{h+mn} + O(z^{j+1}, z^{h+2mn}) \\ &= \lambda z + \lambda z^h + z^{mn+1} + (mn+1)\alpha z^{mn+h} + O(z^{j+1}, z^{mn+2h-1}). \end{aligned}$$

Canceling and collecting terms we get

$$\begin{aligned} a_j z^j + &\alpha(\lambda^h - \lambda) z^h + \alpha[\lambda^{h-1} h - (mn+1)] z^{mn+h} \\ &= 0 + O(z^{j+1}, z^{h+2mn}, z^{mn+2h-1}). \end{aligned}$$

From this we see if $j \neq qmn+1$ for any $q \in \mathbb{N}$ then choosing $h = j$ we get the homological equation $a_j + \alpha(\lambda^h - \lambda) = 0$ which has a unique solution α, and therefore we can solve (1.3).

In case $j = qmn + 1$ for some $q \in \mathbb{N}$, the choice $h = qmn + 1$ does not allow to solve the corresponding homological equation. However, if we let $h = pmn + 1$ for some $p \in \mathbb{N}$ to be chosen later, we obtain

$$a_{qmn+1} z^{qmn+1} + \alpha(p-1)mn z^{(p+1)mn+1} = 0 + O(z^{qmn+2}).$$

In order to solve such an equation we need to set $p + 1 = q$. This leads us to solve the linear equation in α

$$a_{qmn+1} + \alpha(p-1)mn = 0,$$

which has a unique solution if and only if $p \neq 1$, namely $q \neq 2$.

Summing up, we proved that if j is not a resonance degree for λ than we can dispose of a_j. Also, if $j = qmn+1$ with $q > 2$ then we can dispose of a_{qmn+1}. But, if $j = 2mn+1$ then we cannot dispose of a_{2mn+1} which is thus an invariant, let denote it with $a \in \mathbb{C}$.

Therefore, we can conjugate $\widehat{f}$ to a germ of the form

$$\widehat{f}_2(z) = \lambda z + z^{mn+1} + az^{2mn+1} + a_j z^j + O(z^{j+1})$$

with $a_j \neq 0$, $j > 2mn+1$. Now, arguing as before, in case $j \neq qmn+1$ for any $q \in \mathbb{N}$ we can use a germ of the form $z+\alpha z^j$ to dispose of a_j. If $j = qmn+1$ for some $q > 2$ we can use a germ of the form $z + \alpha z^{(q-1)mn+1}$ to dispose of a_j. Continue this way, composing the (infinite) conjugations, we are done.

Now we need to show that n and a depend only on the class of formal conjugation of $\widehat{f}$. Indeed, by construction, the normal form of $\widehat{f}$ is unique and since conjugation is transitive, it depends only on the class of conjugation of $\widehat{f}$.

Finally, given $\widehat{f}(z) = \lambda z + z^{r+1} + az^{2r+1}$, for some $r \geq 1$, we have

$$\frac{1}{\lambda z - f(z)} = -\frac{1}{z^{r+1}}\frac{1}{1+az^{2r+1}} = -\frac{1}{z^{r+1}}(1 - az^{2r+1} + o(|z|^{2r+1})$$
$$= \frac{-1}{z^{r+1}} + \frac{a}{z} + h(z),$$

where $h(z)$ is holomorphic in a neighborhood of the origin. From this it follows that

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{\lambda z - f(z)} = \frac{1}{2\pi i}\int_\gamma [\frac{-1}{z^{r+1}} + \frac{a}{z} + h(z)] = a,$$

as claimed. □

REMARK 1.14. From the proof of Theorem 1.13 it follows that if $f(z) := \lambda z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$ is a parabolic germ with $\lambda^m = 1$ and $\lambda^j \neq 1$ for $j = 1, \dots, m-1$ with $f^{\circ m} \neq \mathsf{id}$, then for every fixed $t >> 1$, f is *holomorphically* conjugated to

$$\lambda z + z^{mn+1} + az^{2mn+1} + O(|z|^t).$$

DEFINITION 1.15. Let $f(z) := \lambda z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$ be a parabolic germ. The number

$$\iota(f, 0) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{\lambda z - f(z)},$$

is called the *parabolic index* of f at 0.

As follows from Theorem 1.13, the parabolic index of a parabolic germ is a formal invariant hence it is a holomorphic invariant, namely, if f, g are two parabolic germs which are holomorphically (hence formally) conjugated, then the parabolic index is the same. This fact can be showed directly (see [**19**] or [**1**]).

REMARK 1.16. As the reader familiar with complex dynamics could recognize, our definition of parabolic index coincides with the usual notion of holomorphic index (see, *e.g.*, [**19**]) of a germ of holomorphic map only in case $\lambda = 1$. Indeed, the *holomorphic index* of $f \in \mathsf{Diff}(\mathbb{C}, O)$ is defined as

$$o(f, 0) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - f(z)}.$$

This number is equal to $1/(1-\lambda)$ provided $\lambda \neq 1$ while it equals $\iota(f, 0)$ in case $\lambda = 1$. See [**19**] for details. For what we are concerned about, we just need the parabolic index introduced above.

The parabolic index is useful in the study of rational (and transcendental) dynamics to estimate the number of non-repelling periodic cycles in terms of the number of critical points (see [**16**], [**4**], [**8**]).

DEFINITION 1.17. A parabolic germ $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ is *tangent to the identity* if $\widehat{f}(z) = z + O(z^2)$.

REMARK 1.18. If $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2}) \in \mathsf{Diff}(\mathbb{C}, O)$ with $a_{k+1} \neq 0$, then $k+1$ is the first non-resonant term and, arguing as in the proof of Theorem 1.13, we can find a holomorphic conjugation to a holomorphic germ of the form $z \mapsto z - z^{k+1} + az^{2k+1} + O(z^h)$ with h as big as we like.

For $k \in \mathbb{N}$ and $a \in \mathbb{C}$, let us denote

$$f_{k,a}(z) := z + z^{k+1} + az^{2k+1}. \tag{1.4}$$

Also, for $\lambda \in \mathbb{R}$ we denote

$$R_\lambda(z) := e^{2\pi i\lambda} z \tag{1.5}$$

By Theorem 1.13 every germ of diffeomorphism tangent to the identity is formally conjugated at one (and only one) $f_{k,a}$ for some $k \in \mathbb{N}$ and $a \in \mathbb{C}$.

REMARK 1.19. Let $\widehat{f}(z) = z + z^{k+1} + az^{2k+1} + O(z^{2k+2}) \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$. From the proof of Theorem 1.13 it follows that $\widehat{f}$ is formally conjugated to $f_{k,a}$, since the conjugation exploited to dispose of the tail $O(z^{2k+2})$ does not effect the previous terms.

Also, Proposition 1.11 and Theorem 1.13 can be rephrased as follows:

COROLLARY 1.20. *Let $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be a parabolic germ of diffeomorphism, with $\widehat{f}(z) = e^{2\pi i p/q}z + O(z^2)$ for some $p/q \in \mathbb{Q}$. Then*

(1) *either $\widehat{f}$ is formally conjugated to $R_{p/q}$, and this is the case if and only if $\widehat{f}^q = \mathsf{id}$,*
(2) *or $\widehat{f}$ is formally conjugated to $R_{p/q} \circ f_{mq,\alpha} = f_{mq,\alpha} \circ R_{p/q}$ for some $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$.*

PROOF. Case (1) follows from Proposition 1.11. Suppose we are in Case (2). By Theorem 1.13, $\widehat{f}$ is formally conjugated to $e^{2\pi i p/q}z + z^{mq+1} + az^{2mq+1}$. Conjugating such a normal form with $D(z) = Az$ and $A^{mq} = \exp(2\pi i p/q)$, we obtain a new normal form given by $e^{2\pi i p/q}(z + z^{mq+1} + ae^{2\pi i p/q}z^{2mq+1})$. Setting $\alpha = ae^{2\pi i p/q}$ we obtain the assertion. □

The decomposition, up to formal conjugation, of $\widehat{f}$ as $R_{p/q} \circ f_{mq,\alpha}$ can be thought of as a Jordan-type normalization. The two germs $R_{p/q}$ and $f_{mq,\alpha}$ commute under composition, one is nilpotent ($R_{p/q}^q = \mathsf{id}$) and the other is tangent to the identity.

1.4. Germs of vector fields and flows: the formal classification revised. Let X be a germ at O of a holomorphic vector field in $\mathbb{C}$. Namely,

$$X(z) = H(z)\frac{d}{dz}$$

with H being a germ of holomorphic function at O (not necessarily invertible). We will use the following result known as *(holomorphic) flow box theorem*

THEOREM 1.21. *Let Ω be a open set in $\mathbb{C}^n$ and $F : \Omega \to \mathbb{C}^n$ be a holomorphic map. For any compact subset $K \subset \Omega$ there exist $\delta > 0$, a open neighborhood U of K and a unique real analytic map $\Phi : (-\delta, \delta) \times U \to \Omega$, such that $z \mapsto \Phi(t, z)$ is holomorphic for all t fixed and*

$$\begin{cases} \frac{\partial}{\partial t}\Phi(z,t) = F(\Phi(z,t)) \\ \Phi(z,0) = z \end{cases} \tag{1.6}$$

By Theorem 1.21, given a germ of holomorphic vector field X there exist two open neighborhoods Ω, U of O and a map $\Phi : (-\delta, \delta) \times U \to \Omega$ which satisfies $\Phi(z, 0) = z$ and $\frac{\partial}{\partial t}\Phi(z,t) = X(\Phi(z,t))$ for all $t \in (-\delta, \delta)$ and $z \in U$.

For a fixed $t \in (-\delta, \delta)$ the map $\Phi_t : z \mapsto \Phi(z,t)$ is called the *time t flow* of X. Multiplying the vector field X by $\delta/2$, the associated flow is given by re-scaling the time, namely, $\Psi(z,t) := \Phi(z, t\delta/2)$ solves $\Psi(z,0) = z$ and $\frac{\partial}{\partial t}\Psi(z,t) = (\delta/2)X(\Psi(z,t))$. Notice that now the map Ψ is defined in an interval $(-2, 2)$. In particular it is well defined the time 1 flow Ψ_1.

Therefore, up to positive constant multiple, *we will always assume that the time one flow is defined*. It is also important to observe that the time one flow is essentially defined as a germ of holomorphic function at O.

If X is a vector field, its time t flow is sometimes denoted by $\exp(tX)$.

PROPOSITION 1.22. *Let X be a germ at O of a holomorphic vector field in $\mathbb{C}$. Then $\exp(tX)$ fixes O for all t if and only if X is singular at O, i.e., $X(O) = 0$.*

PROOF. Since $\exp(tX)$ is the solution of (1.6), it follows at once from the uniqueness of solutions for ordinary differential equations. □

There is a formula which allows to express the time t flow of a holomorphic vector field with an isolated singularity at O with respect to the vector field itself:

THEOREM 1.23. *Let $X = H(z)\frac{d}{dz}$ be a germ at O of a holomorphic vector field in $\mathbb{C}$. Suppose O is an isolated singularity for X, i.e., $X(O) = 0$. Then*

$$\exp(tX)(z) = z + \sum_{n=1}^{\infty} \frac{t^n}{n!} X^n.z, \tag{1.7}$$

where $X^n.z$ is defined by induction as $X.(X^{n-1}.z)$, with $X.z := H(z)$.

PROOF. By Theorem 1.21, we know that the flow of the vector field X, $G(t,z) := \exp(tX)(z)$, is well defined and holomorphic in z (near O) and real analytic in t (for t small). Moreover, it is the unique such a function that has the property that $G(0,z) = z$ and $\frac{\partial}{\partial t}G(t,z) = X.(G(t,z))$. Expanding in Taylor series the previous equality and equation the coefficient with the same degree in z, we come up with infinitely many differential equation which can be solved by recurrence. This shows that actually $G(t,z)$ is unique also in the category of formal power series with coefficient (convergent and smooth) in t.

The (a priori) formal series $F(t,z) := z + \sum_{n=1}^{\infty} \frac{t^n}{n!} X^n.z$ has convergent coefficients in t. It follows that $F(0,z) = z$ and

$$\begin{aligned}\frac{\partial}{\partial t}F(t,z) &= \sum_{n=1}^{\infty} \frac{\partial}{\partial t}\frac{t^n}{n!} X^n.z = \sum_{n=1}^{\infty} n\frac{t^{n-1}}{n!} X^n.z \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} X.(X^n.z) = X.(F(t,z)),\end{aligned}$$

therefore $\exp(tX)(z) = F(t,z)$, as claimed. □

Theorem 1.23 can be used to find the coefficients of the time one flow of a vector field starting with its expansion at O.

Let $X(z) = (Az + O(z^2))\frac{d}{dz}$. Then

$$X^2.z = (Az + O(z^2))\frac{d}{dz}(Az + O(z^2)) = A^2 z + O(z^2),$$

and a simple induction shows that $X^n.z = A^n z + O(z^2)$. Formula (1.7) implies that

$$\exp(X)(z) = z + \sum_{j=1}^{\infty} \frac{A^n}{n!} z + O(z^2),$$

namely, $\exp(X)'(0) = \exp(A)$.

If $X(z) = Az + Bz^2 + O(z^3)$, then arguing by induction, we find

$$X^n.z = A^n z + a_n A^{n-1} B z^2 + O(z^3),$$

with $a_n = 2a_{n-1} + 1$ and $a_1 = 1$. Therefore (1.7) implies

$$\exp(X)(z) = \exp(A)z + \left(1 + \sum_{n=2}^{\infty} \frac{a_n A^{n-1}}{n!}\right) B z^2 + O(z^3) \tag{1.8}$$

Notice that if $A = 0$ then $\exp(X)(z) = z + Bz^2 + O(z^3)$. In fact, if $X(z) = O(z^2)\frac{d}{dz}$ then the expansion of its time one flow can be obtained by polynomial equations in the coefficients of X, more precisely

PROPOSITION 1.24. *Let $X(z) = H(z)\frac{d}{dz}$ be a germ of holomorphic vector field such that $H(0) = H'(0) = 0$. Let $H(z) = \sum_{j=K}^{\infty} A_j z^j$, with $K \geq 2$ and $A_K \neq 0$. Let $f(z) = \exp(X)(z)$. Assume that $f(z) = \sum_{j=0}^{\infty} a_j z^j$. Then*

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_j = 0, \quad j = 2, \dots, K-1 \\ a_K = A_K, \\ a_j = A_j + P_j(A_K, \dots, A_{j-1}), \quad j \geq K+1 \end{cases} \tag{1.9}$$

where $P_j(x_K, \dots, x_{j-1})$ is a polynomial in $x_K, \dots, x_{j-1}$.

PROOF. According to (1.7) we have

$$\sum_{j=0}^{\infty} a_j z^j = z + \sum_{n=1}^{\infty} \frac{1}{n!} X^n.z.$$

Now, writing $H(z) = A_K z^K + O(z^K)$, since $K \geq 2$, we see that $X.(X^{n-1}.z) = O(z^{K+1})$ for $n \geq 2$. Therefore

$$z + \sum_{n=1}^{\infty} \frac{1}{n!} X^n.z = z + X.z + O(z^{K+1}) = z + A_K z^K + O(z^{K+1}).$$

Let $r, s, t \in \mathbb{N}$ with for $K-1 \leq r < s$ and $r \leq t \leq s$. We will denote by $L_r^s(t)$ any polynomial in z of degree greater than or equal to r and less than or equal to s with coefficients given by polynomials in $A_K, \dots, A_t$. With this notation, in order to prove (1.9) it is enough to prove that, for all $j \geq K+1$ we can write

$$X^n.z = L_K^j(j-1) + O(z^{j+1}), \quad n \geq 2. \tag{1.10}$$

First, for $n = 2$ we compute

$$\begin{aligned} X^2.z = X.(X.z) = (A_K z^K + \ldots + A_j z^j \\ + O(z^{j+1}))\frac{d}{dz}(A_K z^K + \ldots + A_j z^j + O(z^{j+1})) \\ = (A_K z^K + \ldots + A_j z^j + O(z^{j+1}))(K A_K z^{K-1} + \ldots + j A_j z^{j-1} + O(z^j)) \\ = L^{2j-3}_{2K-1}(j-1) + (j+K) A_K A_j z^{K+j-1} + O(z^{j+1}). \end{aligned}$$

Since $K \geq 2$, it follows that $z^{K+j-1} = O(z^{j+1})$ and therefore $X^2.z = L^j_K(j-1) + O(z^{j+1})$ as claimed. Now, in order to prove (1.10) by induction, we assume that it holds for n and we prove it is true for $n+1$. We have $X.z := H(z) = L^{j-1}_K(j-1) + A_j z^j + O(z^{j+1})$. Therefore

$$\begin{aligned} X^{n+1}.z = & X.(X^n.z) = (L^{j-1}_K(j-1) + A_j z^j \\ & + O(z^{j+1}))\frac{d}{dz}[L^j_K(j-1) + O(z^{j+1})] \\ & = (L^{j-1}_K(j-1) + A_j z^j + O(z^{j+1})) \cdot (L^{j-1}_{K-1}(j-1) + O(z^j)) \\ & = L^{2j-2}_{2K-1}(j-1) + A_j z^j L^{j-1}_{K-1}(j-1) + O(z^{j+1}) \\ & = L^j_K(j-1) + A_j L^{2j-1}_{K+j-1}(j-1) + O(z^{j+1}) = L^j_K(j-1) + O(z^{j+1}), \end{aligned}$$

which proves (1.10), and we are done. □

DEFINITION 1.25. A *formal vector field* $X(z)$ is given by $X(z) = \sum_{j=0}^{\infty} A_j z^j \frac{d}{dz}$ where $\sum_{j=1}^{\infty} A_j z^j$ is a formal power series. The formal vector field X is *one flat* if $A_0 = A_1 = 0$.

The time one flow of a one-flat formal vector field is defined as

$$\exp(X)(z) := z + \sum_{n=1}^{\infty} \frac{1}{n!} X^n.z, \tag{1.11}$$

where $X^n.z := X.(X^{n-1}.z)$ and, if $\widehat{f}$ is a formal diffeomorphism, $X.\widehat{f}$ is the formal derivation terms by terms of $\widehat{f}$ by X.

PROPOSITION 1.26. *The flow of a one-flat formal vector field is well defined. Moreover, there is a one-to-one correspondence between one-flat formal vector fields and formal diffeomorphisms tangent to the identity.*

PROOF. The proof of Proposition 1.24 applies also to the formal case, and then (1.9) holds. From this it follows that $\exp(X)$ is well defined and that the application $X \mapsto \exp(X)$ can be inverted. □

REMARK 1.27. It is worth noticing that, even if $f \in \mathsf{Diff}(\mathbb{C}, O)$—namely if f is holomorphic and not just formal, then the only vector field X such that $f = \exp(X)$, may not be holomorphic.

THEOREM 1.28. *Let $\widehat{f} \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$ be a germ of diffeomorphism tangent to the identity. Then there exist $k \in \mathbb{N}$ and $a \in \mathbb{C}$ such that $\widehat{f}$ is formally conjugated to the time one flow of the vector field*

$$X_{k,a}(z) := \frac{z^{k+1}}{1 - a z^k} \frac{d}{dz}.$$

Moreover, k and a are univocally determined by $\widehat{f}$ and depend only on its class of conjugation in $\widehat{\mathsf{Diff}}(\mathbb{C},O)$.

PROOF. By Theorem 1.13 there exist k,α such that $\widehat{f}$ is formally conjugated to a diffeomorphism $f_{k,\alpha}$ of the form (1.4). Let $a=\alpha-(k+1)/2$. Expanding $X_{k,a}$ in series, we obtain

$$\begin{aligned}X_{k,a}(z) :=& z^{k+1}(1+az^k+a^2z^{2k}+O(z^{2k+1}))\frac{d}{dz}\\ =& (z^{k+1}+az^{2k+1}+O(z^{2k+2}))\frac{d}{dz}.\end{aligned}$$

Therefore,

$$\begin{aligned}X_{k,a}^2.z &= X_{k,a}.(X_{k,a}.z) = (z^{k+1}+az^{2k+1}+O(z^{2k+2}))((k+1)z^k+O(z^{2k}))\\ &= (k+1)z^{2k+1}+O(z^{3k+1}),\end{aligned}$$

and

$$X_{k,a}^3.z = X_{k,a}.(X_{k,a}^2.z) = (z^{k+1}+az^{2k+1}+O(z^{2k+2}))O(z^{2k}) = O(z^{3k+1}).$$

From this, using induction, it is easy to show that

$$X_{k,a}^n.z = O(z^{nk+1}), \quad n\geq 3. \tag{1.12}$$

By the very definition (1.11) and from (1.12) it follows that

$$\begin{aligned}\exp(X_{k,a})(z) &= z+X_{k,a}.z+\frac{1}{2}X_{k,a}^2.z+O(z^3k+1)\\ &= z+z^{k+1}+az^{2k+1}+\frac{1}{2}(k+1)z^{2k+1}+O(z^{2k+2})\\ &= z+z^{k+1}+(a+\frac{k+1}{2})z^{2k+1}+O(z^{2k+2}).\end{aligned}$$

By Theorem 1.13 and Remark 1.19 the flow $\exp(X_{k,a})$ is formally conjugated to $z+z^{k+1}+(a+\frac{k+1}{2})z^{2k+1}=z+z^{k+1}+\alpha z^{2k+1}$. Therefore, by the uniqueness in Theorem 1.13, $\widehat{f}$ is formally conjugated to $\exp(X_{k,a})$.

Finally, the univocally dependence of k,a on the class of conjugation of $\widehat{f}$ is clear from the previous construction. □

From Theorem 1.6, Corollary 1.20 and Theorem 1.28 we can rephrase the formal classification in the following way:

THEOREM 1.29. *Let $\widehat{f}(z)=\lambda z+O(z^2)\in\widehat{\mathsf{Diff}}(\mathbb{C},O)$. Then*

(1) *either $\widehat{f}$ is formally conjugated to the time one flow of the linear vector field $X_\lambda(z):=\lambda z\frac{d}{dz}$,*
(2) *or $R_\lambda^{-1}\circ\widehat{f}$ is formally conjugated to the time one flow of the holomorphic vector field $X_{k,a}(z):=\frac{z^{k+1}}{1-az^k}\frac{d}{dz}$.*

2. Holomorphic Dynamics

2.1. The hyperbolic case.

THEOREM 2.1. *Let $f(z)=\lambda z+O(z^2)\in\mathsf{Diff}(\mathbb{C},O)$. Suppose $0<|\lambda|<1$. Then there exists a unique $\sigma\in\mathsf{Diff}(\mathbb{C},O)$ with $\sigma'(0)=1$ such that*

$$\sigma\circ f=\lambda\sigma \tag{2.1}$$

PROOF. Let us define

$$\sigma_n(z) := \frac{f^{\circ n}(z)}{\lambda^n}.$$

Then $\{\sigma_n\}$ is a sequence of germs of holomorphic diffeomorphisms. We claim that $\{\sigma_n\}$ converges uniformly on compacta to a holomorphic function σ. Assuming the claim, since $\sigma_n(0) = 0$ and

$$\sigma_n'(0) = \frac{(f^{\circ n})'(0)}{\lambda^n} = \frac{\lambda^n}{\lambda^n} = 1,$$

it follows that $\sigma(0) = 0$ and $\sigma'(0) = 1$. Moreover, by the very definition,

$$\sigma_n \circ f(z) = \lambda \frac{f^{\circ n}(f(z))}{\lambda^{n+1}} = \lambda \frac{f^{\circ n+1}(z)}{\lambda^{n+1}} = \lambda \sigma_{n+1}(z).$$

Hence, taking the limit, $\sigma \circ f = \lambda \sigma$ as needed.

It remains to prove the claim, that is, $\{\sigma_n\}$ converges uniformly on compacta to σ (which must be necessarily holomorphic). This is equivalent to show that the series $\sum_{j=0}^{\infty}[\sigma_{j+1}(z) - \sigma_j(z)]$ converges uniformly on compacta. To this aim, since

$$|f(z) - \lambda z| = O(|z|^2),$$

there exists $\delta > 0$ and $C > 0$ such that

$$|f(z) - \lambda z| \leq C|z|^2, \quad \forall z : |z| \leq \delta. \tag{2.2}$$

Hence, for all z such that $|z| \leq \delta$ it holds

$$|f(z)| \leq |\lambda||z| + C|z|^2 \leq (|\lambda| + C\delta)|z|. \tag{2.3}$$

Since $|\lambda| < 1$, it is possible to choose δ so small that

$$\begin{cases} |\lambda| + C\delta < 1, \\ (|\lambda| + C\delta)^2 < |\lambda| \end{cases} \tag{2.4}$$

In particular, from (2.3) it follows that if $|z| \leq \delta$ then $|f(z)| < \delta$. Thus we can apply (2.3) recursively to obtain, for $|z| \leq \delta$

$$|f^{\circ n}(z)| \leq (|\lambda| + C\delta)|f^{\circ n-1}(z)| \leq \ldots \leq (|\lambda| + C\delta)^n|z|. \tag{2.5}$$

Therefore, for $|z| \leq \delta$,

$$\begin{aligned} |\sigma_{n+1}(z) - \sigma_n(z)| &= \frac{1}{|\lambda|^{n+1}}|f(f^{\circ n}(z)) - \lambda f^{\circ n}(z)| \overset{(2.2)}{\leq} \frac{C}{|\lambda|^{n+1}}|f^{\circ n}(z)|^2 \\ &\overset{(2.5)}{\leq} \frac{1}{|\lambda|^{n+1}}C(|\lambda| + C\delta)^{2n}|z|^2 = \left[\frac{(|\lambda| + C\delta)^2}{|\lambda|}\right]^n \frac{C}{|\lambda|}|z|^2 \end{aligned} \tag{2.6}$$

Let $\epsilon := (|\lambda| + C\delta)^2/|\lambda|$. By (2.4), $\epsilon < 1$. Therefore, by (2.6),

$$\sum_{j=0}^{\infty} |\sigma_{j+1}(z) - \sigma_j(z)| \leq \sum_{j=0}^{\infty} \frac{C\epsilon^j}{|\lambda|}|z|^2 = \frac{C}{(1-\epsilon)|\lambda|}|z|^2,$$

and the series is uniformly convergent on compacta in $|z| \leq \delta$, as claimed.

Finally, the uniqueness of σ follows from Proposition 1.8 in Section 1. □

The functional equation (2.1) is known as *Schröder's equation*. The map σ is known as the *Königs intertwining map*.

REMARK 2.2. From the proof of Theorem 2.1, it follows that if $f_t(z) = \lambda_t z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$ is a family depending on an analytic parameter t, with $0 < \delta_0 < |\lambda_t| < \delta_1 < 1$ for all t and for some $\delta_0, \delta_1 > 0$, then the Königs intertwining map σ_t which solves $\sigma_t \circ f_t = \lambda_t \sigma_t$ depends analytically on t as well. Indeed, from (2.6) it follows that the series which converges to σ_t is uniformly convergent in t as well.

Theorem 2.1 has a straightforward corollary:

COROLLARY 2.3. *Let $f(z) = \lambda z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$. Suppose $|\lambda| > 1$. Then there exists a unique $\psi \in \mathsf{Diff}(\mathbb{C}, O)$ with $\psi'(0) = 1$ such that*

$$\psi \circ f = \lambda \psi. \tag{2.7}$$

PROOF. The inverse of f is $f^{-1}(z) = \frac{1}{\lambda} z + O(z^2)$. By Theorem 2.1 there exists a unique $\sigma \in \mathsf{Diff}(\mathbb{C}, O)$ with $\sigma'(0) = 1$ such that $\sigma \circ f^{-1} = \lambda^{-1}\sigma$. Therefore $\lambda\sigma = \sigma \circ f$. □

2.2. The elliptic case: Cremer, Siegel diffeomorphisms and small divisors. In the previous subsection we already proved that elliptic diffeomorphisms are always *formally* linearizable. As we will see, the holomorphic linearization is not always possible. Proposition 1.9 gives a first hint on the underlying reason: the coefficients of the expansion of the intertwining map are multiples of $|\lambda^n - \lambda|^{-1}$. Therefore, in order to make the series converging, the factor $|\lambda^n - \lambda|^{-1}$ should not tend to zero "too fast". This problem is known as the *small divisors problem*.

Let $f_\theta(z) = e^{2\pi i\theta} z + O(z^2)$, with $\theta \in [0, 1)$. We will see that for almost all (with respect to the Lebesgue measure) θ, the germ f_θ is holomorphically linearizable. On the other hand, for a *generic*[1] choice of θ, one can find a germ f_θ which is not holomorphically linearizable.

DEFINITION 2.4. Let $f \in \mathsf{Diff}(\mathbb{C}, O)$ be a germ of an elliptic diffeomorphism. We say that f is a *Siegel diffeomorphism at O* if f is holomorphically linearizable. On the other hand, we say that f is a *Cremer diffeomorphism at O* if f is not holomorphically linearizable.

We start proving existence of Cremer's germs, using a geometric criterion. Before that, we need some terminology.

DEFINITION 2.5. Let $f \in \mathsf{Diff}(\mathbb{C}, O)$ be elliptic. A *small cycle* of f is a finite set of points $\{z_0, \ldots, z_m\}$ such that $z_j \neq z_k$ for $j \neq k$ and $f(z_j) = z_{j+1}$ for $j \in \{0, \ldots, m-1\}$ and $f(z_m) = z_0$. The number m is called the *length* of the small cycle.

REMARK 2.6. Assume that $f(z) = \lambda z + O(z^2)$ is a Siegel diffeomorphism. Then there exists $\sigma \in \mathsf{Diff}(\mathbb{C}, O)$, such that $\sigma \circ f = \lambda\sigma$. Since the map $\mathbb{C} \ni \zeta \mapsto \lambda\zeta$ has no small cycles because $\lambda^n \neq 1$ for all $n \in \mathbb{N}$ since f is elliptic, it follows that there exists a neighborhood of the origin which contains no small cycles of f.

THEOREM 2.7. *There exists a countable intersection $\mathcal{L}$ of dense subsets of irrational numbers in $[0, 1)$ such that for every $\theta \in \mathcal{L}$ there exists a Cremer diffeomorphism $f_\theta(z) = e^{2\pi i\theta} z + O(z^2)$ with the property that for any open neighborhood U of the origin, there exist infinitely many small cycles of f_θ contained in U.*

[1] namely, for θ chosen in a countable intersection of dense subsets of irrational numbers in $[0, 1)$

PROOF. Let $g(z) = e^{2\pi i p/q} z + z^k$, with p and q relatively prime, $k \geq 2$. Then O is a fixed point of $g^{\circ q}$ with multiplicity k because $g^{\circ q}(z) - z = O(z^k)$. Let $r > 0$ and $\mathbb{D}(O, r)$ be the disc of center O and radius r. Let $\epsilon = \epsilon(r) > 0$ be a number, to be suitably chosen later and $t \mapsto \mu_t := e^{2\pi i (p+t)/q}$ for $t \in [-\epsilon, \epsilon]$. Finally, set

$$g_t(z) := \mu_t z + z^k.$$

For $t \approx 0$ it follows that $|g_t - g| << 1$. Therefore, if ϵ is chosen small enough, then $|g_t^{\circ q}(z) - g^{\circ q}(z)| < |g^{\circ q}(z) - z|$ for all $t \in [0, \epsilon]$ and $|z| = r$. Thus by Rouché theorem, $g^{\circ q}$ and $g_t^{\circ q}$ have the same number of fixed points (counting multiplicity) in $\mathbb{D}(0, r)$. For t irrational, $g_t^{\circ q}(z) - z$ has multiplicity 1 at O, thus there exist $k - 1$ small cycles of g_t of length (at most) q in $\mathbb{D}(0, r)$.

Let $\mathcal{L}_r$ be the set of irrational numbers θ in $[0, 1)$ such that $e^{2\pi i \theta} z + z^k$ has a small cycle in $\mathbb{D}(0, r)$. By the previous argument, for each p, q relatively prime, there exists a open neighborhood $U_{p/q}$ such that for all irrational numbers $t \in U_{p/q}$ the germ $e^{2\pi i t} z + z^k$ has small cycles in $\mathbb{D}(O, r)$. Therefore, $\mathcal{L}_r$ is open and dense in the set of irrational numbers in $[0, 1)$. The set $\mathcal{L} := \cap_{r>0, r\in\mathbb{Q}} \mathcal{L}_r$ is dense by Baire's theorem and every germ of the form $e^{2\pi i \theta} z + z^k$ with $\theta \in \mathcal{L}$ has the property stated in the theorem. □

The proof of the previous theorem shows that every irrational number t which is "well approximated" by rational numbers has the property that $e^{2\pi i t} z + z^2$ has small cycles which accumulate to the origin (and in particular it is a Cremer diffeomorphism). The arithmetic properties of the number t play a fundamental role in the distinction between Cremer and Siegel diffeomorphisms. Before seeing some instance of Siegel's diffeomorphisms, we give another criterion for Cremer's diffeomorphisms.

As a matter of notation, if $x \in \mathbb{R}$ and $[x]$ denotes the integer part of x then we will denote with

$$\{x\} := x - [x].$$

THEOREM 2.8. *Let $a \in [0, 1] \setminus \mathbb{Q}$ be such that*

$$\limsup_{n\to\infty} (\{na\})^{-1/n} = \infty. \tag{2.8}$$

Then there exists $f(z) = e^{2\pi i a} z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$ which is a Cremer diffeomorphism.

PROOF. Let $\lambda := e^{2\pi i a}$. Then

$$|\lambda^n - 1|^2 = |\cos(2\pi a n) + i \sin(2\pi a n) - 1|^2 = 4 \sin^2(\pi a n),$$

hence,

$$\begin{aligned} |\lambda^n - 1| &= 2|\sin(\pi a n)| = 2|\sin(\pi[an] + \pi\{an\})| \\ &= 2|\sin(\pi\{an\})| = 2\{an\} + o(\{an\}). \end{aligned}$$

Therefore (2.8) is equivalent to

$$\limsup_{n\to\infty} |\lambda^n - 1|^{-1/n} = \infty. \tag{2.9}$$

Let us now define $f(z) = \lambda z + \sum_{j\geq 2} a_j z^j$, with $a_j := e^{2\pi i \theta_j}$. We let $\theta_2 := 0$, $\theta_3 := \arg(a_2)$ and more generally, we let

$$\theta_n := \arg \left[\sum_{j=2}^{n-1} (a_j \sum_{k_1+\ldots+k_j=n} b_{k_1} \cdots b_{k_j}) \right],$$

where b_n is defined by (1.2) in Section 1 (so that $b_1 = 1, b_2 = (\lambda^2 - \lambda)^{-1}a_2$, $b_3 = (2a_2b_2 + a_3)/(\lambda^3 - \lambda)$ and so on).

Let $g(z) = z + \sum_{j=2}^{\infty} b_j z^j \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$. By the very definition and by Proposition 1.1.9, it follows that $g^{-1} \circ f = \lambda g^{-1}$. We claim that

$$|b_n| \geq \frac{1}{|\lambda^{n-1} - 1|}. \tag{2.10}$$

Indeed, by we have

$$(\lambda^n - \lambda)b_n = \sum_{j=2}^{n-1}(a_j \sum_{k_1+\ldots+k_j=n} b_{k_1}\cdots b_{k_j}) + a_n =: A_n + a_n,$$

and, by construction, $\arg A_n = \arg a_n$. Namely, a_n and A_n belongs to the same half line from the origin. Therefore

$$|A_n + a_n| \geq |a_n| = 1.$$

From this we obtain

$$|b_n| = \frac{|A_n + a_n|}{|\lambda^n - \lambda|} \geq \frac{|a_n|}{|\lambda||\lambda^{n-1} - 1|} = \frac{1}{|\lambda^{n-1} - 1|},$$

and (2.10) holds.

Inequality (2.10), together with (2.9), implies that

$$\limsup_{n\to\infty} |b_n|^{1/n} = +\infty,$$

which means that the radius of convergence of the series $\sum_{j=2}^{\infty} b_n z^n$ is 0; that is $g \in \widehat{\mathsf{Diff}}(\mathbb{C}, O) \setminus \mathsf{Diff}(\mathbb{C}, O)$.

On the other hand, it is clear that $f \in \mathsf{Diff}(\mathbb{C}, O)$. By Proposition 1.1.8 it follows that any diffeomorphism h which linearizes f must be of the type $h(z) = ag(z)$ for some $a \in \mathbb{C} \setminus \{0\}$ and therefore h cannot be holomorphic, hence f is a Cremer diffeomorphism. □

At this moment, we have two criterions, one geometrical and the other analytical, to say whether an elliptic germ is a Cremer diffeomorphism, but no instance of Siegel diffeomorphisms. The well renowned theorem of Yoccoz states that *for almost all* $\theta \in \mathbb{R}\setminus\mathbb{Q}$ the diffeomorphism $f(z) = e^{2\pi i\theta}z + O(z^2)$ is (holomorphic) linearizable. We will give a qualitative proof of Yoccoz's theorem. In order to provide as many details as possible we need first to study the parabolic case.

2.3. The parabolic case: the Leau-Fatou flowers theorem.

DEFINITION 2.9. Let $f \in \mathsf{Diff}(\mathbb{C}, O)$ be such that $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ with $a_{k+1} \neq 0$. We say that $v \in \partial\mathbb{D}$ is an *attracting direction* if $\frac{a_{k+1}}{|a_{k+1}|}v^k = -1$.

We say that $v \in \partial\mathbb{D}$ is a *repelling direction* if $\frac{a_{k+1}}{|a_{k+1}|}v^k = 1$.

Clearly there exist exactly k attracting and k repelling directions.

REMARK 2.10. The attracting directions of f are the repelling directions of f^{-1} and conversely the repelling directions of f are the attracting directions of f^{-1}.

DEFINITION 2.11. Let $f \in \mathsf{Diff}(\mathbb{C}, O)$ be such that $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ with $a_{k+1} \neq 0$. An *attracting petal* centered at an attracting direction v is a simply connected open set P_v such that

(1) $O \in \partial P_v$,
(2) $f(P_v) \subseteq P_v$,

(3) $\lim_{n\to\infty} f^{\circ n}(z) = O$ and $\lim_{n\to\infty} \frac{f^{\circ n}(z)}{|f^{\circ n}(z)|} = v$ for all $z \in P_v$.

A *repelling petal* centered at a repelling direction v is an attracting petal for f^{-1} centered at the attracting direction v (for f^{-1}).

As a matter of notation, let $f \in \mathrm{Diff}(\mathbb{C}, O)$ be such that $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ with $a_{k+1} \neq 0$. We write $v_1^+, \ldots, v_k^+$ for the attracting directions of f and $v_1^-, \ldots, v_k^-$ for the repelling directions of f, ordered so that starting from 1 and moving counterclockwise on $\partial\mathbb{D}$ the first point we meet is v_1^+, then v_1^-, then v_2^+ and so on.

THEOREM 2.12 (Leau-Fatou). *Let $f \in \mathrm{Diff}(\mathbb{C}, O)$ be such that $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ with $a_{k+1} \neq 0$. Let $\{v_1^+, \ldots, v_k^+, v_1^-, \ldots, v_k^-\}$ be the ordered attracting and repelling directions of f. Then*

(1) *For any $v_j^{\pm}$ there exists an attracting/repelling petal $P_{v_j^{\pm}}$ centered at $v_j^{\pm}$.*
(2) *The union $\cup_{j=1}^k P_{v_j^+} \cup_{j=1}^k P_{v_j^-} \cup \{O\}$ is an open neighborhood of O.*
(3) *$P_{v_j^+} \cap P_{v_l^+} = \emptyset$ and $P_{v_j^-} \cap P_{v_l^-} = \emptyset$ for $j \neq l$.*
(4) *$P_{v_j}^-$ intersects only $P_{v_j}^+$ and $P_{v_{j+1}}^+$, $j = 1, \ldots, k$ (with the convention that $v_{k+1}^{\pm} = v_1^{\pm}$).*
(5) *For any attracting petal $P_{v_j^+}$ the function $f|_{P_{v_j^+}}$ is holomorphically conjugated to $\zeta \mapsto \zeta + 1$ defined on $\{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta > C\}$ for some $C > 0$.*

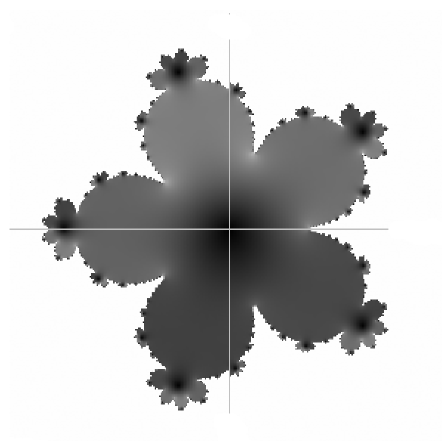

FIGURE 1. Five petals for $z \to z + z^6$.

PROOF. According to Remark 1.18 up to conjugation we can assume that $f(z) = z - z^{k+1} + az^{2k+1} + O(z^h)$ with $h >> k$, so that the k attracting directions are exactly the k-th roots of 1.

Let $\delta > 0$, $\delta << 1$ and consider the set $\{z \in \mathbb{C} : |z^k - \delta| < \delta\}$. This set is made of k connected components (that we call $P_1, \ldots, P_k$), each one centered at a different root of 1. We are going to show that these are the attracting petals of f.

Let $\psi(z) := \frac{1}{kz^k}$. Fix j and let $H_\delta := \psi(P_j)$. We claim that

$$H_\delta = \{w \in \mathbb{C} : \mathrm{Re}\, w > \frac{1}{2k\delta}\}.$$

Indeed, let $w = \frac{1}{kz^k}$, then $\mathsf{Re}\, w = \frac{1}{k}\mathsf{Re}\, \frac{z^k}{|z|^{2k}} = \frac{1}{k|z|^{2k}}\mathsf{Re}\, z^k$. If $z \in P_j$ then $|z^k - \delta|^2 < \delta^2$, namely $|z|^{2k} - 2\delta \mathsf{Re}\, z^k < 0$, that is $\mathsf{Re}\, z^k > \frac{|z|^{2k}}{2\delta}$, thus $\mathsf{Re}\, w = \frac{1}{k|z|^{2k}}\mathsf{Re}\, z^k > \frac{1}{2k\delta}$. Conversely, if $\mathsf{Re}\, w > \frac{1}{2k\delta}$, let $z = (kw)^{-1/k}$ (where the k-th root is chosen so that for $R > 0$ the root $R^{-1/k}$ is in the attracting direction v_j^+). Then $\psi(z) = w$ and we need to show that $z \in P_j$. But

$$\mathsf{Re}\, w > \frac{1}{2\delta k} \Leftrightarrow \frac{1}{k} - 2\delta \mathsf{Re}\, w < 0 \Leftrightarrow \frac{1}{k^2|w|^2} - \frac{2\delta}{k}\mathsf{Re}\, \frac{w}{|w|^2} < 0$$
$$\Leftrightarrow \left|\frac{1}{kw} - \delta\right|^2 < \delta^2 \Leftrightarrow \left|((kw)^{-1/k})^k - \delta\right|^2 < \delta^2,$$

proving the claim.

Note that $\psi : P_j \to H_\delta$ is invertible, and $\psi^{-1}(z) = (kz)^{-1/k}$ where the k-th root is chosen so that for $R > 0$ the root $R^{-1/k}$ is in the semi-straight line from the origin and containing v_j^+. Now we compute $\varphi := \psi \circ f|_{P_j} \circ \psi^{-1} : H_\delta \to \mathbb{C}$. Then

$$f|_{P_j} \circ \psi^{-1}(z) = (kz)^{-1/k} - ((kz)^{-1/k})^k + a((kz)^{-1/k})^{2k+1} + O(((kz)^{-1/k})^h)$$

and

$$\begin{aligned} &\psi \circ f|_{P_j} \circ \psi^{-1}(z) \\ &= \frac{1}{k[(kz)^{-1/k} - ((kz)^{-1/k})^{k+1} + a((kz)^{-1/k})^{2k+1} + O(((kz)^{-1/k})^h)]^k} \\ &= \frac{1}{k[(kz)^{-1/k}(1 - (kz)^{-1} + a(kz)^{-2} + O(|z|^{(1-h)/k})]^k} \\ &= \frac{z}{[1 - (kz)^{-1} + a(kz)^{-2} + O(|z|^{(1-h)/k})]^k} \end{aligned} \tag{2.11}$$

Now, for $|z| >> 1$ we have $|-(kz)^{-1} + a(kz)^{-2} + O(|z|^{(1-h)/k})| < 1$ and therefore (recalling that $(1+x)^k = 1 + kx + O(x^2)$ for $|x| < 1$) we have

$$\begin{aligned} &[1 - (kz)^{-1} + a(kz)^{-2} + O(|z|^{(1-h)/k})]^k \\ &\qquad = 1 + k(-(kz)^{-1} + a(kz)^{-2} + O(|z|^{(1-h)/k})) + O(\frac{1}{z^2}) \\ &\qquad = 1 - \frac{1}{z} + O(\frac{1}{z^2}). \end{aligned}$$

Substituting in (2.11) we obtain

$$\varphi(z) = \frac{z}{1 - \frac{1}{z} + O(\frac{1}{z^2})} = z(1 + \frac{1}{z} + O(\frac{1}{z^2})) = z + 1 + \frac{b}{z} + \dots$$

We claim that $\varphi(H_\delta) \subseteq H_\delta$. To see this we have to show that if $\mathsf{Re}\, z > \frac{1}{2k\delta}$ then

$$\mathsf{Re}\, \varphi(z) > \frac{1}{2k\delta}. \tag{2.12}$$

Since $|z| > \mathsf{Re}\, z > \frac{1}{2k\delta}$, if $\delta << 1$ we have

$$\frac{1}{2} < 1 + \mathsf{Re}\, [\frac{b}{z} + O(\frac{1}{z^2})] < 2. \tag{2.13}$$

Therefore

$$\mathsf{Re}\, \varphi(z) = \mathsf{Re}\, z + 1 + \mathsf{Re}\, [\frac{b}{z} + O(\frac{1}{z^2})] > \mathsf{Re}\, z > \frac{1}{2k\delta},$$

from which (2.12) follows. Therefore $\varphi(H_\delta) \subseteq H_\delta$ which implies that $f(P_j) \subset P_j$.

Moreover, we note that $\varphi^{\circ n}(z) = z + n + O(1/n)$, thus for $z \in H_\delta$

$$f^{\circ n} \circ \psi^{-1}(z) = \psi^{-1}(z + n + O(1/n)) = (k(z + n + O(1/n)))^{-1/k}$$

which, for $n \to \infty$, tends to O tangentially to the direction v_j^-.

Next we show that φ is holomorphically conjugated to $z \mapsto z + 1$ on H_δ.

First we estimate the orbits of φ. For all $n \geq 1$ and $z \in H_\delta$ we have

$$\frac{n}{2} \leq |\varphi^{\circ n}(z)| \leq |z| + 2n. \tag{2.14}$$

The upper estimate follows by induction. Indeed, by (2.13),

$$|\varphi(z)| \leq |z| + |1 + O(\frac{1}{z})| \leq |z| + 2.$$

Assume the upper estimate in (2.14) holds for n, we prove it holds for $n+1$ concluding the induction:

$$|\varphi^{\circ(n+1)}(z)| = |\varphi^{\circ(n)}(\varphi(z))| \leq |\varphi(z)| + 2n \leq |z| + 2 + 2n = |z| + 2(n+1).$$

As for the lower estimate in (2.14), we claim that for all $n \geq 1$ and $z \in H_\delta$

$$\mathsf{Re}\, \varphi^{\circ n}(z) > \mathsf{Re}\, z + \frac{n}{2}. \tag{2.15}$$

Assuming (2.15) we have

$$|\varphi^{\circ n}(z)| > \mathsf{Re}\, \varphi^{\circ n}(z) > \mathsf{Re}\, z + \frac{n}{2} > \frac{n}{2},$$

and (2.14) holds. In order to prove (2.15) we argue again by induction. For $n = 1$ by (2.13) we have

$$\mathsf{Re}\, \varphi(z) = \mathsf{Re}\, z + 1 + \mathsf{Re}\,(\frac{b}{z} + O(\frac{1}{z^2})) > \mathsf{Re}\, z + \frac{1}{2}.$$

Assuming (2.15) holds for n, we prove it for $n+1$ concluding the induction:

$$\mathsf{Re}\, \varphi^{\circ(n+1)}(z) > \mathsf{Re}\, \varphi(z) + \frac{n}{2} > \mathsf{Re}\, z + \frac{n+1}{2}.$$

Fix a compact set $K \subset\subset H_\delta$. By (2.14) we have that for all $z \in K$

$$|\varphi^{\circ n}(z)| \sim n \quad \text{for } n \to \infty. \tag{2.16}$$

Hence

$$\begin{aligned} \varphi^{\circ(k+1)}(z) &= \varphi^{\circ k}(z) + 1 + \frac{b}{\varphi^{\circ k}(z)} + O\left(\frac{1}{|\varphi^{\circ k}(z)|^2}\right) \\ &= \varphi^{\circ k}(z) + 1 + \frac{b}{\varphi^{\circ k}(z)} + O(\frac{1}{k^2}). \end{aligned} \tag{2.17}$$

Now we define for $z \in H_\delta$

$$\sigma_n(z) := \varphi^{\circ n}(z) - n - b \log n.$$

Note that if $\sigma_n(z) = \sigma_n(z_0)$ then $\varphi^{\circ n}(z) = \varphi^{\circ n}(z_0)$, but, being φ univalent it follows $z = z_0$. Hence σ_n's are univalent. We see that

$$\begin{aligned}\sigma_{k+1}(z) - \sigma_k(z) &= \varphi^{\circ(k+1)}(z) - (k+1) \\ &\quad - b\log(k+1) - \varphi^{\circ k}(z) + k + b\log k \\ &= \varphi^{\circ k}(z) + 1 + \frac{b}{\varphi^{\circ k}(z)} + O(\frac{1}{k^2}) \\ &\quad - 1 - b\log(k+1) - \varphi^{\circ k}(z) + b\log k \\ &= \frac{b}{\varphi^{\circ k}(z)} + b\log\frac{k}{k+1} + O(\frac{1}{k^2}) \overset{(2.16)}{=} O(\frac{1}{k}).\end{aligned}$$

Hence

$$|\sigma_n(z)| \leq |z| + |\sigma_1(z) - z| + \sum_{k=1}^{n-1} |\sigma_{k+1}(z) - \sigma_k(z)| = O(\log n).$$

Summing up, we proved that for all $n \geq 1$ and $z \in K \subset\subset H_\delta$,

$$|\varphi^{\circ n}(z)| = O(n), \quad |\sigma_n(z)| \leq O(\log n). \tag{2.18}$$

Now we prove that $\{\sigma_n\}$ is uniformly convergent on compacta. Indeed

$$\begin{aligned}\sigma_{n+1}(z) - \sigma_n(z) &= \varphi^{\circ(n+1)}(z) - b\log(n+1) - \varphi^{\circ n}(z) + b\log n - 1 \\ &= b\left[\frac{1}{\varphi^{\circ n}(z)} - \log\frac{n+1}{n}\right] + O(\frac{1}{n^2}) \\ &= b\left[\frac{1}{\sigma_n(z) + n + b\log n} - \log(1 + \frac{1}{n})\right] + O(\frac{1}{n^2}) \\ &= b\left[\frac{1}{\sigma_n(z) + n + b\log n} - (\frac{1}{n} + O(\frac{1}{n^2}))\right] + O(\frac{1}{n^2}) \\ &= b\left[\frac{-\sigma_n(z) - b\log n}{n^2(\frac{\sigma_n(z)}{n} + 1 + \frac{b\log n}{n})}\right] + O(\frac{1}{n^2}) \\ &\overset{(2.18)}{=} \frac{b}{n^2} O(|\sigma_n(z) + b\log n|) \overset{(2.18)}{=} O\left(\frac{\log n}{n^2}\right).\end{aligned}$$

Therefore the telescopic series $\sum(\sigma_{n+1}(z) - \sigma_n(z))$ is uniformly convergent on compacta, thus $\sigma_n \to \sigma \in \mathsf{Hol}(H_\delta, \mathbb{C})$. Note that

$$\begin{aligned}\sigma_n \circ \varphi(z) &= \varphi^{\circ(n+1)}(z) - n - b\log n \\ &= \varphi^{\circ(n+1)}(z) - (n+1) - b\log(n+1) + 1 + b\log\frac{n+1}{n} \\ &= \sigma_{n+1}(z) + 1 + O(\frac{1}{n}),\end{aligned}$$

hence taking the limit for $n \to \infty$ we obtain

$$\sigma \circ \varphi(z) = \sigma(z) + 1,$$

hence σ is not constant and, being the limit of univalent functions, it is univalent.

Similar arguments hold for f^{-1}.

The petals constructed so far are not exactly the ones whose existence is stated in the theorem. In fact they do not form a full neighborhood around O. In order to do this, one needs to "enlarge" a little bit the petals described before. We leave details to the reader. □

In the last part of the proof we actually proved the following result:

PROPOSITION 2.13. *Let $R \geq 0$ and let $H_R = \{w \in \mathbb{C} : \mathsf{Re}\, w > R\}$. Let $\varphi : H_R \to H_R$ be a holomorphic map such that $\varphi(w) = w + 1 + \frac{b}{w} + O(\frac{1}{w^2})$ for $|w| >> 1$. Then there exists $\sigma : H_R \to \mathbb{C}$ holomorphic such that $\sigma \circ \varphi = \sigma + 1$.*

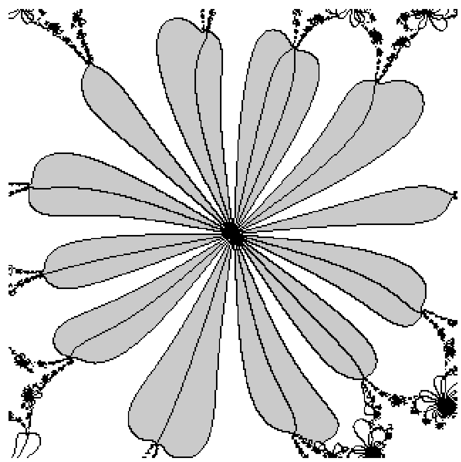

FIGURE 2. 12 petals for $z \to e^{2\pi i 3/4} z + z^4$.

Now the general case follows at once:

COROLLARY 2.14. *Let $f(z) = \lambda z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$ with $\lambda^r = 1$, $\lambda^t \neq 1$ for $t = 1, \dots, r-1$. If $f^{\circ r} \neq \mathsf{id}$ there exist m attracting petals $P_1, \dots, P_m$ for $f^{\circ r}$ such that $m = kr$ for some $k \in \mathbb{N}$ and f permutes $P_1, \dots, P_m$ in cycle of length r.*

PROOF. The map $f^{\circ r}$ is tangent to the identity of the form $z + \alpha z^{m+1} + \dots$ with $\alpha \neq 0$. Apply Theorem 2.12 to $f^{\circ r}$. Then there exist m attracting petals for $f^{\circ r}$. Let P_1 be one of such petals centered at the attracting direction v. Then clearly $P_2 := f(P_1)$ is another attracting petal for $f^{\circ r}$ with attracting direction $f'(0)v$. Hence $P_2 \cap P_1 = \emptyset$. Define $P_3 := f(P_2)$. Then P_2 is centered at the attracting direction $[f'(0)]^2 v$, and so on. After r steps, $P_{r+1} = P_1$. Therefore f acts as a permutation of length r on the attracting cycles. Hence $m = kr$ for some $k \in \mathbb{N}$. □

2.4. Écalle-Voronin holomorphic classification of germs tangent to the identity. First of all we show that classifying germs tangent to the identity is enough to get the classification of parabolic germs. It is obvious that the multiplier of a holomorphic germ is a holomorphic invariant, and we have

PROPOSITION 2.15. *Let $f, g \in \mathsf{Diff}(\mathbb{C}, O)$ be parabolic with $f'(0) = g'(0) = \exp(2\pi i p/q)$ for some $p, q \in \mathbb{N}$. Then f and g are holomorphically conjugated if and only if $f^{\circ q}$ and $g^{\circ q}$ are holomorphically conjugated.*

PROOF. If f, g are holomorphically conjugated, so clearly are $f^{\circ q}$ and $g^{\circ q}$. Conversely, if $f^{\circ q}$ and $g^{\circ q}$ are holomorphically conjugated by the holomorphic map φ, it follows that

$$g^{\circ q} = \varphi \circ f^{\circ q} \circ \varphi^{-1} = (\varphi \circ f \circ \varphi^{-1})^{\circ q}.$$

Hence, since f is holomorphically conjugated to $\varphi \circ f \circ \varphi^{-1}$, up to replace f with $\varphi \circ f \circ \varphi^{-1}$, we can assume that $f^{\circ q} = g^{\circ q}$. Then it is easy to see that the germ $h(z) := \sum_{j=0}^{q-1} g^{\circ(q-j)} \circ f^{\circ j}$ is a biholomorphism conjugating f to g. □

Therefore, we concentrate on the holomorphic classification of germs tangent to the identity. We briefly and roughly sketch the construction of the holomorphic invariants, referring the reader to the original papers by Écalle [**13, 14**] and Voronin [**23**] (see also [**1**], [**17**]). What follows is taken essentially by [**10**].

By Remark 1.14 on page 9, we can, and we will, assume that f is normalized as

$$f(z) = z + z^{r+1} + az^{2r+1} + O(|z|^{2r+2}).$$

The Leau-Fatou Theorem 2.12 guarantees the existence of r attracting petals $P_1^+, \ldots, P_r^+$ centered at the r-th roots of -1 and r repelling petals $P_1^-, \ldots, P_r^-$ (attracting for f^{-1}) centered at the r-th roots of 1 (numbered counterclockwise). On each such petal $P_j^\pm$ there exists a biholomorphic map $\varphi_j^\pm$ (called *Fatou coordinates*) which conjugates $f|_{P_j^\pm}$ to a translation $z \mapsto z+1$ (on a right half-plane on attracting petals and on a left half-plane on repelling petals).

If $r = 1$, we let U_1^+ be the connected component of $P_1^+ \cap P_1^-$ contained in the upper half-plane and let U_1^- be the connected component of $P_1^+ \cap P_1^-$ contained in the lower half-plane. In case $r > 1$ we let $U_j^+ = P_{j+1}^+ \cap P_j^-$ and $U_j^- = P_j^+ \cap P_j^-$, for $j = 1, \ldots, r$ where as customary, $P_{r+1}^\pm = P_1^\pm$. We define

$$S_j^\pm := \cup_{m\in\mathbb{Z}} f^{\circ m}(U_j^\pm).$$

The sets $S_j^\pm$ are totally f-invariants by construction and they are disjoint each other.

It is possible to extend the Fatou coordinates φ_j^+ to $P_j^+ \cup S_j^- \cup S_{j-1}^+$ by

$$\varphi_j^+(z) := \varphi_j^+(f^{\circ m}(z)) - m,$$

where $m \in \mathbb{N}$ is such that $f^{\circ m}(z) \in P_j^+$ (and it can be easily checked that the definition does not depend on the m chosen).

In a similar way, one can extend the repelling Fatou coordinates φ_j^- to all $P_j^- \cup S_j^- \cup S_j^+$ via

$$\varphi_j^-(z) := \varphi_j^-(f^{\circ -m}(z)) + m,$$

where $m \in \mathbb{N}$ is such that $f^{\circ -m}(z) \in P_j^-$.

Let $V_j^- := \varphi_j^-(S_j^-)$, $V_j^+ := \varphi_j^-(S_j^+)$, $W_j^- := \varphi_j^+(S_j^-)$, $W_j^+ := \varphi_{j+1}^+(S_j^+)$. Then one can define two holomorphic maps, called the *lifted horn maps*, as follows:

$$H_j^- := \varphi_j^+ \circ (\varphi_j^-)^{-1}|_{V_j^-} : V_j^- \longrightarrow W_j^-$$

and

$$H_j^+ := \varphi_{j+1}^+ \circ (\varphi_j^-)^{-1}|_{V_j^+} : V_j^+ \longrightarrow W_j^+.$$

The lifted horn maps are uniquely defined up to pre and post composing with a translation because the Fatou coordinates are so. Also, the open sets $V_j^\pm$ and $W_j^\pm$ are invariant by $z \mapsto z+1$ and V_j^+, W_j^+ contains a upper half-plane while V_j^-, W_j^- contains a lower half-plane. Hence, using the projection $z \mapsto \exp(2\pi i z)$ the image of V_j^+ and W_j^+ are transformed into punctured neighborhoods of the origin, say A_j^+, B_j^+ while V_j^- and W_j^- are transformed into punctured neighborhoods of ∞, say A_j^-, B_j^-.

Since $H_j^{\pm}(z+1) = H_j^{\pm}(z) + 1$ (because the Fatou coordinates do), the lifted horn maps project via $z \mapsto \exp(2\pi i z)$ to holomorphic maps, called *horn maps*

$$h_j^{\pm} : A_j^{\pm} \to B_j^{\pm}.$$

Since the lifted horn maps are uniquely defined up to pre and post composing with a translation, the horns maps are unique up to pre and post multiplication by a constant.

We have the following result:

PROPOSITION 2.16. *Let $f, g \in \mathsf{Diff}(\mathbb{C}, O)$ be two germs tangent to the identity. Let $\{h_j^{\pm}\}$ be horn maps for f and let $\{k_j^{\pm}\}$ be horn maps for f. If f and g are holomorphically conjugated then then they have the same multiplicity, the same parabolic index and there exist $\alpha_j, \beta_j \in \mathbb{C}^*$ such that, up to a cyclic permutation of the horn maps it follows*

$$k_j^-(z) = \alpha_j h_j^-(\beta_j z), \quad k_j^+(z) = \alpha_{j+1} h_j^+(\beta_j z). \tag{2.19}$$

PROOF. Let $\varphi_j^{\pm}$ be the Fatou coordinates for f. Then if $f = \psi \circ g \circ \psi^{-1}$, it follows that $\varphi_j^{\pm} \circ \psi$ are Fatou coordinates for g. The uniqueness up to additive constants of the Fatou coordinates allows then quite easily to prove the statement. □

The converse of the previous result is also true and it is the content of the Écalle-Voronin theorem. In order to describe it we need to define a relation on the space of horn maps.

Looking at the way the Fatou coordinates have been defined, one can show that

$$H_j^+(z) = z + O(1),$$

from which it follows that h_j^+ has a removable singularity at 0 and can be extended holomorphically by defining $h_j^+(0) = 0$. Similarly, defining $h_j^-(\infty) = \infty$ the horn map h_j^- is a holomorphic germ at ∞. Let $\lambda_j^{\pm}$ be the multiplier of $h_j^{\pm}$ at 0 (or ∞). It can be proved that

$$\prod_{j=1}^{r} \lambda_j^+ \lambda_j^- = \exp\left[4\pi^2 \left(\frac{r+1}{2} - \iota(f,0)\right)\right], \tag{2.20}$$

where $\iota(f,0) = a$ is the parabolic index of f.

DEFINITION 2.17. Let M_r denote the set whose elements are $\mathbf{h} := \{h_1^{\pm}, \ldots, h_r^{\pm}\}$, where the h_j^+'s are germs of holomorphic maps in a neighborhood of 0, the h_j^-'s are germs of holomorphic maps in a neighborhood of ∞ and such that their multipliers $\lambda_j^{\pm}$ satisfy (2.20).

We set an equivalence relation on M_r saying that two elements $\mathbf{h}, \mathbf{k} \in M_r$ are equivalent if, up to a cyclic permutation of the indices, they satisfy (2.19) for suitable $\alpha_j, \beta_j \in \mathbb{C}^*$.

The set of equivalence classes is denoted by $\mathcal{M}_r$.

As we described before, to any germ tangent to the identity f it is possible to associate a set of horn maps which, since every map is clearly conjugated to itself, by Proposition 2.16 defines uniquely an element $\mu_f \in \mathcal{M}_r$ called the *sectorial invariant* of f.

THEOREM 2.18 (Écalle-Voronin). *Let $f, g \in \mathsf{Diff}(\mathbb{C}, O)$ be two germs tangent to the identity. Then f and g are holomorphically conjugated if and only if they have the same multiplicity, the same parabolic index and the same sectorial invariant.*

Moreover, for any $r \geq 1$, $a \in \mathbb{C}$ and $\mu \in \mathcal{M}_r$ there exists a germ $f \in \mathsf{Diff}(\mathbb{C}, O)$ tangent to the identity such that f has multiplicity $r+1$, parabolic index a and sectorial invariant μ.

2.5. Stability versus Linearizability. Stability is a topological property of orbits. Roughly speaking the orbits of a germ $f \in \mathsf{Diff}(\mathbb{C}, O)$ are stable if they stay bounded near O. Such a condition implies (and is rather trivially implied by) linearizability. Here is one of the possible formal definition:

DEFINITION 2.19. Let $f \in \mathsf{Diff}(\mathbb{C}, O)$. The point O is *stable* for f if there exists an open neighborhood U of O, such that for all $z \in U$ and $n \in \mathbb{N}$ the map $z \mapsto f^{\circ n}(z)$ is well defined and $|f^{\circ n}(z)| < 1$.

THEOREM 2.20. *Let $f \in \mathsf{Diff}(\mathbb{C}, O)$ with $|f'(0)| \leq 1$. Then the point O is stable for f if and only if f is (holomorphically) linearizable.*

PROOF. Let first assume $|f'(0)| < 1$. Then we already saw in Theorem 2.1 that f is both linearizable and O is stable for f (indeed $|f(z)| \leq |\lambda|(|z| + C|z|^2)$).

Assume now that $|f'(0)| = 1$. Suppose that f is linearizable. Then there exists $h \in \mathsf{Diff}(\mathbb{C}, O)$ such that $h \circ f \circ h^{-1}(z) = \lambda z$. Let W be a neighborhood of O such that both f and h are defined and univalent on W. Let $r > 0$ be such that $\mathbb{D}_r \subset h(W)$ and h^{-1} is defined on $\mathbb{D}_r$. Let $U := h^{-1}(\mathbb{D}_r) \subset \mathbb{D}$. Then

$$f^{\circ n}(U) = h^{-1} \circ \lambda^n h(U) = h^{-1}(\lambda^n \mathbb{D}_r) = h^{-1}(\mathbb{D}_r) = U,$$

therefore O is stable for f.

Conversely, assume that O is stable for f and define

$$K := \cap_{n \in \mathbb{N}} f^{\circ\{-n\}}(\mathbb{D}).$$

Such a set is contained in $\mathbb{D}$ (being $f^{\circ 0}(\mathbb{D}) = \mathbb{D}$). Let U be the connected component of K which contains O. Since O is stable for f, $U \neq \emptyset$. Moreover, by construction, $f(U) = U$. We claim that U is simply connected. Indeed, let D be any compact set with Jordan boundary whose boundary ∂D is contained in U. Since $|f(z)| < 1$ for all $z \in \partial D$, by the maximum principle for holomorphic function, $|f(z)| < 1$ for all $z \in D$, hence $D \subset U$, proving that U is simply connected.

By the Riemann mapping theorem there exists a univalent map $g : U \to \mathbb{D}$ and we can assume that $g(O) = O$. Thus $g \circ f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$ is a holomorphic self-map of $\mathbb{D}$ which fixes O and such $g(f(g^{-1}))'(O) = \lambda$, hence by the Schwarz lemma, $g \circ f \circ g^{-1}(z) = \lambda z$ proving that f is linearizable. □

2.6. Diffeomorphisms of the circle. In this subsection we introduce an invariant, called the *rotation number* for orientation preserving homeomorphisms of the circle and we show that it is invariant under conjugation in the same class. Let $\mathbb{S}^1 := \partial\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. Recall that the map $\exp(2\pi i \cdot) : \mathbb{R} \to \mathbb{S}^1$ is the covering map from the universal covering $\mathbb{R}$ of $\mathbb{S}^1$ to $\mathbb{S}^1$. Therefore, if $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism of the circle there exists a continuous map $g : \mathbb{S}^1 \to \mathbb{R}$ such that

$$\exp(2\pi i g(\theta)) = f(\theta).$$

The map g is unique once fixed the value at one point, say 1. All the others liftings of f are of the form $g + N$ with $N \in \mathbb{Z}$. Fixing such a lifting g, we have a continuous map $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) := g(\exp(2\pi i t)),$$

which makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ {\scriptstyle \exp(2\pi i t)}\big\downarrow & & \big\downarrow{\scriptstyle \exp(2\pi i t)} \\ \mathbb{S}^1 & \xrightarrow{f} & \mathbb{S}^1 \end{array}$$

By construction, it follows that $F(t+1) = F(t) + N$ for some fixed $N \in \mathbb{Z}$, which we may assume to be 1. We call such a map F a *lifting associated to* f.

DEFINITION 2.21. An *orientation preserving homeomorphism* of $\mathbb{S}^1$ is a homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ such that the associated map F is increasing in t.

Clearly the previous definition implies that f is a orientation preserving homeomorphism if it preserves the counterclockwise orientation of $\mathbb{S}^1$.

PROPOSITION 2.22. *Let f be an orientation preserving homeomorphism of $\mathbb{S}^1$ and let F be the associated lifting. Then*

$$\alpha(F) := \lim_{n\to\infty} \frac{F^{\circ n}(0)}{n}$$

exists. Moreover the number

$$\rho(f) := \alpha(F) \mod 1$$

is independent of the associated lifting F chosen to define it.

PROOF. If $F(0) = 0$ then $\alpha(F)$ is well defined. Assume that $F(0) > 0$. Let $0 \le t \le 1$. Then $F(0) \le F(t) \le F(1) = F(0) + 1$. Fix $s > 0$ and let $j = [s]$ (integer part of s). Then

$$\begin{aligned} s + F(0) - 1 &\overset{s-1\le j}{\le} j + F(0) = F(j) \overset{F\nearrow}{\le} F(s) \\ &\le F(j+1) = F(0) + j + 1 \le s + F(0) + 1 \end{aligned}$$

By induction we obtain

$$s + h(F(0) - 1) \le F^{\circ h}(s) \le s + h(F(0) + 1) \quad s > 0, h \ge 1. \tag{2.21}$$

Let now $p \ge 1$ and let m denote the least integer such that $F^{\circ m}(0) > p$. Then $F^{\circ(m-1)}(0) \le p \le F^{\circ m}(0)$ and, again by induction

$$F^{\circ k(m-1)}(0) \le kp \le F^{\circ km}(0), \quad k \ge 1. \tag{2.22}$$

Let now $0 \le q < m$ and write $n = km + q$. From (2.21) with $s = kp$, $h = q$ we obtain

$$\begin{aligned} kp + q(F(0) - 1) \le F^{\circ q}(kp) &\overset{(2.22)}{\le} F^{\circ q}(F^{\circ km}(0)) = F^{\circ(km+q)}(0) = F^{\circ n}(0) \\ &= F^{\circ(q+k)}(F^{\circ(k(m-1))}(0)) \overset{(2.22)}{\le} F^{\circ(q+k)}(kp) \\ &\overset{(2.21)}{\le} kp + (q+k)(1 + F(0)). \end{aligned}$$

From this

$$\frac{kp}{n} + \frac{q(F(0)-1)}{n} \le \frac{F^{\circ n}(0)}{n} \le \frac{kp}{n} + \frac{q+k}{n}(1 + F(0)).$$

For $n \to \infty$ since $\frac{k}{n} = \frac{1}{m} - \frac{q}{n}$, it follows $\frac{k}{n} \to \frac{1}{m}$. Hence for all p

$$\frac{p}{m} \le \liminf_{n\to\infty} \frac{F^{\circ n}(0)}{n} \le \limsup_{n\to\infty} \frac{F^{\circ n}(0)}{n} \le \frac{p}{m} + \frac{1 + F(0)}{m},$$

Letting $p \to \infty$ (then $m \to \infty$) we obtain

$$\liminf_{n\to\infty} \frac{F^{\circ n}(0)}{n} = \limsup_{n\to\infty} \frac{F^{\circ n}(0)}{n},$$

proving the first claim. Now, we already saw that the liftings F associated to f differ by integer numbers, thus $\alpha(F) \mod 1$ does not depend on F. □

DEFINITION 2.23. Let $f\ \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism of the circle. The number $\rho(f)$ is called the *rotation number* of f.

THEOREM 2.24 (Poincaré). *Let $f\ \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism of the circle. The rotation number $\rho(f)$ is invariant under conjugation with orientation preserving homeomorphisms of the circle.*

PROOF. Let F be a lifting associated to f. First of all we note that for all $t \in \mathbb{R}$ we have

$$\alpha(F) = \lim_{n\to\infty} \frac{F^{\circ n}(t)}{n}. \tag{2.23}$$

Indeed, for $0 \le t \le 1$ we have

$$\frac{F^{\circ n}(0)}{n} \le \frac{F^{\circ n}(t)}{n} \le \frac{F^{\circ n}(0)}{n} + \frac{1}{n},$$

and letting $n \to \infty$ we have (2.23). If $t > 1$, writing $t = [t] + t'$ with $0 \le t' \le 1$, then $F(t) = F(t' + [t]) = F(t') + [t]$ and $F^{\circ 2}(t) = F(F(t' + [t])) = F(F(t') + [t]) = F^{\circ 2}(t') + [t]$ and more generally

$$F^{\circ n}(t) = F^{\circ n}(t') + [t].$$

From this it follows that $\lim_{n\to\infty} F^{\circ n}(t)/n = \lim_{n\to\infty} F^{\circ n}(t')/n$ and (2.23) holds.

Now let g be an orientation preserving homeomorphism of $\mathbb{S}^1$ and let $G : \mathbb{R} \to \mathbb{R}$ be the associated lifting so that $G(t+1) = G(t) + 1$. Then $G \circ F \circ G^{-1}$ lifts $g \circ f \circ g^{-1}$ and we only need to show that $\alpha(G \circ F \circ G^{-1}) = \alpha(F)$. But

$$\begin{aligned}
\alpha(G \circ F \circ G^{-1}) &\overset{(2.23)}{=} \lim_{n\to\infty} \frac{(G \circ F \circ G^{-1})^{\circ n}(G(0))}{n} = \lim_{n\to\infty} \frac{G(F^{\circ n}(0))}{n} \\
&= \lim_{n\to\infty} \left\{ \frac{G(F^{\circ n}(0) - [F^{\circ n}(0)])}{n} + \frac{[F^{\circ n}(0)]}{n} \right\} \\
&= \lim_{n\to\infty} \frac{F^{\circ n}(0)}{n}
\end{aligned}$$

because $|G(F^{\circ n}(0) - [F^{\circ n}(0)])| \le \max_{t\in[0,1]} |G(t)| \le C < \infty$ and $|F^{\circ n}(0) - [F^{\circ n}(0)]| \le 1$. □

2.7. Pérez-Marco's construction. In this subsection we roughly examine Pérez--Marco's construction which gives rise to the so called *hedgehogs* and will be useful to (sketchy) prove the Naishul theorem in next subsection. More details are in [**20**]. First of all we recall Koebe's 1/4-theorem. As a matter of notation, $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and $\mathbb{D}_r := \{\zeta \in \mathbb{C} : |\zeta| < r\}$ for $r > 0$.

THEOREM 2.25 (Koebe 1/4-theorem). *Let $f : \mathbb{D} \to \mathbb{C}$ be univalent and such that $f(0) = 0$ and $f'(0) = 1$. Then $\mathbb{D}_{1/4} \subset f(\mathbb{D})$.*

As a consequence which will be useful later we have

COROLLARY 2.26. *Let $r > 0$ and let $\mathbb{D}_r := \{\zeta \in \mathbb{C} : |\zeta| < r\}$. Let $f : \mathbb{D}_r \to \mathbb{C}$ be univalent and such that $f(0) = 0$ and $f'(0) = 1$. Then $\mathbb{D}_{r/4} \subset f(\mathbb{D}_r)$.*

PROOF. Let use define $g(z) := \frac{f(rz)}{r}$. Then g satisfies the hypothesis of Theorem 2.25. Hence

$$\mathbb{D}_{1/4} \subset \frac{1}{r}(r\mathbb{D}) = \frac{1}{r}\mathbb{D}_r,$$

from which the result follows. □

By Koebe's 1/4 Theorem 2.25, if $f : \mathbb{D} \to \mathbb{C}$ is univalent and $f(0) = 0$, $|f'(0)| = 1$ it follows that f^{-1} is defined at least on $\mathbb{D}_{1/4}$.

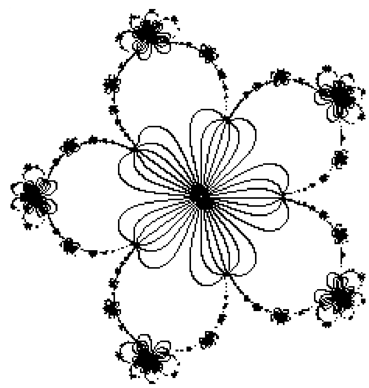

FIGURE 3. Invariant petals forming the Siegel compacta inside attracting petals.

THEOREM 2.27 (Pérez-Marco). *Let $f(z) = \lambda z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$ with $|\lambda| = 1$. Suppose that f, f^{-1} are defined and univalent on a neighborhood of the closed disc $\overline{\mathbb{D}}_r$. Then there exists a set $K \subset \overline{\mathbb{D}}_r$ with the following properties:*

1. *K is compact, connected and full (namely $\mathbb{C} \setminus K$ is connected)*
2. *$O \in K \subset \overline{\mathbb{D}}_r$*
3. *$K \cap \partial\mathbb{D}_r \neq \emptyset$*
4. *$f(K) = K, f^{-1}(K) = K$.*

Moreover, if $f^{\circ m} \neq \mathsf{id}$ for all $m \in \mathbb{N}$ then f is linearizable if and only if $O \in \overset{o}{K}$.

SKETCH OF THE PROOF. The last sentence follows at once from Theorem 2.20. The proof of the theorem goes as follows:

1. Let $\mathcal{F}_r$ be the set of holomorphic function $g : \overline{\mathbb{D}}_r \to \mathbb{C}$ which satisfy the hypotheses of the theorem. Let endow $\mathcal{F}_r$ with the topology τ_{uc} of uniform convergence on compacta. The space $(\mathcal{F}_r, \tau_{uc})$ is closed.

2. Using Leau-Fatou's flowers theorem 2.12 one can show that parabolic germs $f(z) = e^{2\pi i p/q}z + O(z^2)$ with $p, q \in \mathbb{N}$ satisfy the hypotheses of the theorem and therefore they belong to the family $\mathcal{F}_r$. The set K in such a case is the union of petals contained inside the attracting petals (see Figure 3), more precisely such petals are the intersection between attracting and repelling petals. The family of parabolic germs is dense in $\mathcal{F}_r$ and since this set is closed the result follows. □

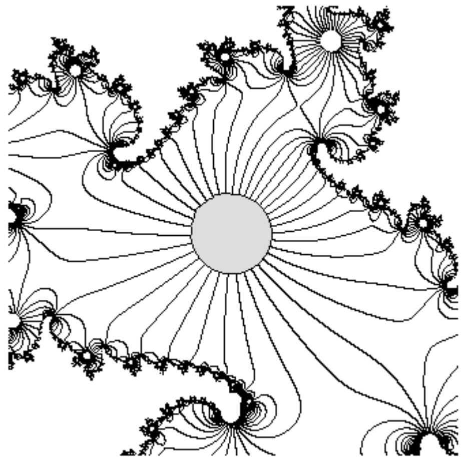

FIGURE 4. Bounded basin of attraction around a Siegel compacta, with a non-empty linearization domain (grey disc), under the iterates of a linearizable germ of the form $z \mapsto e^{2\pi i\theta}z + z^2$.

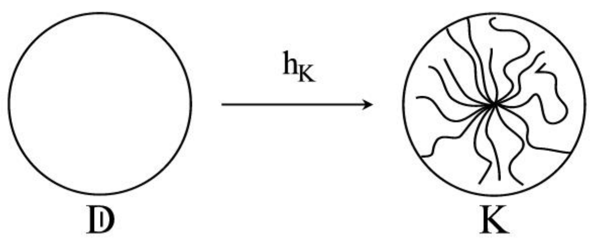

FIGURE 5. The map h_K in Pérez-Marco's construction.

The set K defined in Theorem 2.27 is called a *Siegel compacta* and it is called a *hedgehog* in case the germ f is not linearizable.

Let $f(z) = \lambda z + O(z^2)$ with $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$. Up to rescaling, we can assume that f is univalent on an open neighborhood of the disc $\mathbb{D}$. By Koebe's 1/4 theorem 2.25 f^{-1} is defined and univalent on a neighborhood of the closed disc $\overline{\mathbb{D}}_{1/4}$ of radius $1/4$. Let K be the Siegel compacta for $\mathbb{D}_{1/4}$ defined in Theorem 2.27. By construction $\mathbb{CP}^1 \setminus K$ is simply connected and therefore there exists a univalent map $h_K : \mathbb{CP}^1 \setminus \overline{\mathbb{D}} \to \mathbb{CP}^1 \setminus K$, such that $h_K(\infty) = \infty$.

Let $g_K := h_K^{-1} \circ f \circ h_K$. Such a map is defined and holomorphic in an annulus $A := \{\zeta \in \mathbb{C} : 1 < |\zeta| < r\}$ for some $r > 1$. Moreover, since $f(K) = f^{-1}(K) = K$ it follows $f(\mathbb{D}_{1/4} \setminus K) \subset \mathbb{C} \setminus K$, hence the image $g_K(A)$ is contained in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Moreover, if $\{z_n\} \subset A$ and $A \ni z_n \to z_0 \in \partial\mathbb{D}$ it follows that $g_K(z_n) \to \partial\mathbb{D}$. By Schwarz reflection principle the map g_K extends to a univalent map (which we still denote by g_K) on $\{\zeta \in \mathbb{C} : 1/r < |\zeta| < r\}$. In particular $g_K : \mathbb{S}^1 = \partial\mathbb{D} \to \mathbb{S}^1$ is an orientation preserving diffeomorphism of $\mathbb{S}^1$.

We call g_K the *orientation preserving diffeomorphism of the circle associated to* (f, K).

LEMMA 2.28. *Let $f = e^{2\pi i\theta}z + O(z^2)$ with $\theta \in \mathbb{R}$. Let K be a Siegel compacta for f and let g_K be the orientation preserving diffeomorphism of the circle associated to (f, K). Then the rotation number $\rho(g_K) = \theta$.*

The proof of such a lemma is omitted. We only note here that the basic remark underlying such a lemma is that the diffeomorphism g_K is nothing but the action of f on the space of *prime ends* of $\mathbb{CP}^1 \setminus K$. With this in mind the result is firstly proved for parabolic germs, then extended by density to the space of all non-hyperbolic germs.

2.8. Naishul's theorem. In this subsection we sketch Pérez-Marco's proof for the topological invariance of the multiplier for non hyperbolic germs:

THEOREM 2.29 (Naishul). *Let $f_1(z) = e^{2\pi i\theta_1}z + O(z^2)$ and $f_1(z) = e^{2\pi i\theta_2}z + O(z^2)$ with $\theta_1, \theta_2 \in \mathbb{R}$. Assume that there exists a germ of an orientation preserving homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$, $\varphi(O) = O$ such that $\varphi \circ f_1 \circ \varphi^{-1} = f_2$. Then $\theta_1 = \theta_2$.*

PROOF. Let K be a Siegel compacta for f_1 defined by Theorem 2.27. We may choose K so that it is contained in the domain of definition of φ. Let g_1 be the orientation preserving diffeomorphism of the circle associated to (f_1, K). The set $\varphi(K)$ is a Siegel invariant for f_2, and we let g_2 be the orientation preserving diffeomorphism of the circle associated to $(f_2, \varphi(K))$. We also denote by h_1 the Riemann mapping from $\mathbb{CP}^1 \setminus \mathbb{D} \to \mathbb{CP}^1 \setminus K$ and by h_2 the Riemann mapping from $\mathbb{CP}^1 \setminus \mathbb{D} \to \mathbb{CP}^1 \setminus \varphi(K)$. Recall from Pérez-Marco's construction that $g_j = h_j^{-1} \circ f_j \circ h_j$, $j = 1, 2$. Let us define $\psi := h_2^{-1} \circ \varphi \circ h_1$. On $\partial\mathbb{D}$ we have

$$\begin{aligned}
\psi \circ g_1 &= (h_2^{-1} \circ \varphi \circ h_1) \circ (h_1^{-1} \circ f_1 \circ h_1) \\
&= h_2^{-1} \circ \varphi \circ f_1 \circ h_1 = h_2^{-1} \circ f_2 \circ \varphi \circ h_1 \\
&= (h_2^{-1} \circ f_2 \circ h_2) \circ (h_2^{-1} \circ \varphi \circ h_1) = g_2 \circ \psi.
\end{aligned}$$

The map φ is uniformly continuous on a neighborhood of K. This implies that φ defines a homeomorphism from the space of prime ends of $\mathbb{CP}^1 \setminus K$ to the space of prime ends of $\mathbb{CP}^1 \setminus \varphi(K)$. Hence ψ is an orientation preserving homeomorphism of the circle which conjugates g_1 and g_2. By Lemma 2.28 and Theorem 2.24 it follows

$$\theta_1 = \rho(g_1) = \rho(g_2) = \theta_2,$$

and we are done. □

2.9. Douady-Hubbard's Straightening Theorem. In this subsection we give a sketch of the proof of Douady Hubbard's Straightening Theorem. More details can be found in [**11**] (see also [**2**, p.131]).

THEOREM 2.30 (Douady-Hubbard Straightening Theorem). *Let $F(z) = \lambda z + z^2 + \psi(z)$ with $|\lambda| = 1$, $\psi(0) = \psi'(0) = 0$ be defined on the disc $\mathbb{D}_R$ with $R > 5$ and assume $|\psi(z)| << 1$. Then there exist $r > 0$ and a homeomorphism $h : \mathbb{D}_r \to \mathbb{C}$ such that $h^{-1} \circ F \circ h(z) = \lambda z + z^2$.*

As a consequence, $F(z)$ is topologically conjugated to $\lambda z + z^2$. In order to prove Douady-Hubbard's theorem we need a few auxiliary results.

LEMMA 2.31. *Let $|\lambda| > 0$ and $a \in \mathbb{C} \setminus \{0\}$. The polynomials $\lambda z + z^2$ and $\lambda z + az^2$ are holomorphically conjugated.*

PROOF. Simply conjugate with $z \mapsto az$. □

Let

$$ds^2_{\mathbb{CP}^1} := \left(\frac{2|dz|}{1+|z|^2}\right)^2, \tag{2.24}$$

be the standard metric on $\mathbb{CP}^1$.

THEOREM 2.32 (Measurable Riemann Mapping Theorem). *Let $d\sigma^2 = \rho(z)|dz + \mu(z)d\overline{z}|^2$ be a metric on $\mathbb{CP}^1$ such that ρ, μ are L^∞ functions with $\rho(z) > 0$ (almost everywhere) and $\|\mu\|_\infty < 1$. Then there exists a quasiconformal mapping $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ (in particular h is a homeomorphism and it is differentiable almost everywhere) such that $h(0) = 0$, $h(\infty) = \infty$ and $h^*(ds^2_{\mathbb{CP}^1}) = \theta(z)d\sigma^2$ (almost everywhere) for some positive L^∞ function θ.*

For a proof see [**2**, Thm. 3 Ch. V]. Also, for definitions and properties of quasiconformal mappings see [**2**].

LEMMA 2.33 (Shishikura's surgery principle). *Let $K \geq 1$. Let $g : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a non-constant continuous map which is locally the composition of a holomorphic and a K-quasiconformal mapping (such a map is called* quasiregular*). Let $X \subset \mathbb{CP}^1$ be a measurable set with the properties that*

(1) *there exists a positive L^∞ function ρ such that $g^*(ds^2_{\mathbb{CP}^1}) = \rho(z)ds^2_{\mathbb{CP}^1}$ almost everywhere on $\mathbb{CP}^1 \setminus X$,*
(2) *for all $z \in X$ it holds $g^{\circ n}(z) \notin X$ for all $n = 1, 2, \ldots$.*

Then there exists a quasiconformal map $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ (in particular h is a homeomorphism) such that $h(0) = 0$, $h(\infty) = \infty$ and $h \circ g \circ h^{-1}$ is a rational map of $\mathbb{CP}^1$.

PROOF. Note that g is open (since it is locally open, being locally the composition of open mappings by hypothesis). Hence, $g(\mathbb{CP}^1)$ is a connected compact open subset of $\mathbb{CP}^1$, therefore $g(\mathbb{CP}^1) = \mathbb{CP}^1$. Let $Y := \{z \in \mathbb{CP}^1 : g^{\circ n}(z) \notin X, n = 0, 1, 2, \ldots\}$. By hypothesis $g(X) \subset Y$. Moreover,

$$\mathbb{CP}^1 = \cup_{n\geq 0} g^{\circ(-n)}(Y). \tag{2.25}$$

Define

$$d\sigma^2_z := (g^{\circ n})^*(ds^2_{\mathbb{CP}^1, g^{\circ n}(z)})$$

for $g^{\circ n}(z) \in Y$. It is easy to see that it does not depend on n. By (2.25), $d\sigma^2$ is a metric (defined almost everywhere) on $\mathbb{CP}^1$. By hypothesis and definition of $d\sigma^2$ it follows that $g^*(d\sigma^2) = \rho(z)d\sigma^2$ for some positive L^∞ function ρ. Write $d\sigma^2_z = \nu(z)|dz + \mu(z)d\overline{z}|^2$. Since g is locally the composition of a K-quasiconformal mapping and a holomorphic mapping, by the very definition of $d\sigma^2$ it can be proved that both ν and μ are L^∞ and moreover $\|\mu\|_\infty < 1$. Thus we can apply the Measurable Riemann Mapping Theorem 2.32 to come up with a quasiconformal mapping $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $h(0) = 0$, $h(\infty) = \infty$ and $h^*(ds^2_{\mathbb{CP}^1}) = \theta(z)d\sigma^2$ (almost everywhere) for some positive θ. Letting $f := h \circ g \circ h^{-1}$ we have (almost everywhere)

$$f^*(ds^2_{\mathbb{CP}^1}) = (h^{-1})^* \circ g^* \circ h^*(ds^2_{\mathbb{CP}^1}) = (h^{-1})^*(\rho\theta d\sigma^2) = \rho\theta ds^2_{\mathbb{CP}^1}.$$

Therefore f is a continuous function which is locally conformal. By Riemann's removable singularities theorem f is then a holomorphic map from $\mathbb{CP}^1$ into itself (hence a rational map). □

PROOF OF THEOREM 2.30. 1. Since $|\psi(z)| << 1$, the map F is a *polynomial-like mapping* on $\mathbb{D}_4$. Namely, $\overline{\mathbb{D}}_4 \subset F(\mathbb{D}_4)$ and $F : \mathbb{D}_4 \to F(\mathbb{D}_4)$ is a $2:1$ (branched) covering map. Indeed, for every $w \in \mathbb{D}_4$ we have for $z \in \partial\mathbb{D}_4$

$$|F(z) - w - (\lambda z + z^2 - w)| = |\psi(z)| < |\lambda z + z^2 - w|.$$

Hence by Rouché theorem, $F(z) - w$ and $\lambda z + z^2 - w$ have the same number of zeros in $\mathbb{D}_4$, thus $\overline{\mathbb{D}}_4 \subset F(\mathbb{D}_4)$, and F is a $2:1$ branched covering map from $\mathbb{D}_4$ onto its image.

2. *Straightening of polynomial-like mappings.* Up to change 4 with $4-\epsilon$, we can assume that F is regular on $\partial\mathbb{D}_4$. Hence $F(\partial\mathbb{D}_4)$ is an analytic regular curve which bounds an unbounded region U in $\mathbb{C}$. Let $V = \mathbb{C}\setminus\overline{U}$. Let $T > 1$ be such that $\overline{V} \subset \{\zeta \in \mathbb{C} : |\zeta| < T\}$. The region U is simply connected in $\mathbb{CP}^1$, and we let $\Phi : \mathbb{C}\setminus\overline{V} \to \{\zeta \in \mathbb{C} : |\zeta| < T^2\}$ be a univalent mapping such that $\Phi(\infty) = \infty$. Since $F(\partial\mathbb{D}_4)$ is real analytic, $\Phi : \partial V \to \{|\zeta| = T^2\}$ is real analytic. Now $F : \partial\mathbb{D}_4 \to \partial V$ is $2:1$ and $z \mapsto z^2$ is $2:1$ from $\{|\zeta| = T\}$ onto $\{|\zeta| = T^2\}$. Therefore there exists a homeomorphism $\kappa : \partial\mathbb{D}_4 \to \{|\zeta| = T\}$ such that the following diagram commutes:

$$\begin{array}{ccc} \partial\mathbb{D}_4 & \xrightarrow{F} & \partial V \\ \kappa\downarrow & & \downarrow\Phi \\ \{|\zeta| = T\} & \xrightarrow{z\mapsto z^2} & \{|\zeta| = T^2\} \end{array}$$

Hence $\kappa^2 = \Phi \circ F$. Let us define $\Phi(z) := \kappa(z)$ for $z \in \partial\mathbb{D}_4$. It is then possible to extend Φ to a *quasiconformal* map (bearing the same name) $\Phi : V \setminus \overline{\mathbb{D}}_4 \to \{T < |\zeta| < T^2\}$. In particular Φ is a homeomorphism. Let then define

$$g(z) := \begin{cases} F(z) & z \in \mathbb{D}_4 \\ \Phi^{-1}((\Phi(z))^2) & z \in \mathbb{CP}^1 \setminus \mathbb{D}_4 \end{cases}$$

Since $F(z) = \Phi^{-1}((\Phi(z))^2)$ on $\partial\mathbb{D}_4$, the map g extends continuously as a map $g : \mathbb{CP}^1 \to \mathbb{CP}^1$.

On $\mathbb{CP}^1 \setminus (\overline{V} \setminus \mathbb{D}_4)$

$$g^*(ds^2_{\mathbb{CP}^1}) = \left(|g'(z)|\frac{1+|z|^2}{1+|g(z)|^2}\right)^2 ds^2_{\mathbb{CP}^1}$$

Therefore $g^*(ds^2_{\mathbb{CP}^1})$ and $ds^2_{\mathbb{CP}^1}$ are in the same class of conformality[2] outside the compact set $\overline{V} \setminus \mathbb{D}_4$.

Moreover, if $z \in \overline{V} \setminus \mathbb{D}_4$ then $g(z) = \Phi(z) \in \{T < |\zeta| < T^2\}$. By construction $\overline{V} \subset \{|\zeta| < T\}$, hence $g^{\circ n}(z) \notin \overline{V} \setminus \mathbb{D}_4$ for all $n = 1, 2, \ldots$.

We can thus apply Shishikura's surgery principle (Lemma 2.33) and we find a quasiconformal map $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ (in particular h is a homeomorphism) such that $h(0) = 0$, $h(\infty) = \infty$ and $G(z) := h^{-1} \circ g \circ h$ is a rational map of $\mathbb{CP}^1$. But such a map G fixes ∞ since h, g do. Thus G is a polynomial. On $z \notin \mathbb{D}_4$ the map $g(z) = \Phi^{-1}((\Phi(z))^2)$ with Φ univalent. Hence g has "degree two" at infinity, therefore $G(z)$ is a polynomial of degree two. By Naishul's theorem 2.29 it follows that $G(z) = \lambda z + z^2$. □

[2] Two metrics ω_1, ω_2 on $\mathbb{CP}^1$ are in the same class of conformality if there exists a positive function $p(z)$ such that $\omega_1 = p\omega_2$.

2.10. Yoccoz's proof of the qualitative version of the Siegel-Bruno-Yoccoz theorem. In this subsection we prove the following theorem:

THEOREM 2.34. *Let* $f(z) = e^{2\pi i\theta}z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$. *Then for almost all* $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *(with respect to the Lebesgue measure),* f *is holomorphically linearizable.*

Such a theorem in this form goes back to Siegel [**22**] who gave conditions on θ (roughly speaking saying that θ is badly approximated by rational numbers) which hold almost everywhere. Then Bruno [**6**], [**7**] gave some refined conditions on θ which later on Yoccoz [**24**] proved to be sharp. In this subsection we are going to give a qualitative proof of Theorem 2.34 due a very ingenious construction of J.-C. Yoccoz [**24**] (see also [**18**]).

In what follows we will need one of the Koebe distortion formula (see, *e.g.* [**9**, p.3]), which we recall here for the reader convenience: if $h : \mathbb{D} \to \mathbb{C}$ is univalent then for all $z \in \mathbb{D}$

$$|h'(0)|\frac{|z|}{(1+|z|)^2} \leq |h(z) - h(0)| \leq |h'(0)|\frac{|z|}{(1-|z|)^2}. \tag{2.26}$$

Let us denote by

$$P_\lambda(z) := \lambda z + z^2.$$

THEOREM 2.35. *Let* $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *and let* $\lambda := e^{2\pi i\theta}$. *If* P_λ *is linearizable then every germ* $f \in \mathsf{Diff}(\mathbb{C}, O)$ *with* $f'(0) = \lambda$ *is linearizable.*

PROOF. Let $f(z) = \lambda z + a_0 z^2 + \psi(z)$ with $\lambda = e^{2\pi i\theta}$ and $\psi(0) = \psi'(0) = \psi''(0) = 0$. Let $f_a(z) = \lambda z + az^2 + \psi(z)$. For $a \neq 0$, let $h(z) := z/a$. Then $h^{-1} \circ f_a \circ h(z) = \lambda z + z^2 + a\psi(z/a)$. Since $\psi(z) = O(z^3)$ it follows that for $|a| >> 1$ the function $\psi(z/a)$ is defined on a neighborhood of the closed disc of radius 10 and $|a\psi(z/a)| << 1$. By the Douady-Hubbard straightening theorem 2.30 the map $h^{-1} \circ f_a \circ h$ for $|a| >> 1$ is topologically conjugated to P_λ. But P_λ is linearizable by hypothesis and then Theorem 2.20 implies that O is stable for P_λ. Hence O is stable for $h^{-1} \circ f_a \circ h$ and then again by Theorem 2.20 the map $h^{-1} \circ f_a \circ h$ is holomorphically linearizable for $|a| >> 1$, say $|a| \geq R$ for some $R >> 1$. Hence f_a is holomorphically linearizable for $|a| \geq R$. If $|a_0| \geq R$ then we are done, so we can assume $|a_0| < R$.

By Theorem1.6 in Section 1, for all $a \in \mathbb{C}$ the germ f_a is formally linearizable, with intertwining map $g_a(z) = z + \sum_{j=2}^{\infty} b_{a,j} z^j$ whose coefficients $b_{a,j}$ are given by (1.2) in Section 1. Therefore, as a simple induction proves, the coefficient $b_{a,j}$ is a polynomial in a of degree j. Hence the map $(a, z) \mapsto g_a(z)$ is both a power series in z with coefficients depending on a and a power series in a with coefficients depending on z. For $|a| = R$ the map $z \mapsto g_a(z)$ is holomorphic. Hadamard's formula (root's criterion) for the radius of convergence of power series implies that the radius of convergence of g_a depends linearly on $1/|a|$. In particular, for all $|a| = R$ the domains of definition of g_a contain a fixed disc, say $|z| \leq r$ for some $r > 0$. By Lemma 1.10 in Section 1 the maps g_a are univalent on $\mathbb{D}_r$.

Therefore by Koebe's distortion formula (2.26) there exists a constant $K > 0$ such that $|g_a(z)| < K$ for all $|a| = R$ and $|z| < r$. Hence, $|b_{a,j}| \leq K/r^j$, by the maximum principle this holds for all $|a| < R$, hence $g_a(z)$ is convergent for all a. □

By Theorem 2.35 it is enough to study linearization of the quadratic polynomials P_λ. If $\lambda = e^{2\pi i\theta}$ with $\theta \notin \mathbb{Q}$ (possibly $\theta \in \mathbb{C}$, namely $|\lambda| \neq 1$), by Theorem 1.6 in Section 1 the polynomial P_λ is formally linearizable, namely, there exists $g_\lambda \in \widehat{\mathsf{Diff}}(\mathbb{C}, O)$, $g'_\lambda(0) = 1$ such that

$$P_\lambda(g_\lambda(z)) = g_\lambda(\lambda z). \tag{2.27}$$

Let us denote by $r(\lambda) \in [0, +\infty]$ the radius of convergence of g_λ.

PROPOSITION 2.36. *Let* $0 < |\lambda| < 1$. *Then*

(1) $0 < r(\lambda) \leq 2$.
(2) *The map* g_λ *extends continuously on* $\partial \mathbb{D}_{r(\lambda)}$. *Moreover* $g_\lambda : \overline{\mathbb{D}_{r(\lambda)}} \to \mathbb{C}$ *is injective and satisfies* $P_\lambda \circ g_\lambda(z) = g_\lambda(\lambda z)$.
(3) *The map* g_λ *has a unique singular point on* $\partial \mathbb{D}_{r(\lambda)}$ *which is denoted* $u(\lambda)$.
(4) $g_\lambda(u(\lambda)) = -\lambda/2$ *and* $(g_\lambda(z) + \lambda/2)^2$ *is holomorphic at* $z = u(\lambda)$.

PROOF. By Theorem 2.1, $r(\lambda) > 0$. The map g_λ is univalent in $\mathbb{D}_{r(\lambda)}$. Indeed, if $g_\lambda(z_1) = g_\lambda(z_2)$ for some $z_1, z_2 \in \mathbb{D}_{r(\lambda)}$, for all $n \in \mathbb{N}$ it follows by (2.27) that

$$P_\lambda^{\circ n}(g_\lambda(z)) = g_\lambda(\lambda^n z) \tag{2.28}$$

hence $g_\lambda(\lambda^n z_1) = g_\lambda(\lambda^n z_2)$. But g_λ is univalent in a neighborhood of O, therefore, since $|\lambda| < 1$, for some $n >> 1$ it holds $\lambda^n z_1 = \lambda^n z_2$, hence $z_1 = z_2$ and g_λ is univalent. In particular g_λ has no critical points in $\mathbb{D}_{r(\lambda)}$.

Now, note that P_λ has a unique critical point at $c_\lambda = -\lambda/2$ with critical value $v_\lambda = -\lambda^2/4$. We claim that $c_\lambda \notin g_\lambda(\mathbb{D}_{r(\lambda)})$. Indeed, if it were $g_\lambda(z_0) = c_\lambda$ for some $z_0 \in \mathbb{D}_{r(\lambda)}$, differentiating (2.27) and evaluating at z_0 we would obtain

$$0 = P'_\lambda(c_\lambda) = P'_\lambda(c_\lambda) g'_\lambda(z_0) = P'_\lambda(g_\lambda(z_0)) g'_\lambda(z_0) = \lambda g'_\lambda(\lambda z_0),$$

hence $g'_\lambda(\lambda z_0) = 0$. Thus g_λ would have a critical point in its domain of definition, contradiction. Let $r := \min\{100, r(\lambda)\}$. Then $g_\lambda : \mathbb{D}_r \to \mathbb{C}$ satisfies the hypothesis of Corollary 2.26. Thus $\mathbb{D}_{r/4} \subset g_\lambda(\mathbb{D}_r)$. But $c_\lambda \notin g_\lambda(\mathbb{D}_r)$, hence $c_\lambda \notin \mathbb{D}_{r/4}$. Thus

$$\frac{1}{2} \geq \frac{|\lambda|}{2} = |c_\lambda| \geq \frac{r}{4},$$

which implies that $r(\lambda) \leq r \leq 2$. This proves (1).

Next, we note that if $g_\lambda(z) \neq c_\lambda$ then P_λ is invertible at $g_\lambda(z)$, hence from (2.27)

$$g_\lambda(z) = P_\lambda^{-1}(g_\lambda(\lambda z)). \tag{2.29}$$

This implies in particular that g_λ can be analytically continued until its image reaches c_λ. Hence $c_\lambda \in \partial(g_\lambda(\mathbb{D}_{r(\lambda)}))$ and there exists a sequence $\{z_j\} \subset \mathbb{D}_{r(\lambda)}$ such that $g_\lambda(z_j) \to c_\lambda$. Up to extracting subsequences we can assume that $z_j \to u(\lambda)$ with $|u(\lambda)| = r(\lambda)$ (for otherwise if $|u(\lambda)| < r(\lambda)$ then $g_\lambda(u(\lambda)) = c_\lambda$ against $c_\lambda \notin g(\mathbb{D}_{r(\lambda)})$). By (2.27)

$$g_\lambda(\lambda u(\lambda)) = \lim_{j\to\infty} g_\lambda(\lambda z_j) = \lim_{j\to\infty} P_\lambda(g_\lambda(z_j)) = P_\lambda(c_\lambda) = v_\lambda.$$

Therefore, by the injectivity of g_λ, such a $u(\lambda)$ is uniquely defined, namely, if $\{z_j\} \subset \mathbb{D}_{r(\lambda)}$ is such that $g_\lambda(z_j) \to c_\lambda$ then $\{z_j\}$ have to converge to $u(\lambda)$. Hence, if we define

$$\begin{cases} g_\lambda(w) := P_\lambda^{-1}(g_\lambda(\lambda w)) & w \in \partial\mathbb{D}_{r(\lambda)} \setminus u(\lambda) \\ g_\lambda(u(\lambda)) := c_\lambda \end{cases}$$

such a function is continuous and injective on $\partial\mathbb{D}_{r(\lambda)}$ and by construction it satisfies the functional equation. This proves (2) and (3).

Finally, since $P_\lambda^{-1}(z) = -\frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda^2 + 4z}$, for $z \in \overline{\mathbb{D}_{r(\lambda)}} \setminus \{u(\lambda)\}$, by (2.29) we have

$$(g_\lambda(z) + \lambda/2)^2 = (P_\lambda^{-1}(g_\lambda(\lambda z)) + \lambda/2)^2 = \frac{\lambda^2 + 4 g_\lambda(\lambda z)}{4},$$

which defines a holomorphic function on a neighborhood of $u(\lambda)$, proving (4). □

DEFINITION 2.37. The map $\mathbb{D}^* \ni \lambda \mapsto u(\lambda)$ is called the *Yoccoz function*.

We are going to see that, quite surprisingly, the Yoccoz function is holomorphic (and extends holomorphically to 0).

THEOREM 2.38. *The Yoccoz function* $u : \mathbb{D}^* \to \mathbb{C}$ *has a holomorphic extension, still denoted* $u : \mathbb{D} \to \mathbb{D}_2$.

PROOF. For $\lambda \in \mathbb{D}^*$, let $u_n(\lambda) := \frac{P_\lambda^{\circ n}(-\lambda/2)}{\lambda^n}$. The sequence $\{u_n\}$ is composed of holomorphic functions on $\mathbb{D}^*$. We are going to show that $\{u_n\}$ converges uniformly on compacta of $\mathbb{D}^*$ to u, which will prove that $u : \mathbb{D}^* \to \mathbb{C}$ is holomorphic. Then, by Proposition 2.36 we have that $|u(\lambda)| = r(\lambda) \leq 2$, hence $u : \mathbb{D}^* \to \mathbb{C}$ is bounded and by the Riemann removable singularity's theorem u extends holomorphically to 0.

Let us then show that $\{u_n\}$ converges uniformly on compacta. Let $h(z) := \frac{g_\lambda(zu(\lambda))}{u(\lambda)}$. The function $h : \mathbb{D} \to \mathbb{C}$ is univalent, and by Koebe's distortion formula (2.26), recalling that $g_\lambda(0) = 0$ and $g'_\lambda(0) = 1$,

$$\left|\frac{g_\lambda(zu(\lambda))}{u(\lambda)}\right| = |h(z) - h(0)| \leq |h'(0)|\frac{|z|}{(1-|z|)^2} = \frac{|z|}{(1-|z|)^2}. \tag{2.30}$$

By (2.28) with $z = u(\lambda)$ and since $g_\lambda(u(\lambda)) = -\lambda/2$, we have

$$P_\lambda^{\circ n}(-\lambda/2) = g_\lambda(\lambda^n u(\lambda)). \tag{2.31}$$

Thus by (2.30) with $z = \lambda^n$,

$$|P_\lambda^{\circ n}(-\lambda/2)| = |u(\lambda)| \left|\frac{g_\lambda(\lambda^n u(\lambda))}{u(\lambda)}\right| \leq r(\lambda)\frac{|\lambda^n|}{(1-|\lambda^n|)^2} \leq 2\frac{|\lambda^n|}{(1-|\lambda|)^2}.$$

This implies that

$$|u_n(\lambda)| = \left|\frac{P_\lambda^{\circ n}(-\lambda/2)}{\lambda^n}\right| \leq \frac{2}{(1-|\lambda|)^2}$$

and therefore $\{u_n\}$ is uniformly bounded on compacta of $\mathbb{D}^*$, hence it is a normal family. Let $\{u_{n_k}\}$ be a converging subsequence. Now

$$\begin{aligned}\lim_{k\to\infty} u_{n_k}(\lambda) &= \lim_{k\to\infty} \frac{P_\lambda^{\circ n_k}(-\lambda/2)}{\lambda^{n_k}} \\ &\overset{(2.31)}{=} \lim_{k\to\infty} \frac{g_\lambda(\lambda^{n_k}u(\lambda))}{\lambda^{n_k}} = \frac{d}{dz}(g_\lambda(u(\lambda)z))|_{z=0} = u(\lambda).\end{aligned}$$

This proves both that the sequence $\{u_n\}$ is converging and the limit is u. □

Now we relate the Yoccoz function to the radius of convergence of the formal intertwining map for an elliptic germ:

PROPOSITION 2.39. *Let* $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *and let* $\lambda := e^{2\pi i\theta}$. *Then*

$$r(\lambda) \geq \limsup_{z\to\lambda} |u(z)|.$$

PROOF. Let $r := \limsup_{z\to\lambda} |u(z)|$. Assume $r > 0$. Let $\{\eta_n\} \subset \mathbb{D}$ be such that $\eta_n \to \lambda$ and $|u(\eta_n)| \to r$. The family $\{g_{\eta_n}\}$ is a family of univalent functions each of which is defined on a disc $\mathbb{D}_{r(\eta_n)}$ with $r(\eta_n) = |u(\eta_n)|$. Hence, for all $m \in \mathbb{N}$, the disc $\mathbb{D}_{r-1/m} \subset \mathbb{D}_{r(\eta_n)}$ for $n >> 1$. The family $\{g_{\eta_n}|_{\mathbb{D}_{r-1/m}}\}$ is thus a normal family by Koebe's distortion formula (2.30). Therefore, up to extracting subsequences, the sequence $\{g_{\eta_n}\}$ is converging uniformly on compacta to a function $g : \mathbb{D}_r \to \mathbb{C}$ which, by Hurwitz theorem, is either constant or univalent. Since $g_{\eta_n}(0) = 0$ and $g'_{\eta_n}(0) = 1$ then $g(0) = 0$ and $g'(0) = 1$, proving that g is univalent on $\mathbb{D}_r$. Also, since clearly $P_{\eta_n} \to P_\lambda$ and

$P_{\eta_n}(g_{\eta_n}(z)) = g_{\eta_n}(\eta_n z)$ for all n, then $P_\lambda(g(z)) = g(\lambda z)$. By the uniqueness of the intertwining map it follows that $g_\lambda|_{\mathbb{D}_r} = g$, proving that $r(\lambda) \geq r$. □

Now we are able to prove Theorem 2.34:

PROOF OF THEOREM 2.34. The Yoccoz function $u : \mathbb{D} \to \mathbb{C}$ is holomorphic and bounded, thus Fatou's lemma (see, *e.g.*, [**21**]) implies that u has radial limit almost everywhere at $\partial\mathbb{D}$. Since $u \not\equiv 0$, such radial limits must be $\neq 0$ for almost all points. Hence for almost all $\lambda \in \partial\mathbb{D}$

$$\limsup_{z\to\lambda} |u(z)| \geq \lim_{r\to 1} |u(r\lambda)| > 0,$$

and by Proposition 2.39 it follows $r(\lambda) > 0$ proving that P_λ is holomorphically linearizable for almost all $\lambda \in \partial\mathbb{D}$. □

2.11. Arithmetic forms of Siegel-Bruno-Yoccoz's theorem. In this subsection we will discuss (without proof) the arithmetic form of the Siegel-Bruno-Yoccoz's theorem 2.34. For details and proofs see, *e.g.*, [**19**]. First, we start introducing Siegel's theorem [**22**] (see also [**9**]).

Let $\lambda = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

DEFINITION 2.40. The number $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is *Diophantine* (or satisfies the *Siegel condition*) if there exist $K > 0$ and $0 < t < \infty$ such that for all $m, n \in \mathbb{N}$ and $n \neq 0$ it follows

$$\left|\theta - \frac{m}{n}\right| \geq \frac{K}{n^t}.$$

The set of Diophantine numbers is dense in $\mathbb{R}$.

THEOREM 2.41 (Siegel, 1942). *Let $f(z) = \lambda z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$ with $\lambda = e^{2\pi i\theta}$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. If θ is Diophantine then f is holomorphically linearizable.*

Next we recall briefly how continued fractions are defined, in order to introduce *Bruno's numbers*.

The *Gauss map* is defined by

$$A(x) := \left\{\frac{1}{x}\right\}$$

for $x \in \mathbb{R}$. We let $x_0 = x - [x]$, $a_0 = [x]$. Then we define by induction for $n \geq 0$

$$x_{n+1} = A(x_n) = \left\{\frac{1}{x_n}\right\}, \quad a_{n+1} = \left[\frac{1}{x_n}\right] \geq 1.$$

Then $\frac{1}{x_n} = a_{n+1} + x_{n+1}$. Therefore we have

$$x = a_0 + x_0 = a_0 + \frac{1}{a_1 + x_1} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + x_2}} = \ldots$$

As customary, we write $x = [a_0, a_1, \ldots]$ to denote the continued fraction expansion of x. We also let

$$\frac{p_n}{q_n} := [a_0, \ldots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots + \frac{1}{a_n}}}.$$

The sequence $\{\frac{p_n}{q_n}\}$ is the best approximation sequence of rational numbers for x.

DEFINITION 2.42. We say that $x \in \mathbb{R} \setminus \mathbb{Q}$ is a *Bruno number* if

$$\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty,$$

where $\{\frac{p_n}{q_n}\}$ is the best approximation sequence of rational numbers for x.

One can prove that every Diophantine number is a Bruno number but the converse is not true. Bruno [**6**], [**7**] (see also [**18**, section 5.1]) proved the following result:

THEOREM 2.43 (Bruno, 1965). *Let $f(z) = \lambda z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$ with $\lambda = e^{2\pi i\theta}$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. If θ is a Bruno number then f is holomorphically linearizable.*

Finally, in 1985 J.-C. Yoccoz [**24**] proved that Bruno's condition is sharp, namely:

THEOREM 2.44. *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be a non-Bruno number. Then the quadratic polynomial $e^{2\pi i\theta} z + z^2$ is not holomorphically linearizable.*

3. Topological normal forms

We say that two germs of diffeomorphisms f, g are *topologically conjugated* if there exists a homeomorphism φ such that $\varphi \circ f \circ \varphi^{-1} = g$.

3.1. The hyperbolic case.

THEOREM 3.1. *Let $f(z) = \lambda z + O(|z|^2) \in \mathsf{Diff}(\mathbb{C}, O)$ with $|\lambda| \neq 1$. Then*

(1) *If $|\lambda| < 1$ then f is topologically conjugated to $z \mapsto \frac{1}{2}z$.*
(2) *If $|\lambda| > 1$ then f is topologically conjugated to $z \mapsto 2z$.*

PROOF. 1. According to Theorem 2.1 the map f is holomorphically conjugated to $z \mapsto \lambda z$, so it is enough to show that λz and $\frac{1}{2}z$ are topologically conjugated. Fix $\epsilon > 0$. For $0 < r < R$ we denote by $A(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$. Let $\varphi : A(\frac{\epsilon}{2}, \epsilon) \to A(|\lambda|\epsilon, \epsilon)$ be a homeomorphism such that $\varphi(\frac{1}{2}z) = \lambda\varphi(z)$ for $|z| = \frac{\epsilon}{2}$ and $\varphi(z) = z$ for $|z| = \epsilon$. Extend by induction for $k \in \mathbb{N}$ the map $\varphi : A(\frac{\epsilon}{2^{k+1}}, \frac{\epsilon}{2^k}) \to A(|\lambda|^{k+1}\epsilon, |\lambda|^k\epsilon)$, defining inductively

$$\varphi(\frac{1}{2}z) := \lambda\varphi(z), \quad z \in A(\frac{\epsilon}{2^{k+1}}, \frac{\epsilon}{2^k}).$$

Then set $\varphi(0) = 0$. The map φ is the searched homeomorphism. The proof of 2. is similar. $\square$

3.2. The parabolic case: Camacho's theorem.

THEOREM 3.2 (Camacho). *Let $f(z) = \lambda z + O(z^2) \in \mathsf{Diff}(\mathbb{C}, O)$, $\lambda^n = 1$ for some $n \in \mathbb{N}$ and, if $n > 1$ assume $\lambda^m \neq 1$ for $1 \leq m < n$. Then*

(i) *either $f^n(z) = z$ for all z,*
(ii) *or there exists $k \in \mathbb{N}$ such that f is topological conjugate to $z \mapsto \lambda z(1 + z^{nk})$.*

REMARK 3.3. From the proof it will follow that if $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ with $a_{k+1} \neq 0$ then f is topological conjugate to $z \mapsto z + z^{k+1}$.

The idea of the proof is to look at f as a diffeomorphism of a suitable Riemann surface in such a way that f behaves like an automorphism of such a surface and it is actually topologically conjugated to it. To see how this idea comes out, we make some digressions.

Dynamics. The map $T_{\lambda,kn} : z \mapsto \lambda z(1 + z^{nk})$ preserves the union of kn lines given by $\{z : z^{kn} \in \mathbb{R}\}$. These lines divide $\mathbb{C}$ into $2nk$ sectors $\{V_j\}$, which we can enumerate counterclockwise. Thus, *e.g.*, $V_1 = \{z : 0 < \arg z < \pi/nk\}$ and more generally $V_j = \{z : (j-1)\pi/nk < \arg z < j\pi/nk\}$, $j = 1, \ldots, 2nk$. If $\lambda = 1$ then each $V_j^\delta := V_j \cap \{|z| < \delta\}$, $0 < \delta << 1$ is mapped into V_j by $T_{1,k}$. If $\lambda^n = 1$ then $T_{\lambda,kn}$ acts as a permutation on the V_j^δ (in the sense that maps V_j^δ in some other V_h). More precisely, if $\lambda = e^{2\pi i q/n}$ with $(q, n) = 1, 0 < q < n$, then $z \mapsto \lambda z$ is a rotation which maps V_j into V_{j+2kq}, modulo $2nk$. Therefore $T_{\lambda,nk}$ permutes the V_j's in cycles with length n, and there are exactly $2k$ cycles. Let $S_j = V_{2j-1} \cup V_{2j} \cup L_j^+$, where $L_j^+ = \partial V_{2j-1} \cap \partial V_{2j} \setminus \{0\}$, for $j = 1, \ldots, 2nk$. Note that each S_j contains exactly one Leau-Fatou petal for $T_{\lambda,kn}$. It is clear that if $f - T_{\lambda,nk}(z) = O(z^{nk+1})$ then f moves the sectors S_j essentially as $T_{\lambda,nk}$ does.

The idea is now to consider each sector $S_j^\delta = S_j \cap \{|z| < \delta\}$ as a chart of a Riemann surface in such a way that on each chart f is conjugated to some automorphism and such conjugations glue together in a good way on the Riemann surface.

Riemann surfaces of multivalued functions. Let us consider the holomorphic function $z \mapsto z^{-kn}$. This function has the property that each sector S_j is mapped to $\mathbb{C} \setminus [0, \infty]$ (except for $kn = 1$ where the image is $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). For $nk = 1$ we define $\mathcal{S}_1 = \mathbb{C}^*$. Now we assume $nk > 1$ and we are going to define a Riemann surface $\mathcal{S}_{nk}$ which will be a nk-th covering of $\mathbb{C}^*$. Let $U_1, \ldots, U_{nk}$ be nk-th copies of $\mathbb{C} \setminus [0, \infty]$. Glue U_1 along the upper boundary of the cut $[0, \infty]$ with U_2 along the lower boundary of the cut $[0, \infty]$. Proceed this way gluing U_j along the upper boundary with U_{j+1} along the lower boundary, $j = 1, \ldots, nk-1$ and finally glue U_{nk} along the upper boundary with U_1 along the lower boundary. Call $\mathcal{S}_{nk}$ such a (topological) surface. Now we define a one-to-one map $\pi_{nk} : \mathcal{S}_{nk} \to \mathbb{C}^*$ as follows. On $\mathbb{C} \setminus [0, \infty]$ one can define nk-th branches of the inverse function of $z \mapsto z^{-kn}$. Let us denote by $z \mapsto B_j(z)$ these branches, according to $B_j(\mathbb{C} \setminus [0, \infty]) = S_j$, $j = 1, \ldots, nk$. Then let $\pi_{nk}|_{U_j} := B_j$. By definition π_{nk} extends continuously to all $\mathcal{S}_{nk}$ and is clearly an homeomorphism on $\mathbb{C}^*$. Declaring π_{nk} to be a biholomorphism we give $\mathcal{S}_{nk}$ the structure of a Riemann surface. It is naturally a nk-th covering of $\mathbb{C}^*$ by the map $P : \mathcal{S}_{nk} \to \mathbb{C}^*$ defined on each U_j by $U_j \ni x \mapsto x \in \mathbb{C}^*$, extended obviously on all of $\mathcal{S}_{nk}$. The map P is holomorphic for one can check that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}_{nk} & = & \mathcal{S}_{nk} \\ {\scriptstyle \pi_{nk}}\downarrow & & \downarrow{\scriptstyle P} \\ \mathbb{C}^* & \xrightarrow{z \mapsto z^{-nk}} & \mathbb{C}^* \end{array}$$

Indeed we can check this using $(U_j, \pi_{nk}|_{U_j})$ as a chart, and then in local coordinates (S_j, ζ) it follows that $P(\zeta) = P(\pi_{nk}^{-1}(\zeta)) = \zeta^{-nk}$. To be precise, $\{(U_j, \pi_{nk}|_{U_j})\}$ is not an atlas for $\mathcal{S}_{nk}$ for it misses some half-lines. However one can define an atlas by constructing $\mathcal{S}_{nk}$ starting from open sets of the form $U_j' = \mathbb{C} \setminus i[0, \infty]$ (and then, instead of sectors of the form S_j one must use sectors of the form $S_j' = V_{2j} \cup V_{2j+1}$, $j = 1, \ldots, nk-1$, $S_{nk}' = V_1 \cup V_{2nk}$. Then $\{U_j, \pi_{nk}|_{U_j}\} \cup \{U_j', \pi_{nk}|_{U_j'}\}$ is the wanted atlas: we leave the details to the reader. Alternatively one can first endow $\mathcal{S}_{nk}$ with the (unique) structure of Riemann surface which makes the covering map P holomorphic, and then show that π_{nk} is a biholomorphism. From this point of view it is much more natural to consider the atlas of $\mathcal{S}_{nk}$ given by $\{U_j, \varphi_j\} \cup \{U_j', \varphi_j'\}$, where $\varphi_j(\zeta) = \zeta$, $\varphi_j'(\zeta) = \zeta$. In such local coordinates one sees that for $\zeta \in U_j$, $\pi_{kn} \circ \varphi_j^{-1}(\zeta) = B_j(\zeta) = \zeta^{-1/kn}$ where the branch is chosen so that $i^{-1/kn} \in S_j$.

PROOF OF THEOREM 3.2. By Remark 1.14 of Section 1 we can assume that

$$f(z) = \lambda z(1 + z^{nk} + O(z^{nk+1})) \tag{3.1}$$

Let $\mathbb{C}^*_r = \{z \in \mathbb{C}^* : |z| < r\}$ for a small $r > 0$. Let $\mathcal{S}^r_{nk} = \pi^{-1}_{nk}(\mathbb{C}^*_r)$. Then we can well define a holomorphic injective map $F : \mathcal{S}^r_{nk} \to \mathcal{S}_{nk}$ as

$$F = \pi^{-1}_{nk} \circ f \circ \pi_{nk}.$$

Assume that $x \in U_j$ and $F(x) \in U_l$. In the local coordinates $(U_j, \pi_{nk}|_{U_j})$ and $(U_l, \pi_{nk}|_{U_l})$ one can see that $F = f$. However if we use the local coordinates (U_j, φ_j) and (U_l, φ_l) to write down a local expression of F we obtain that, for $\zeta \in \mathbb{C} \setminus [0, \infty]$,

$$\begin{aligned} F_{jl}(\zeta) &= \varphi_l \circ F \circ \varphi_j^{-1}(\zeta) = \varphi_l \circ f(\zeta^{-1/kn}) \\ &= [f(\zeta^{-1/kn})]^{-kn} = \zeta - kn + c\zeta^{-1/kn} + \dots, \end{aligned}$$

where the branch of $\zeta^{-1/kn}$ is chosen so that $i^{-1/kn} \in S_j$. Note that, for what we said about dynamics, if r is sufficiently small then F^n maps each $U_j \cap \mathcal{S}^r_{nk}$ *essentially* into U_j (with this we mean that the image is almost all in U_j, and the rest is in $U'_{j-1} \cup U'_j$, counted modulo kn).

We define an injective holomorphic map $G : \mathcal{S}^r_{nk} \to \mathcal{S}_{nk}$ in the following way. If $x \in U_j$ and $F(x) \in U_l$ then

$$G(x) := \varphi_l^{-1}(\varphi_j(x) - kn).$$

Similarly if $x \in U'_j$ and $F(x) \in U'_l$ then we define $G(x) := (\varphi'_l)^{-1}(\varphi'_j(x) - kn)$. We have only to check that the map $G(x)$ is well defined if $x \in U_j \cap U'_j$, which follows at once from the definition of φ_j, φ'_j. By the very definition it follows that in local coordinates

$$G_{jl}(\zeta) = \varphi_l \circ G \circ \varphi_j^{-1}(\zeta) = \zeta - kn.$$

The upshot is to show that F is topologically conjugated to G on S^r_{nk}, which will imply that f is topologically conjugated to $g := \pi_{nk} \circ G \circ \pi^{-1}_{nk}$ on $\mathbb{C}^*_r$. Since also $\lambda z(1 + z^{nk})$ is topologically conjugated to g this will prove the theorem.

We define a new C^∞ diffeomorphism $K : \mathcal{S}^r_{nk} \to \mathcal{S}_{nk}$ by gluing together F and G. Such a map K is better defined on $\mathbb{C}^*_r = \pi_{nk}(\mathcal{S}^r_{nk})$. Let $0 < r_2 < r_1 < r < 1$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $0 \le \rho \le 1$, $\rho|_{[-\infty,0]} \equiv 0$, $\rho|_{[1,+\infty]} \equiv 1$. Then we define

$$k(z) = g(z) + \rho\left(\frac{r_1 - |z|}{r_1 - r_2}\right)[f(z) - g(z)].$$

The map K is then given by $K = \pi^{-1}_{nk} \circ k \circ \pi_{nk}$. We have to show that K is a diffeomorphism. To see this, we notice that $F_{jl}(\zeta) - G_{jl}(\zeta)$ goes to zero as ζ tends to infinity, for any j, l. This means that $|f(z) - g(z)|$ is very small as $r << 1$. In particular then k is C^1-close to the diffeomorphism g on $\mathbb{C}^*_r$ and therefore it is a diffeomorphism, and hence K is.

Let $\mathcal{S}^{r_1}_{nk} = \pi^{-1}_{nk}(\mathbb{C}^*_{r_1})$ and similarly define $\mathcal{S}^{r_2}_{nk}$. Then $\mathcal{S}^{r_1}_{nk} \cap U_j$ is given by $\{\zeta \in U_j : |\zeta| > r_1^{-kn}\}$, while $\mathcal{S}^{r_2}_{nk} \cap U_j = \{\zeta \in U_j : |\zeta| > r_2^{-kn}\}$. By definition $K \equiv G$ on $B := \mathcal{S}^r_{nk} \setminus \mathcal{S}^{r_1}_{nk}$, while $K \equiv F$ on $\mathcal{S}^{r_2}_{nk}$.

It is now enough to show that K is topologically conjugated to G. The idea is to define a conjugation H on a set E, called *exaggerated fundamental domain*, such that for any $x \in \mathcal{S}_{nk}$ there exists $a \in \mathbb{Z}$ such that $G^a(x) \in E$, and then extend the conjugation by means of the relation $H \circ G \circ H^{-1} = K$.

We let L_t be defined as $L_t \cap U_t := \{\zeta \in \mathcal{S}^r_{nk} : \mathrm{Re}\zeta = 0\}$, for $t = 1, \ldots, k$. By definition we know that $G^n(L_t) \subset U_t$, and actually $G^n(L_t)$ is a line in $S^r_{nk} \cap U_t$ given by the translation of L_t. We let $L'_t = G^n(L_t)$, $t = 1, \ldots, k$. We define the exaggerated fundamental set E to be the union of B and the $2k$-th semi-strips bounded by $L_t \cap U_t$, $L'_t \cap U_t$ for $t = 1, \ldots, k$. This set E is clearly fundamental for G, for G^n is a translation on each U_j with step given by the distance between L_1 and L'_1; also, if $k > 1$ then G permutes cyclically the n charts contained into the k-th cycles, as explained when talking about dynamics. Now we define $H|_B = Id$, $H|_{L_t} = Id$, while we define $H|_{L'_t} := K|_{L'_t}$ for $t = 1, \ldots, k$. Clearly H conjugates G to K on $B \cup_t L_t \cup_t L'_t$. Now we simply extend H as a diffeomorphism into the interior of each semi-strip between L_t and L'_t. For $x \in \mathcal{S}^r_{nk}$ we can define $H(x)$ by means of $H(x) := K^{-a} \circ H \circ G^a(x)$, where $a \in \mathbb{Z}$ is the minimum (in modulus) integer such that $G^a(x) \in E$. For this definition to make sense, we have to be sure that $K^{-b}(H(G^a(x)) \in \mathcal{S}^r_{nk}$ for $b = 1, \ldots, a$ if $a > 0$ ($b = -1, \ldots, -a$ if $a < 0$). Indeed $K(B) \not\subset \mathcal{S}^r_{nk}$ (this corresponds dynamically to the existence of repelling directions). However, from the fact that $H|_B = Id$ one can easily see that the definition is well posed. Finally, we note that H is, by construction, a diffeomorphism. □

REMARK 3.4. The proof shows that, if $f^n(z) \neq z$, then actually f is C^∞-conjugated to $\lambda z(1 + z^{kn})$ outside 0.

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