Dynamical Systems and an example: Billiards

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Contents

A foreword

These are partial notes for courses held in Maryland in the Fall of 2024 and Rome Fall 2025. The purpose here is to provide a basic understanding for people with essentially no prior knowledge of the subject. First, I will discuss basic examples and concepts. Then, how to establish hyperbolicity and how to establish ergodicity. As the field is very broad, I will focus on the idea and techniques rather than attempting to present an exhaustive overview of the field. Additionally, I will attempt to present the ideas in a clear manner, illustrating how they can be applied to other relevant dynamical systems (e.g., cones and hyperbolicity, Hopf argument for ergodicity). The notes are both more and less extensive than the lectures. I apologize for that, but writing notes is a rather time-consuming activity for someone slow like. Additionally, as I wrote them in a hurry, they may contain errors. So read at your own risk, and apologies again.

Chapter 1

Instability everywhere

Although this is not the proper occasion for a historical excursus, it is worthwhile to stress that the first Dynamical Systems widely investigated have been the planetary motions. Not surprisingly, the main emphasis in such investigations was accurate prediction of future positions. Nevertheless, exactly from the effort of predicting accurately future motions stemmed the consciousness of the existence of very serious obstructions to such a program. Specifically, in the work of Poincaré [60] appeared for the first time the phenomena of instability with respect to initial conditions, a central concept in the understanding of modern Dynamical Systems. In fact, we will see briefly that such instability phenomena can be already observed in very simple systems–such as a periodically forced pendulum–that exhibit a so called "homoclinic tangle" [56, 58].

The realization that many relevant systems are very sensitive with respect to the initial conditions dealt a strong blow to the idea that it is always possible to predict the future behavior of a system,[1] yet the work of many physicists (and we must mention at least Maxwell, Boltzmann, and Gibbs) and mathematicians (in particular, the so called *Russian School* with people like Kolmogorov, Anosov, Sinai, but also some Western mathematicians, like Birkhoff, Smale, Ruelle and Bowen made important contributions) led to the understanding that, although precise predictions were not possible, it was possible and, at times, even easy to make statistical predictions. The concept of statistical properties of a Dynamical System will be addressed

[1] Without going to the extreme of some authors of the eighteenth century arguing that, given the present state of the universe, a sufficiently powerful mind (maybe God) could predict all the future. Think, more modestly, of an isolated system and imagine to use some numerical scheme to try to solve the equations of motion for an arbitrarily long time with an arbitrary precision.

in the following chapters. This chapter is dedicated to making precise, in a simple example, the nature of the above mentioned instability.

1.1 A pendulum–The model and a question

We will study a seemingly trivial example: a forced pendulum. To be more concrete, let us imagine a pendulum of length $l = 1$ meter, mass $m = 1$ kilogram and remember that the gravitational constant (on the earth's surface) is approximately $g = 9.8$ meters per second squared. The Hamiltonian of the system reads [36]

$$H = \frac{1}{2l^2m}p^2 - mgl\cos\theta, \tag{1.1.1}$$

where θ is the angle, counted counterclockwise, formed by the pendulum with the vertical direction ($\theta = 0$ corresponds to the configuration in which the pendulum assumes the lowest possible position) and $p = l^2m\dot{\theta}$ is the associated momentum. Thus, (θ, p) are the coordinates of the pendulum. The phase space $\mathcal{M}$ where the motion takes place consists of $\mathbb{T}^1 \times \mathbb{R}$.

The equations of motion associated with the Hamiltonian (1.1.1) represent the motion of an ideal pendulum in the vacuum, feeling only the force of gravity. Clearly, this is a highly idealized situation with no counterpart in reality. Every system interacts with the rest of the universe. Thus, the only hope for the idea of *isolated systems* to be fruitful is that the interaction with the exterior does not significantly affect the behavior of the system. Let us try to see what this can mean in reality.

The first issue is clearly friction. Let us imagine that we have set up the pendulum in a reasonable vacuum and reduced the friction at the suspension point so that the loss of energy is negligible on the time scale of a few minutes. Does such a system behave as an isolated pendulum within such a time frame? One problem is that the suspension point is still in contact with the rest of the world. If the pendulum is in a lab not so distant from a street (a rather common situation), then the traffic will induce some vibrations. It is then natural to ask: what happens if the suspension point of the pendulum vibrates?

In fact, nothing much happens for small pendulum oscillations (this is a consequence of Komogorv-Arnold-Moser theory, a highly non trivial fact), but if we start close to the vertical configuration, it is conceivable that a motion that would be oscillatory for the unperturbed pendulum could gather enough energy from the external force as to change its nature and become rotatory, this would create a substantial difference between the unperturbed (ideal) and the perturbed (more realistic) case.

This is exactly the question we want to address:

Question: *Can we really predict the motion for a reasonable time if the initial condition is close to the vertical ?*

We will assume that the frequency of vibration ω is of the order of one hertz[2] and the amplitude of the oscillations is very, very small. Hence, as good mathematicians, we will call such an amplitude ε. In other words, the suspension point moves vertically according to the law $\varepsilon \cos \omega t$.

The Hamiltonian of the vibrating pendulum is then given by (see Problem 1)

$$H_\varepsilon(\theta, p, t) = \frac{1}{2l^2 m} p^2 - mgl\cos\theta - \varepsilon m\omega^2 l \cos\omega t \cos\theta. \tag{1.1.2}$$

Accordingly, the equation of motion are (see Problem 1)[3]

$$\begin{aligned} \dot{\theta} &= \frac{\partial H_\varepsilon}{\partial p} = \frac{p}{l^2 m} \\ \dot{p} &= -\frac{\partial H_\varepsilon}{\partial \theta} = -mgl\sin\theta - \varepsilon m\omega^2 l\cos\omega t \sin\theta. \end{aligned} \tag{1.1.3}$$

It is well known that the function H is an integral of motion for the solutions of (1.1.3) for $\varepsilon = 0$, that is: H computed along the solutions of the associated equations of motion is constant.[4] The physical meaning of H is the energy of the system. Clearly, the energy H_ε is not constant in general since the vibration can add or subtract energy to the pendulum.

1.2 Instability–unperturbed case

Let us first recall a few basic facts about the unperturbed pendulum. The equations of motion are given by the (1.1.3) setting $\varepsilon = 0$. It is obvious that there exist two fixed points: $(0, 0)$ which corresponds to the pendulum at

[2]One hertz corresponds to one oscillation every second, and it can be the order of magnitude for the frequency of a vibration transmitted through the ground (R waves) at a reasonable distance. Thus we are assuming $\omega = 2\pi$.

[3]Here we write the Hamilton equations associated to the Hamiltonian, see [4, 36] for the general theory.

[4]See [4, 36] for this general fact or do Problem 4 for the simple case at hand.

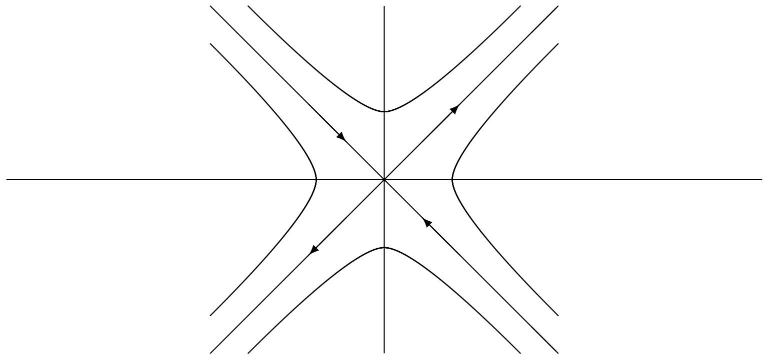

Figure 1.1: Unstable fixed point (phase portrait)

rest and is clearly stable, and $(\pi, 0)$, which corresponds to the pendulum in the vertical position and is certainly unstable. Our interest here is to analyze the motions that start close to the unstable equilibrium and to make more precise what it is meant by *instability*.

1.2.1 Unstable equilibrium

If we want to have an idea of how the motion looks like near a fixed point the natural first step is to study the linearization of the equation of motion near such a point. In our case, using the coordinates $(\theta_0, p) = (\theta - \pi, p)$, they look like

$$\begin{aligned} \dot{\theta}_0 &= \frac{p}{l^2 m} \\ \dot{p} &= mgl\theta_0. \end{aligned} \tag{1.2.4}$$

Let $\omega_p = \sqrt{\frac{g}{l}}$, the general solution of (1.2.4) is

$$(\theta_0(t), p(t)) = (\alpha e^{\omega_p t} + \beta e^{-\omega_p t}, ml^2\omega_p\{\alpha e^{\omega_p t} - \beta e^{-\omega_p t}\}),$$

where α and β are determined by the initial conditions. Note that if the initial condition has the form $\alpha(1,\, ml\sqrt{gl})$ it will evolve as $\alpha e^{\omega_p t}(1,\, ml\sqrt{gl})$. While if the initial condition is of the form $\beta(1,\, -ml\sqrt{gl})$ it will evolve as $\beta e^{-\omega_p t}(1,\, -ml\sqrt{gl})$. In other words the directions $(1,\, ml\sqrt{gl})$ and $(1,\, -ml\sqrt{gl})$ are invariant for the linear dynamics. The first direction is expanded (and because of this is called *unstable direction*) while the second is contracted (*stable direction*).

Let us imagine to start the motion from an initial condition of the type $(\pi + \theta_0, 0)$, $\theta_0 \in [-\delta, \delta]$, where $\delta \leq 10^{-4}$ represents the precision with which we are able to set the initial condition (one tenth of a millimeter); what will happen under the linear dynamics?

Our initial condition correspond to choosing, at time zero, $\alpha = \beta \leq \frac{\delta}{2}$. As time goes on the coefficient of β becomes exponentially small while the coefficient of α increases exponentially, thus a good approximation of the position of the pendulum after some time is given by

$$\theta_0(t) \approx \alpha e^{\omega_p t}. \tag{1.2.5}$$

Since $\omega_p \approx 3.13$ seconds^{-1}, it follows that after about 2.5 seconds the position of the pendulum can be anywhere up to a distance of about 10 centimeters from the unstable position.

This means that the unstable position is really unstable and if we tray, as best as we can, to put the pendulum in the unstable equilibrium (always imagining that the friction has been properly reduced) it will typically fall after few seconds and it will fall in a direction that we are not able to predict (since it depend on the sign of δ, our unknown mistake). Nevertheless, after the ideal pendulum starts falling in one direction the subsequent motion is completely predictable, as we will see shortly.

An obvious objection to the above analysis is that I did not show that the linearized equation describes a motion really close to the one of the original equations. The answer to this question is particularly simple in this setting and is addressed in the next subsection.

1.2.2 The unstable trajectories (separatrices)

Given the already noted fact that, for $\varepsilon = 0$, H is a constant of motion, the phase space $\mathcal{M}$ is naturally foliated in the level curves of H, on which the motion must take place. This allows us to obtain a fairly accurate picture of the motions of the unperturbed pendulum. In fact, the level curves are given by the equations

$$\frac{p^2}{2l^2 m} - mgl\cos\theta = E$$

where E is the energy of the motion. It is easy to see that $E = -mgl$ corresponds to the stable fixed point $(\theta, p) = (0, 0)$; $-mgl < E < mgl$ corresponds to oscillations of amplitude $\arccos\left[\frac{E}{mgl}\right]$; $E > mgl$ corresponds to rotatory motions of the pendulum. The last case $E = mgl$ is of particular

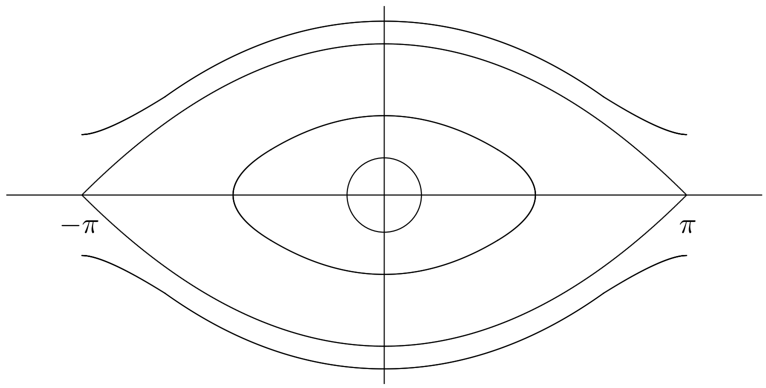

Figure 1.2: Unperturbed pendulum (phase portrait)

interest to us: obviously it corresponds to the unstable fixed point $(\pi, 0)$, yet there are other two solution that travel on the two curves

$$p = \pm ml\sqrt{2lg(1+\cos\theta)}.$$

This two curves are the ones that separate the oscillatory motions from the rotatory ones and, for this reasons, are called *separatrices*. It is very important to understand the motion along such trajectories, luckily the two differential equations

$$\dot{\theta} = \pm\sqrt{2\frac{g}{l}(1+\cos\theta)}. \tag{1.2.6}$$

can be integrated explicitly (see Problem 5) yielding, for $\theta(0) = 0$,

$$\theta(t) = 4\arctan e^{\pm\omega_p t} - \pi. \tag{1.2.7}$$

This orbits are asymptotic to the unstable fixed point both at $t \to +\infty$ and at $-\infty$ and, for $|t|$ large, agree with the linear behaviour of section 1.2.1. This situation is somewhat atypical as we will see briefly.

1.3 The perturbed case

1.3.1 Reduction to a map

The motion of the above system takes place on the cylinder $\mathcal{M} = S^1 \times \mathbb{R}$. By the theorem of existence and uniqueness for the solutions of differential equations follows immediately the possibility to define the maps $\phi_\varepsilon^t : \mathcal{M} \to \mathcal{M}$ associating to the point (θ, p) the point reached by the solution of (1.1.3) at time t, when starting at time 0 from the initial condition (θ, p). In such a way, we define the *flow* ϕ_ε^t associated to the (1.1.3).

Clearly $\phi_\varepsilon^0(\theta, p) = (\theta, p)$, that is, the map corresponding to time zero is the identity. Moreover, if $\varepsilon = 0$ the system is autonomous (the vector field does not depend on the time), hence the flow defines a group: for each $t, s \in \mathbb{R}$

$$\phi_0^{t+s}(\theta, p) = \phi_0^t(\phi_0^s(\theta, p)).$$

This corresponds to the obvious fact that the motion for a time $t + s$ can be obtained first as the motion from time 0 to time s, and then pretending that the time s is the initial time and follows the motion for time t.

Of course, the above fact does not hold anymore when $\varepsilon \neq 0$. In this case, the maps ϕ_ε^t depend on our choice of the initial time (if we define them by starting from time 1 instead then time 0, in general, we obtain different maps). Nevertheless, because the external force is periodic, some of the above nice properties can be retained.

Let us define the map $T_\varepsilon : \mathcal{M} \to \mathcal{M}$ by

$$T_\varepsilon = \phi_\varepsilon^{\frac{2\pi}{\omega}},$$

then (see Problem 3), for each $n \in \mathbb{Z}$,

$$T_\varepsilon^n = \phi_\varepsilon^{\frac{2n\pi}{\omega}}. \tag{1.3.8}$$

The interest of (1.3.8) is that, for many purposes, we can study the map T_ε instead than the more complex object ϕ_ε^t. Morally, it means that if we look at the system *stroboscopically*, that is only at the times $\frac{2\pi}{\omega} n$ with $n \in \mathbb{Z}$, then it behaves like an autonomous (time independent) system.[5] Another interesting fact is that the flow ϕ_ε^t (and hence also the map T_ε) is area preserving (see Problem 7).[6]

1.3.2 Perturbed pendulum, $\varepsilon \neq 0$

The situation for the case $\varepsilon \neq 0$ is more complex and no easy way exists to study these motions.

[5] Another instance of a very simple case of a very fruitful and general strategy: to look at the system only when some special event happens–in our case, at each time in which the suspension point has its maximum height.

[6] This also is a special instance of a more general fact: the Hamiltonian nature of the system, see [4, 36] if you want to know more.

As a general strategy, to study the behavior of a system (in our case the map T_ε) it is a good idea to start by investigating simple cases and then move on from there. In our systems the simplest motion consists of the equilibrium solutions. These are the time independent solutions.[7] Because of the special type of perturbation chosen the fixed points of the system for the case $\varepsilon = 0$ remain unchanged when $\varepsilon \neq 0$ (see Problem 8 for a brief discussion of a more general case).

Next, we can study the infinitesimal nature of the fixed points. It is natural to expect that the nature of the two fixed points does not change if ε is small, yet to verify this requires some checking. We will discuss explicitly only the fixed point $(\pi, 0)$.

The first step is to make precise the sense in which the case $\varepsilon \neq 0$ is a perturbation of the case $\varepsilon = 0$. This can be achieved by obtaining an explicit estimate on the size of

$$R_\varepsilon = \varepsilon^{-1}(T_0 - T_\varepsilon).$$

Let $z(t) = (z_1(t), z_2(t)) = \phi_0^t(x) - \phi_\varepsilon^t(x)$, then substituting in (1.1.3) and subtracting the general case to the case $\varepsilon = 0$ it yields

$$\begin{aligned} |\dot{z}_1| &\leq \frac{|z_2|}{ml^2} \\ |\dot{z}_2| &\leq mgl|z_1| + \varepsilon m\omega^2 l. \end{aligned}$$

In order to get better estimates it is convenient to define the new variables $\zeta_1 = z_1$ and $ml^2\omega_p\zeta_2 = z_2$. In these new variables the preceding equations read

$$\begin{aligned} |\dot{\zeta}_1| &\leq \omega_p|\zeta_2| \\ |\dot{\zeta}_2| &\leq \omega_p|\zeta_1| + \varepsilon\frac{\omega^2}{\omega_p l}. \end{aligned} \tag{1.3.9}$$

Which implies $\|\dot{\zeta}\| \leq \omega_p\|\zeta\| + \varepsilon m\omega^2 l$. Taking into account that, in our situation, $ml^2\omega_p > 1$, it follows (see Problem 9)

$$\|R\|_{\mathcal{C}^0} \leq \frac{m\omega^2}{l\omega_p}(e^{2\pi\frac{\omega_p}{\omega}} - 1) \leq 69.$$

Unfortunately, the above norm does not suffice for our future needs. We will see quite soon that it is necessary to estimate also the first derivatives of R, that is the $\mathcal{C}^1$ norm.

[7] That is, equilibrium solutions for the map T_ε. These are *periodic* solutions for the flows of period $\frac{2\pi}{\omega}$. In fact, $T_\varepsilon x = x$ means $\phi^{\frac{2\pi}{\omega}} x = x$.

To do so the easiest way is to use the differentiability with respect to the initial conditions of the solutions of our differential equation. Fixing any point $x \in \mathcal{M}$ and calling $\xi^\varepsilon(t) = d_x\phi^t_\varepsilon\xi(0)$ we readily obtain:[8]

$$\begin{aligned} \dot{\xi}_1^\varepsilon &= \frac{\xi_2^\varepsilon}{l^2 m} \\ \dot{\xi}_2^\varepsilon &= -mgl\cos\theta\,\xi_1^\varepsilon - \varepsilon m\omega^2 l\cos\omega t\cos\theta\,\xi_1^\varepsilon \end{aligned} \tag{1.3.10}$$

One can then estimate the $\mathcal{C}^1$ norm of R by estimating $\|\xi^\varepsilon(\frac{2\pi}{\omega}) - \xi^0(\frac{2\pi}{\omega})\|$, since $\xi^\varepsilon(\frac{2\pi}{\omega}) = D_{(\theta,p)}T_\varepsilon\xi^\varepsilon(0)$. Doing so one obtains[9]

$$\|R\|_{\mathcal{C}^1} \leq \frac{2m\omega^2}{l\omega_p}e^{3\pi\frac{\omega_p}{\omega}} := d_1 \leq 690. \tag{1.3.11}$$

1.4 Infinitesimal behavior (linearization)

As a first application of the above considerations let us study the linearization of T_ε at $x_f = (\pi, 0)$. From (1.3.10) follows (see Problem 12)

$$\begin{aligned} D_{x_f}T_0 &= \begin{pmatrix} \cosh\frac{2\pi\omega_p}{\omega} & \frac{\sinh\frac{2\pi\omega_p}{\omega}}{ml^2\omega_p} \\ ml^2\omega_p\sinh\frac{2\pi\omega_p}{\omega} & \cosh\frac{2\pi\omega_p}{\omega} \end{pmatrix} \\ D_{x_f}T_\varepsilon &= D_{x_f}T_0 + \mathcal{O}(d_1\varepsilon) \end{aligned} \tag{1.4.12}$$

The eigenvalues of $D_{x_f}T_\varepsilon$ are then $\lambda_\varepsilon = e^{\frac{2\pi\omega_p}{\omega}} + \mathcal{O}(d_2\varepsilon)$,[10] λ_ε^{-1}, where $d_2 = 2d_1\omega_p ml^2 \simeq 4400$. In addition, calling v_ε, $\langle v_\varepsilon, v_0\rangle = 1$, the eigenvector associate to λ_ε, holds true $\|v_0 - v_\varepsilon\| \leq d_3\varepsilon$, $d_3 = 4\lambda_0^{-1}\omega_p^2\omega^2 l^4 d_1 \simeq 1200$.[11]

Clearly, if ε is sufficiently small, then $\lambda_\varepsilon > 1$. This means that the hyperbolic nature of the unstable fixed point remains unchanged under small

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[8] The vector $\xi_\varepsilon(t)$ is nothing else than the derivative $\frac{d\phi^t_\varepsilon(x+s\xi(0))}{ds}|_{s=0}$, the following equation is then obtained by exchanging the derivative with respect to t with the derivative with respect to s.

[9] The following bounds are not sharp, working more one can obtain better estimates but this would not make much of a difference in the sequel.

[10] In this chapter, we will adopt the strict convention that $\mathcal{O}(x)$ means a quantity bounded, in absolute value, by x.

[11] This follows by the fact that the eigenvalues of $D_{x_f}T_0$ are $e^{\pm\frac{2\pi\omega_p}{\omega}} \simeq (23)^{\pm 1}$, a simple perturbation theory of matrices (see Problems 10, 11) and the already mentioned fact that the map T_ε is area preserving, thus the determinant of its derivative must be one.

perturbations (see Problem 13 for a case when the perturbation is not so small).[12]

If one does a similar analysis at the fixed point $(0,0)$ one finds that the eigenvalues have modulus one: that is the infinitesimal motion is a rotation around the fixed point, exactly as in the $\varepsilon = 0$ case.

Hence the comments made at the end of subsection 1.2.1 for the unperturbed pendulum hold for the perturbed pendulum as well. Only now the is no longer an integral of motion (the energy) that controls globally the behavior of the system.

Imagining that the map is linear (which is clearly false but, as we will see, qualitatively not so wrong) this would mean that the distance between two trajectories can be expanded by almost a factor 23 in a second. Initial conditions that are δ close at time zero will be about 23δ far apart after 1 second. If such a state of affair could persist (and we will see it may) after one minute the two configurations would differ roughly by a factor $10^{80}\delta$, which means that not even knowing the initial condition plus or minus a quark could we predict the final one. This is certainly a rather worrisome perspective but much more work it is needed to decide if this may be indeed the case.

1.5 Local behavior (Hadamard-Perron Theorem)

The next step is to try to go from the above infinitesimal analysis to a local picture in a small neighborhood of the fixed points.

It is natural to expect that the two fixed points are still stable and unstable respectively, yet this is a far from trivial fact.

The stability of the point $(0,0)$ can be proven by invoking the so called KAM Theorem (this exceeds the scope of the present book, and we will not discuss such matters, see [36] for such a discussion).[13]

The study of the local behavior around the point x_f is instead a bit easier and can be performed by applying the Hadamard-Perron Theorem to

[12] As we will see later in detail, hyperbolicity means that there is a direction in which the maps expands (the eigenvector v_ε^u associated to the eigenvalue λ_ε) and a direction in which the map contracts (the eigenvector v_ε^s associated to the eigenvalue λ_ε^{-1})

[13] In some sense, this implies that we can indeed predict the motion for an extremely long time if we consider only oscillations close to the configuration $(0,0)$, so in that case the assumption that the pendulum is isolated is legitimate. Yet, this depends on the precision we are interested in and tends to degenerate if the amplitude of the oscillations is rather large. A complete analysis would be a very complicated matter, but we will have an idea of the type of problems that can arise by considering extremely large oscillations, close to a full rotation of the pendulum.

conclude that, in a neighborhood of $(\pi, 0)$, there exists two curves $x^u_\varepsilon(s) = (\theta^u_\varepsilon(s), p^u_\varepsilon(s))$, $x^s_\varepsilon(s)$ that are invariant with respect to the map T_ε.
Namely, there exists $\delta_\varepsilon > 0$ such that $T_\varepsilon x^s_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon]) \subset x^s_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon])$ and $T_\varepsilon^{-1} x^u_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon]) \subset x^u_\varepsilon([-\delta_\varepsilon, \delta_\varepsilon])$; these are called the local stable and unstable manifold of zero, respectively. Essentially δ_ε is determined by the requirement that the non-linear part of T_ε be smaller than the linear part.

Clearly, for $\varepsilon = 0$ $x^s_0 = x^u_0 = x_0$ and it coincides with the homoclinic orbit of the unperturbed pendulum. In addition, by Hadarmd-Perron and the estimates of the previous section, we can choose δ_ε such that

$$\|x^u_\varepsilon - x_0\| \leq 2d_3\varepsilon\|x_0\|. \tag{1.5.13}$$

and the analogous for the stable manifold. We have obtained a local picture of the behavior of the map T_ε, yet this does not suffice to answer our original question. To do so, we need to follow the motion for at least a full oscillation: this requires global information.

To gain a more global knowledge, we can try to construct a larger invariant set for the map T_ε. A natural way to do so is to iterate: define $W^u = \cup_{n=0}^\infty T_\varepsilon^n x^u([-\delta_\varepsilon, \delta_\varepsilon])$. Since $T_\varepsilon x^u([-\delta_\varepsilon, \delta_\varepsilon]) \supset x^u([-\delta_\varepsilon, \delta_\varepsilon])$, it is clear that each time we iterate, we get a longer and longer curve. The set W^u is then clearly a manifold, and it is called the global unstable manifold.[14]

The global manifold, as the name clearly states, is a global object: it carries information on the dynamics for arbitrarily long times. Yet, the procedure by which it has been defined is far from constructive, and the truth is that, besides the sketchy considerations above, at the moment we know very little of it. The next step is to gain a more detailed understanding of a large portion of W^u.

1.6 A more global understanding (Melnikov)

From the above considerations follows that the stable and unstable manifolds $(\theta^s_\varepsilon(s), p^s_\varepsilon(s))$, $(\theta^u_\varepsilon(s), p^u_\varepsilon(s))$, $|s| \leq \delta_\varepsilon$, of T_ε at 0, are ε close to the homoclinic orbit of the unperturbed pendulum, $(\theta_0(t), p_0(t))$, $\theta_0(0) = 0$.

Note, however, that while $x_0 = (\theta_0, p_0)$ is invariant under the unperturbed flow, the same does not apply to $(\theta^{s,u}_\varepsilon(s), p^{s,u}_\varepsilon(s))$ under ϕ^t_ε. The invariant object is the time-space surface $(\tau, x^{s,u}_\varepsilon(s,\tau)) := (\tau, \phi^\tau_\varepsilon(\theta^s_\varepsilon(s), p^s_\varepsilon(s)))$ where $(s,\tau) \in [-\delta_\varepsilon, \delta_\varepsilon] \times [0, \frac{2\pi}{\omega}]$ and and $\tau = t \mod \frac{2\pi}{\omega}$.[15]

[14] Applying the above procedure to the unperturbed problem yields the full separatrix.
[15] A standard way to bring the present non-autonomous setting into the more familiar

We can choose freely the parameterization of our curves in such a surface, and some are more convenient than others. The separatrix of the unperturbed pendulum is most conveniently parametrized by time, hence $\phi^t(\theta_0(s), p_0(s)) = (\theta_0(s+t), p_0(s+t))$. Note that the separatrix can be visualized as a graph of $(\theta, G(\theta))$. Analogously, for ε small enough, the perturbed unstable manifold of T_ε will be the graph of $(\theta, G^u_\varepsilon(\theta))$, for $\theta \in [0, \frac{3}{2}\pi]$. Given $\theta \in [0, \frac{3}{2}\pi]$, let $S_n = 2\pi\omega^{-1}n$. Let $z_n := (\theta_n, G^u_\varepsilon(\theta_n)) = \phi_\varepsilon^{-S_n}(\theta, G^u_\varepsilon(\theta))$, by Hadamrad-Perron we know that $|G^u_\varepsilon(\theta) - G(\theta)| \leq C\theta$ for $\theta \in [0, \delta]$, also $|\theta_n| \leq Ce^{-an}$ for some $C, a > 0$. The basic idea is to compute[16]

$$\begin{aligned}
H_0(z_n) &= H_0(z) - \int_{-S_n}^0 \frac{d}{ds} H \circ \phi_s^\varepsilon(z) ds \\
&= H_0(z) - \int_{-S_n}^0 \langle \nabla H_0, J\nabla H_\varepsilon \rangle \circ \phi_\varepsilon^s(z) ds \\
&= H_0(z) - \varepsilon \int_{-S_n}^0 \langle \nabla H_0, J\nabla H_1 \rangle \circ \phi_\varepsilon^s(z) ds.
\end{aligned}$$

Note that equation (1.5.13) implies $\|\phi_\varepsilon^s(\theta, G^u_\varepsilon(\theta)) - \phi_0^t(\theta, G(\theta))\| \leq C\varepsilon$, moreover $z_n \to 0$ as $n \to \infty$ and $\nabla H_0(0) = 0$. Thus, taking the limit $n \to \infty$, yields

$$H_0(\theta, G^u_\varepsilon(\theta)) = H_0(0) + \varepsilon \int_{-\infty}^0 \langle \nabla H_0, J\nabla H_1 \rangle \circ \phi_0^s(z) ds + \mathcal{O}(\varepsilon^2).$$

Consequently,

$$\begin{aligned}
G^u_\varepsilon(\theta) - G_0(\theta) &= \frac{G^u_\varepsilon(\theta)^2 - G_0(\theta)^2}{G^u_\varepsilon(\theta) + G_0(\theta)} = \frac{2\ell^2 m(H_0(\theta, G^u_\varepsilon(\theta)) - H_0(\theta, G_0(\theta)))}{G^u_\varepsilon(\theta) + G_0(\theta)} \\
&= \frac{2\ell^2 m(H_0(\theta, G^u_\varepsilon(\theta)) - H_0(0))}{G^u_\varepsilon(\theta) + G_0(\theta)} \\
&= 2\ell^2 m\varepsilon \frac{\int_{-\infty}^0 \langle \nabla H_0, J\nabla H_1 \rangle \circ \phi_0^s(\theta, G(\theta)) ds + \mathcal{O}(\varepsilon)}{G^u_\varepsilon(\theta) + G_0(\theta)}
\end{aligned}$$

autonomous one is to introduce the fake variables $(\varphi, \eta) \in S^1 \times \mathbb{R}$ and the new, time independent, Hamiltonian $\bar{H}_\varepsilon(\theta, p, \varphi, \eta) := H_\varepsilon(\theta, p, \varphi) + \frac{2\pi}{\omega}\eta$. The Hamilton equations yield $\varphi(t) = \frac{2\pi}{\omega}t + \varphi(0)$ and hence the equations for θ, p reduce to (1.1.3). Since $\bar{H}_\varepsilon$ is now conserved under the motion we can restrict the system to the three dimensional manifold $\bar{H}_\varepsilon = 0$. In such a manifold, we have the *weak* stable and unstable manifolds (now flow invariant) $(x_\varepsilon^{s,u}(s, \varphi), \varphi, -\frac{2\pi}{\omega} H_\varepsilon((x_\varepsilon^{s,u}(s, \varphi), \varphi))$.

[16] As usual $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

This allows us to conclude

$$G^u_\varepsilon(\theta) - G_0(\theta) = \varepsilon \ell^2 m \frac{\int_{-\infty}^0 \langle \nabla H_0, J\nabla H_1\rangle \circ \phi_0^s(\theta, G(\theta))ds}{G_0(\theta)} + \mathcal{O}(\varepsilon^2)$$

Arguing analogously for the stable manifold yields

$$G^u_\varepsilon(\theta) - G^s_\varepsilon(\theta) = \varepsilon \ell^2 m \frac{\int_{\mathbb{R}} \langle \nabla H_0, J\nabla H_1\rangle \circ \phi_0^s(\theta, G(\theta))ds}{G_0(\theta)} + \mathcal{O}(\varepsilon^2). \quad (1.6.14)$$

The separatrix of the unperturbed pendulum is most conveniently parametrized by time, hence

$$\phi^t(\theta_0(s), p_0(s)) = (\theta_0(s+t), p_0(s+t)) = (\theta_0(s+t), G(\theta_0(s+t)) =: x_0(s+t).$$

Setting $\Delta(s) = \varepsilon^{-1}\ell^{-2}m^{-1}[G^u_\varepsilon(\theta_0(s)) - G^s_\varepsilon(\theta_0(s))]G_0(\theta_0(s))$, one can compute

$$\Delta(\sigma) = \int_{-\infty}^{\infty} \{H_1,\, H\}_{x_0(t+\sigma)}dt + \mathcal{O}\left(\varepsilon d_4 e^{2\omega_p|\sigma|}\right), \quad (1.6.15)$$

where and explcit computation yields $d_4 \simeq 4\cdot 10^6$, and the curly brackets stand for the so called *Poisson brackets* ($\{f,\, g\}_x = \langle J\nabla_x f,\, \nabla_x g\rangle$).
The integral in (1.6.15) is called *Melnikov integral* and provides an expression, at first order in ε, of the distance between the stable and the unstable manifold. All we are left with is to compute the integrals in (1.6.15). This turns out to be an exercise in complex analysis, and it is left to the reader, the result is:[17]

$$\int_{-\infty}^{\infty} \{H_1(\cdot, t),\, H\}_{x_0(t+\sigma)}dt = 8\pi m l \frac{\omega^4 e^{-\frac{\pi\omega}{2\omega_p}}}{\omega_p^2(e^{\frac{\pi\omega}{\omega_p}} - 1)} \sin\omega\sigma.$$

We have thus gained a very sharp control on the shape of the above

[17] A simple computation yields:

$$\{H_1,\, H\}_{x_0(t+s)} = -\frac{\omega^2}{l} p(t+s)\cos\omega t \sin\theta(t+s).$$

Then, by using (1.2.7) and looking at Problem 6, one readily obtains:

$$\{H_1,\, H\}_{x_0(t)} = 4\frac{\omega^2}{l}\frac{\cos\omega(t-s)\sinh\omega_p t}{(\cosh\omega_p t)^3}.$$

Finally, use Problem 15.

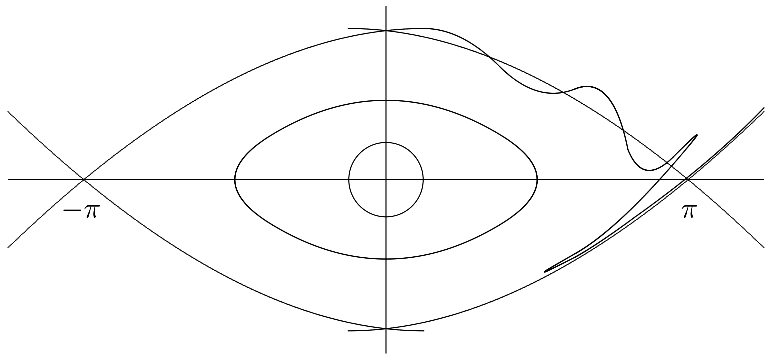

Figure 1.3: Perturbed pendulum

manifolds.[18] In particular, $\Delta(\pm 1/4) \simeq \pm 76 + \mathcal{O}(4 \cdot 10^7 \varepsilon) \neq 0$ provided $\varepsilon \leq 1.5 \cdot 10^{-6}$, that is the two manifolds intersect. To understand a bit better such an intersection (we would like to know that in the region $\sigma \in [-1/4, 1/4]$ there is only one *transversal* intersection) its suffices to notice that (1.6.14) provides a control on the angle between x_ε^u and x_0.

This intersections are called *homoclinic* intersection and their very existence is responsible for extremely interesting phenomena as can be readily seen by trying to draw the stable and unstable manifolds (see Figure 1.3 for an approximate first idea); we will discuss this issue in detail shortly.[19]

We have gained much more global information on the map T_ε, yet it does not suffice to answer to our question. The next section is devoted to obtaining a really global picture. Up to now we have used mainly analytic tools. Next, geometry will play a much more significant rôle.[20]

[18] Note that ε must be exponentially small with respect to ω. In many concrete problems (notably the so-called *Arnold diffusion* it happens that this it is not the case. One can try to solve such an obstacle by computing the next terms of the ε expansion of Δ. In fact, it turns out that it is possible to express Δ as a power series in ε with all the terms exponentially small in ω . Yet this is a quite complex task far beyond our scope.

[19] Note that the intersection corresponds to an homoclinic orbit for the map T_ε (that is, an orbit which approaches the fixed point x_f both in the future and in the past). This is what it is left of the homoclinic orbit of the unperturbed pendulum.

[20] What comes next is the first example in this book of what is loosely called *a dynamical argument*.

1.7 Global behavior (an horseshoe)

We want to explicitly construct trajectories with special properties. A standard way to do so is to start by studying the evolution of appropriate regions and to use judiciously the knowledge so gained. Let us see what this does mean in practice.

The starting point is to note that we understand the shape of the invariant manifold but not very well the dynamics on them, this is our next task. Since points on the unstable manifolds are pulled apart by the dynamics, the estimate must be done with a bit of care. In fact, we will use a way of arguing which it typical when instabilities are present, we will see many other instances of this type of strategy in the sequel.

For each x in the unstable manifold (zero included) let us call $D_x^u T_\varepsilon := D_x T_\varepsilon v^u(x)$, where $v^u(0) = v^u$ and if $x = x_\varepsilon^u(t)$ then $v^u(x) = \|\dot{x}_\varepsilon^u(t)\|^{-1}\dot{x}_\varepsilon^u(t)$, that is the derivative of the map computed along the unstable manifold. A useful idea in the following is the concept of *fundamental domain*. Define $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by $x_\varepsilon^u(t) = x_\varepsilon^u(\alpha(t))$. Then $[t, \alpha(t)]$ is a fundamental domain and has the property that, setting $t_i := \alpha^i(t)$, the sets $\alpha^i[t_0, t_1]$ intersect only at the boundary.

Lemma 1.7.1 (Distortion) *For each x, y in the same fundamental domain of the unstable manifold, $\delta_0 > 0$, and $n \in \mathbb{N}$ such that $\|T_\varepsilon^n x\| \le \delta_0$, holds*[21]

$$e^{-\delta_0 C_2} \le \left|\frac{D_x^u T_\varepsilon^n}{D_y^u T_\varepsilon^n}\right| \le e^{\delta_0 C_2},$$

where $C_2 = \sup_{t\le 0}\left|\frac{\ddot{\alpha}(t)}{\dot{\alpha}(t)}\right|$.

Proof. The proof is a direct application of the chain rule:

$$\left|\frac{D_x^u T_\varepsilon^n}{D_y^u T_\varepsilon^n}\right| = \prod_{i=1}^n \left|\frac{D_{T^i x}^u T_\varepsilon}{D_{T^i y}^u T_\varepsilon}\right| \le \mathrm{Exp}\left[\sum_{i=1}^n |\log(|D_{T^i x}^u T_\varepsilon) - \log(|D_{T^i y}^u T_\varepsilon|)|\right]$$
$$\le \mathrm{Exp}\left[\sum_{i=1}^n C_2\|T^i x - T^i y\|\right] = \mathrm{Exp}\left[\sum_{i=1}^n C_2\|x_\varepsilon^u(t_i) - x_\varepsilon^u(t_{i-1})\|\right] \le e^{C_2\delta_0}.$$

The other inequality is obtained by exchanging the rôle of x and y. □

[21] This quantity is commonly called *Distortion* because it measures how much the map differs from a linear one (notice that if T is linear then $\frac{D_x T}{D_y T} = 1$). Although apparently an innocent quantity, it is hard to overstate its importance in the study of hyperbolic dynamics.

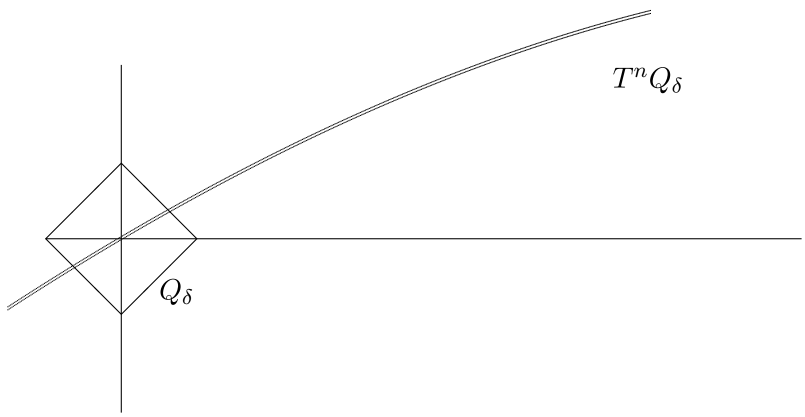

Figure 1.4: The evolution of the small box Q_δ

Next we would like to consider the evolution of a small box constructed around the fix point.

Consider the following small parallelogram: $Q_\delta := \{\xi \in \mathbb{R}^2 \mid \xi = av^u + bv^s \text{ for some } a, b \in [-\frac{\delta}{2}, \frac{\delta}{2}]\}$, $\delta \ll \delta_0$. Next consider the first $n \in \mathbb{N}$ such that $T_\varepsilon^n Q_\delta \cap \{\theta = 0\} \neq \emptyset$. Our first task is to understand the shape of $T_\varepsilon^n Q_\delta$ near $\{\theta = 0\}$. Since a fundamental domain in the latter region is of order one, while at the boundary of Q_ε is of order δ, Lemma 1.7.1 implies that the expansion is proportional to $C\delta^{-1}$. By the area preserving of the map it follows that $T_\varepsilon^n Q_\delta$ must be contained din a $C\delta^2$ neighborhood of the unstable manifold, see Figure 1.4.

By the previous section considerations on the shape of the invariant manifolds $T^n Q_\delta \cap T^n Q_\delta \neq \emptyset$, moreover they intersect *transversally*.[22]

This is all is needed to construct an horseshoe. In particular, in our case it means that $T^{2n_0} Q_\delta \cap Q_\delta \neq \emptyset$, in fact the intersection are transversal and consists of three strips almost parallel to the unstable sides. One contains zero, and it is the lest interesting for us, the other two cross above and below the unstable manifold respectively. The width of such a strip is about δ^{-3}. We will discuss in the next chapters all the implications of this situation, here it suffices to notice that if we have two initial conditions in $T^{-2n_0} Q_\delta \cap Q_\delta$ at a distance h, after $2n_0$ iterations the two points will be in Q_δ again but

[22]The meaning of *transversally* is the following: the square Q_δ has two sides parallel to v^u (the unstable direction), which we will call unstable sides, and two sides parallel to v^s (the stable direction), which we will call stable sides. Then the intersection is transversal if it consists of a region with again four sides: two made of the image of the unstable sides and two made of images of stable sides of Q_δ.

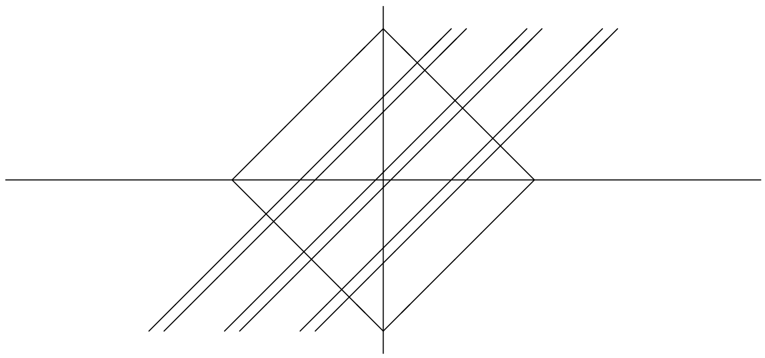

Figure 1.5: Horseshoe construction

at a distance $h\varepsilon^{-1}$. Since to decide if after that there will be a rotation or an oscillation we need to know the final position with a precision of order δ, we need to know the initial position with a precision $\mathcal{O}(\delta\varepsilon) = \mathcal{O}(\delta^3)$.

Note that in the above construction, we have lost almost all the points, only the ones that come back to Q_δ at time $2n_0$ are under control. Nevertheless, we can consider the set $\Lambda := \cup_{k\in\mathbb{Z}} \bigcap T_\varepsilon^{2kn_0} Q_\delta$. This is clearly a measure zero set, yet it is far from empty (it contains uncountably many points) and it is made of points that at times multiples of $2n_0$ are always in Q_δ. When they arrive in Q_δ they will rotate if they are above the separatrices and oscillate otherwise. Let us call this two subsets of Q_δ R and O. Given a point $\xi \in Q_\delta$ we can associate to it the doubly infinite sequence $\sigma \in \{0,1\}^{\mathbb{Z}}$ by the rule $\sigma_i = 1$ iff $T^{2n_0 i}\xi \in R$. The reader can check that the correspondence is onto.

1.8 Conclusion–an answer

If $\varepsilon = 10^{-6}$ and δ is a millimeter then we need to know the initial condition with a precision of 10^{-9} meters if we want to decide if the point will come back or rotate when it will get almost vertical again (this will happen in about 6 seconds). By the same token if we want to answer the same question, but for the second time the pendulum get close to the unstable position, we need to know the initial condition with a precision of the order 10^{-15} meters, and this just to predict the motion for about 12 seconds.[23]

[23]Remark that it is not just a matter of precision on the initial condition, it is also a matter of how one actually does the prediction. If the method is to integrate numerically the equation of motion, then one has to insure that the precision of the algorithm is of the

We can finally answer to our original question:

Answer: *NO!*

Nevertheless, as we mentioned at the beginning, the above answer it is not the end of the story. In fact, there exists many other very relevant questions that can be answered.[24] The rest of the book deals with a particular type of question: can we meaningfully talk about the *statistical behavior* of a system?

Problems

1.1 Derive the Lagrangian, Hamiltonian and equations of motions for a pendulum attached to a point vibrating with frequency ω and amplitude ε. (Hint: see [47, 36] on how to do such things. Remember that two Lagrangian that differ by a total time derivative give rise to the same equation of motions and are thus equivalent.)

1.2 Consider the systems of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ and f smooth and bounded. Prove that the associated flow form a group. (Hint: use the uniqueness of the solutions of the ordinary differential equation)

1.3 Consider the systems of differential equations $\dot{x} = f(x,t)$, $x \in \mathbb{R}^n$ and f smooth, bounded and periodic in t of period τ. Let ϕ^t be the associated flow. Define $T = \phi^\tau$, prove that $T^n = \phi^{n\tau}$.

1.4 Show that the Hamiltonian is a constant of motion for the pendulum. (Hint: Compute the time derivative)

1.5 Prove (1.2.7). (Hint: Write (1.2.6) in the integral form

$$t = \int_0^t \frac{\dot{\theta}(s)}{\sqrt{\frac{2g}{l}(1+\cos\theta(s))}} ds.$$

order of 10^{-15}. This maybe achieved by working in double precision but if one wants to make predictions of the order of one minute it is quite clear that the numerical problem becomes very quickly intractable.

[24] For example: which type of motions are possible? This is a *qualitative* question. Such type of questions give rise to the qualitative theory of Dynamical Systems [58, 40], an extremely important part of the theory of dynamical systems, although not the focus here.

Using some trigonometry and changing variable obtain

$$t = \int_0^{\theta(t)} \frac{1}{2\omega_p \cos\frac{\theta}{2}} d\theta.$$

Thne compute it.)

1.6 If $\theta(t)$ is the motion obtained in the previous problem, show that

$$\sin\theta(t) = 2\frac{\sinh\omega_p t}{(\cosh\omega_p t)^2}; \quad \cos\theta(t) = \frac{2}{(\cosh\omega_p t)^2} - 1;$$
$$\cos^2\frac{\theta(t)+\pi}{4} = \frac{1}{1+e^{2\omega_p t}}.$$

1.7 Consider the systems of differential equations $\dot{x} = f(x,t)$, $x \in \mathbb{R}^n$ and f smooth. Suppose further that $\mathrm{d}ivf = 0$ (that is $\sum_{i=1}^n \frac{\partial f_i}{\partial x_i} = 0$). Show that the associated flow preserves the volume. (Hint: note that this is equivalent to saying that $|\det d\phi^t| = 1$, moreover by the group property and the chain rule for differentiating it suffices to check the property for small t. See that $d\phi^t = \mathbb{1} + Dft + \mathcal{O}(t^2) = e^{Dft+\mathcal{O}(t^2)}$. Finally, remember the formula $\det e^A = e^{\mathrm{Tr}A}$.)

1.8 Let $T, T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth maps such that $T0 = 0$ and $\det(\mathbb{1} - D_0T) \neq 0$. Consider the map $T_\varepsilon = T + \varepsilon T_1$ and show that, for ε small enough, there exists points $x_\varepsilon \in \mathbb{R}^2$ such that $T_\varepsilon x_\varepsilon = x_\varepsilon$. (Hint: Consider the function $F(x,\varepsilon) = x - T_\varepsilon x$ and apply the Implicit Function Theorem to $F = 0$.)

1.9 Let $x(t) \in \mathbb{R}^n$ be a smooth curve satisfying $\|\dot{x}(t)\| \leq a(t)\|x(t)\| + b(t)$, $x(0) = x_0$, $a, b \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}_+)$, prove that

$$\|x(t) - x_0\| \leq \int_0^t e^{\int_s^t a(\tau)d\tau} \left[a(s)\|x_0\| + b(s)\right] ds.$$

(Hint: Note that $\|x(t) - x_0\| \leq \int_0^t \|\dot{x}(s)\| ds$. Transform the differential inequality into an integral inequality. Show that if $z(t) \leq 0$ and $z(t) \leq \int_0^t z(s)ds$, then $z(t) \leq 0$ for each t. Use the last fact to compare a function satisfying the obtained integral inequality with the solution of the associated integral equation. This, essentially, is the Gronwall inequality, which, indeed, you can use directly.)

1.10 Given two by two matrices A, B such that A has eigenvalues $\lambda \neq \mu$, show that the matrix $A_\varepsilon = A + \varepsilon B$, for ε small enough, has eigenvalues $\lambda_\varepsilon, \mu_\varepsilon$ analytic as functions of ε. Show that the same holds for the eigenvectors. (Hint:[25] consider z in the resolvent of A, that is $(z-A)^{-1}$ exists. Then $(z - A_\varepsilon) = (z - A)(\mathbb{1} - \varepsilon(z - A)^{-1}B)$. Accordingly, if ε is small enough, $(z - A_\varepsilon)^{-1} = \left\{\sum_{n=0}^{\infty} \varepsilon^n \left[(z - A)^{-1}B\right]^n\right\}(z - A)^{-1}$. Finally, if γ, γ' are curves on the complex plane containing λ and μ, respectively, verify that

$$\Pi_\varepsilon := \frac{1}{2\pi i}\int_\gamma (z - A_\varepsilon)^{-1}dz \qquad \Pi'_\varepsilon := \frac{1}{2\pi i}\int_{\gamma'} (z - A_\varepsilon)^{-1}dz$$

are commuting projectors and $A_\varepsilon = \lambda_\varepsilon \Pi_\varepsilon + \mu_\varepsilon \Pi'_\varepsilon$. Finally verify that

$$\lambda_\varepsilon \Pi_\varepsilon := \frac{1}{2\pi i}\int_\gamma z(z - A_\varepsilon)^{-1}dz \qquad \mu_\varepsilon \Pi'_\varepsilon := \frac{1}{2\pi i}\int_{\gamma'} z(z - A_\varepsilon)^{-1}dz.$$

The statement follows then from the fact that the right hand side of the above equalities is written as a power series in ε.[26])

1.11 Given two by two matrices A, B such that A has eigenvalues $\lambda \neq \mu$, show that the matrix $A_\varepsilon = A + \varepsilon B$ has eigenvalues $\lambda_\varepsilon, \mu_\varepsilon$ such that $|\lambda_\varepsilon - \lambda| \leq C\varepsilon\|B\|$ and $|\mu_\varepsilon - \mu| \leq C\varepsilon\|B\|$. Compute C. (Hint: By Problem 10 we know that $\lambda_\varepsilon, \mu_\varepsilon$ are differentiable function of ε and the same holds for the corresponding eigenvector $v_\varepsilon, \tilde{v}_\varepsilon$. Let us discuss λ_ε since the other eigenvalues can be treated in the same way. One possibility is to use the above formula for $\lambda_\varepsilon \Pi_\varepsilon$ to obtain the wanted estimates.

In alternative, let v, w, $\langle w, v\rangle = 1$ and $\|v\| = 1$, be the eigenvectors of A, with eigenvalue λ and of A^*, with eigenvalue $\bar{\lambda}$, respectively. Hence $\Pi_0 = v \otimes w$ and $\|\Pi_0\| = \|w\|$. Normalize v_ε such that $\langle v_\varepsilon, w\rangle = 1$. Differentiate then the above constraint and the defining equation $(A + \varepsilon B)v_\varepsilon = \lambda_\varepsilon v_\varepsilon$ obtaining (the prime refers to the derivative with respect to ε)

$$\begin{aligned} &Av'_\varepsilon + Bv_\varepsilon + \varepsilon Bv'_\varepsilon = \lambda'_\varepsilon v_\varepsilon + \lambda_\varepsilon v'_\varepsilon \\ &\langle v'_\varepsilon, w\rangle = 0. \end{aligned}$$

[25] Of course, for matrices one could argue more directly by looking at the characteristic polynomial. Yet the strategy below has the advantage to work even in infinitely many dimensions (that is, for operators over Banach spaces).

[26] This is a very simple case of the very general problem of perturbation of point spectrum, see [39] if you want to know more.

Multiplying the first for w yields $\lambda'_\varepsilon = \langle w, Bv_\varepsilon\rangle + \varepsilon\langle w, Bv'_\varepsilon\rangle$. Setting $\tilde{A} := A - \lambda\Pi_0$ we have

$$v'_\varepsilon = (\lambda - \tilde{A})^{-1}\left[Bv_\varepsilon + \varepsilon Bv'_\varepsilon - \lambda'_\varepsilon v_\varepsilon - (\lambda - \lambda_\varepsilon)v'_\varepsilon\right].$$

Next, consider ε_0 such that, for $\varepsilon < \varepsilon_0$ holds

$$\|v'_\varepsilon\| \le 4\|(\lambda - \tilde{A})^{-1}\|\,\|B\|\,\|w\| = 4\|(\lambda - \tilde{A})^{-1}\|\,\|B\|\,\|\Pi_0\| =: C_0, \tag{1.8.16}$$

then $\|v_\varepsilon - v\| \le \varepsilon C_0$ and $|\lambda'_\varepsilon| \le \|B\|\,\|w\|(1 + 2\varepsilon C_0)$. If $4\varepsilon_0 C_0 < 1$, then, indeed, (1.8.16) holds true.)

1.12 Compute D_0T. (Hint: solve (1.3.10) for $\varepsilon = 0$, $\theta = \pi$, $p = 0$ and $t = \frac{2\pi}{\omega}$.)

1.13 Compute D_0T_ε and see that, if ω is sufficiently large, the eigenvalues have modulus one (the unstable point becomes stable!). (Hint: setting $\xi := \xi_1$ equation (1.3.10) yields $\ddot{\xi} = \omega_p^2\xi + \varepsilon\frac{\omega^2}{l}\cos\omega t\xi$. It is then convenient to write $\xi := \bar{\xi} + \varepsilon\eta + \varepsilon^2\zeta$ where $\ddot{\bar{\xi}} = \omega_p^2\bar{\xi}$ and $\ddot{\eta} = \omega_p^2\eta + \frac{\omega^2}{l}\cos\omega t\,\bar{\xi}$. One can look for a solution of the latter equation of the form

$$\bar{\eta} = Ae^{\omega_p t}\cos\omega t + Be^{\omega_p t}\sin\omega t + Ce^{-\omega_p t}\cos\omega t + De^{-\omega_p t}\sin\omega t.$$

This allows to compute $D_0T_\varepsilon(\alpha,\beta) = (\xi_1(\frac{2\pi}{\omega}), \xi_2(\frac{2\pi}{\omega})) + \mathcal{O}(\varepsilon^2)$, where $(\xi_1(0), \xi_2(0)) = (\alpha,\beta)$. Finally one can verify that, for ε small and ω large enough the eigenvalues of D_0T_ε are imaginary, hence the equilibrium is linearly stable.)

1.14 Given an Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$, for each solution $x(t)$ of the associated equations of motion show that $\langle\nabla_{x(t)}H, \dot{x}(t)\rangle = 0$.

1.15 Compute the following integrals (1.6.15):

$$\int_{\mathbb{R}} e^{iat}(\cosh t)^{-n}\sinh t\,dt,$$

$a \in \mathbb{R}$ and $n \in \mathbb{N}$, $n > 1$. (Hint: By a change of variable, one can consider only the case $a > 0$. Consider the integral on the complex plane, and show that the integral on the half circle $Re^{i\phi}$, $\phi \in [0,\pi]$, goes to zero as $R \to \infty$, then check that the poles of the integrand, on the complex plane, lie on the imaginary axis, finally use the residue theorem to compute the integrals. The result, for $a > 0$, is:

$$\int_{\mathbb{R}} e^{iat}(\cosh t)^{-n} \sinh t = 2\pi i \sum_{k=0}^{\infty} \frac{\phi_{n,k}^{(n-1)}(i\frac{2k+1}{2}\pi)}{(n-1)!},$$

where

$$\phi_{n,k}(z) = e^{iza} \sinh z \left(\frac{z - i\frac{2k+1}{2}\pi}{\cosh z} \right)^n .$$

For $n = 3$ the above formula yields

$$\int_{\mathbb{R}} e^{iat}(\cosh t)^{-3} \sinh t = \pi a^2 e^{-\frac{\pi}{2}a}(1 - e^{-\pi a})^{-1}. \quad)$$

1.16 Do the same analysis carried out for the pendulum with a vibrating suspension point in the case of a pendulum subject to an external force $\varepsilon \cos \omega t$ and in presence of a small friction $-\varepsilon^2 \gamma \dot{\theta}$.

Notes

As already mentioned in the text, the first to realize that the motions arising from differential equations can be very complex was probably Poincaré [60]. At the time the main problem in celestial mechanics (the famous n-body problem) was to find all the integral of motion. Dirichlet and Weierstrass worked on this problem, but Poincaré was the first to rise serious doubt on the existence of such integrals (which would have implied regular motions). For more historical remarks see [56]. In fact, all the content of this chapter is inspired by the more sophisticated, but more qualitative, analysis in [56].

Chapter 2

General facts and definitions

This chapter discusses some general facts concerning (measurable) dynamical systems. It is intended for readers with no previous knowledge of Dynamical Systems.

The chapter contains few basic facts, some of which will be used in the following while others are meant to provide a wider context to the material actually discussed. For a much more complete discussion of the relevant concepts the reader is referred to [55], [40].

2.1 Basic Definitions and examples

Definition 1 *By Dynamical System*[1] *with discrete time we mean a triplet* (X, T, μ) *where* X *is a measurable space,*[2] μ *is a probability measure and* T *is a measurable map from* X *to itself that preserves the measure (i.e.,* $\mu(T^{-1}A) = \mu(A)$ *for each measurable set* $A \subset X$*).*

[1] To be really precise this is the definition of "Measurable Dynamical Systems," hopefully the reader will excuse this abuse of language. More generally a Dynamical System can be defined as a set X together with a map $T : X \to X$ or, even more generally, an algebra $\mathcal{A}$ (e.g., the algebra of the functions on X) and an isomorphism $\tau : \mathcal{A} \to \mathcal{A}$ (e.g., $\tau f := f \circ T$). This last definition is so general as to include Stochastic Processes and Quantum Systems. A further generalization consists in realizing that the above setting can be viewed as the action of the semigroup $\mathbb{N}$ (or the group $\mathbb{Z}$ if T is invertible) on the algebra $\mathcal{A}$. One can then consider other groups (already in the next definition the group is $\mathbb{R}$), for example, $\mathbb{Z}^n$ or $\mathbb{R}^n$, this goes in the direction of the Statistical Mechanics and it has received a lot of attention lately. Yet, such a generality is excessive for the task at hand.

[2] By measurable space we simply mean a set X together with a σ-algebra that defines the measurable sets.

An equivalent characterization of invariant measure is $\mu(f\circ T) = \mu(f)$ for each $f \in L^1(X,\mu)$ since, for each measurable set A, $\mu(\chi_A\circ T) = \mu(\chi_{T^{-1}A}) = \mu(T^{-1}A)$, where χ_A is the characteristic function of the set A.[3]

Remark 2.1.1 *In this book we will always assume $\mu(X) < \infty$ (and quite often $\mu(X) = 1$, i.e. μ is a probability measure). Nevertheless, the reader should be aware that there exists a very rich theory pertaining to the case $\mu(X) = \infty$, see [1].*

Definition 2 *By Dynamical System with continuous time we mean a triplet (X, ϕ_t, μ) where X is a measurable space, μ is a measure and ϕ_t is a measurable group ($\phi^t(x)$ is a measurable function for a.e. $x \in X$, $\phi^t(x)$ is a measurable function of t for almost all $x \in X$; $\phi^0 =$identity and $\phi^t \circ \phi^s = \phi^{t+s}$ for each $t, s \in \mathbb{R}$) or semigroup ($t \in \mathbb{R}^+$) from X to itself that preserves the measure (i.e., $\mu(\phi_t^{-1}A) = \mu(A)$ for each measurable set $A \subset X$).*

The above definitions are very general; this reflects the wideness of the field of Dynamical Systems. In the present book, we will be interested in much more restricted situations.

In particular, X will always be a topological compact space. The measures will always belong to the class $\mathcal{M}(X)$ of probability Borel measures on X.[4] For future use, given a topological space X and a map T let us define $\mathcal{M}_T$ as the collection of all Borel measures that are T invariant.[5]

Tipically, X will consist of finite unions of smooth manifolds (eventually with boundaries). Analogously, the dynamics (the map or the flow) will be almost surely differentiable on X.

Let us see few examples to get a feeling of how a Dynamical System can look like.

2.1.1 Examples

Rotations

–Let $\mathbb{T}$ be $\mathbb{R}$ mod 1. By this we mean $\mathbb{R}$ quotiented with respect to the equivalence relations $x \sim y$ if and only if $x - y \in \mathbb{Z}$. $\mathbb{T}$ can be though as the interval [0, 1] with the points 0 and 1 identified. We put on it the topology

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[3] We use the notation, for each measurable function f, $\mu(f) = \int_X f(x)\mu(dx)$.

[4] Remember that a Borel measure is a measure defined on the Borel σ-algebra, that is the σ-algebra generated by the open sets. A probability measure gives measure one to the full space X.

[5] Obviously, for each $\mu \in \mathcal{M}_T$, (X, T, μ) is a Dynamical Systems.

induced by the topology of $\mathbb{R}$ via the defined equivalence relation. Such a topology is the usual one on $[0, 1]$, apart from the fact that each open set containing 0 must contain 1 as well. Clearly, from the topological point of view, $\mathbb{T}$ is a circle. We choose the Borel σ-algebra. By μ we choose the Lebesgue measure m, while $T : \mathbb{T} \to \mathbb{T}$ is defined by

$$Tx = x + \omega \mod 1,$$

for some $\omega \in \mathbb{R}$. In essence, T translates, or rotates, each point by the same quantity ω. It is easy to see that the measure μ is invariant (Problem 2.4).

Bernoulli shift

–A Dynamical System needs not live on some differentiable manifold, more abstract possibilities are available.

Let $\mathbb{Z}_n = \{1, 2, ..., n\}$, then define the set of two sided (or one-sided) sequences $\Sigma_n = \mathbb{Z}_n^{\mathbb{Z}}$ ($\Sigma_n^+ = \mathbb{Z}_n^{\mathbb{Z}_+}$). This means that the elements of Σ_n are sequences $\sigma = \{..., \sigma_{-1}, \sigma_0, \sigma_1, ...\}$ ($\sigma = \{\sigma_0, \sigma_1, ...\}$ in the one-sided case) where $\sigma_i \in \mathbb{Z}_n$. To define the measure and the σ-algebra it is necessary a bit of care. To start with, consider the *cylinder sets*, that is the sets of the form

$$A_i^j = \{\sigma \in \Sigma_n \mid \sigma_i = j\}.$$

Such sets will be our basic objects and can be used to generate the algebra $\mathcal{A}$ of the cylinder sets via unions and intersections. We can then define a topology on Σ (the product topology, if $\{1, \dots, n\}$ is endowed by the discrete topology) by declaring the above algebra made of open sets and a basis for the topology. To define the σ-algebra we could take the minimal σ-algebra containing $\mathcal{A}$, yet this it is not a very constructive definition, neither a particular useful one, it is better to invoke the Carathèodory construction.

Let us start by defining a measure on $\mathbb{Z}_n$, that is n numbers $p_i > 0$ such that $\sum_{i=1}^n p_i = 1$. Then, for each $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_n$,

$$\mu(A_i^j) = p_j.$$

Next, for each collection of sets $\{A_{i_l}^{j_l}\}_{l=1}^s$, with $i_l \neq i_k$ for each $l \neq k$, we define

$$\mu(A_{i_1}^{j_1} \cap A_{i_2}^{j_2} \cap ... \cap A_{i_s}^{j_s}) = \prod_{l=1}^{s} p_{j_l}.$$

We now know the measure of all finite intersection of the sets A_i^j. The measure of the union of two sets A, B obviously must satisfy $\mu(A \cup B) =$

$\mu(A) + \mu(B) - \mu(A \cap B)$. We have so defined μ on $\mathcal{A}$. It is easy to check that such a μ is σ-additive on $\mathcal{A}$; namely: if $\{A_i\} \subset \mathcal{A}$ are pairwise disjoint sets and $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. The following step is to define the outer measure[6]

$$\mu^*(A) := \inf_{\substack{B \in \mathcal{A} \\ B \supset A}} \mu(A) \quad \forall A \subset \Sigma.$$

Finally, we can define the σ-algebra as the collection of all the sets that satisfy the *Carathèodory's criterion*, namely A is measurable (that is belongs to the σ-algebra) iff

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \forall E \subset \Sigma.$$

The reader can check that the sets in $\mathcal{A}$ are indeed measurable.

The Carathèodory Theorem then asserts that the measurable sets form a σ-algebra and that on such a σ-algebra μ^* is σ-additive, thus we have our measure μ.[7] The σ-algebra so obtained is nothing else that the completion with respect to μ of the minimal σ-algebra containing $\mathcal{A}$ (all the sets with zero outer measure are measurable).

The map $T : \Sigma_n \to \Sigma_n$ (usually called *shift*) is defined by

$$(T\sigma)_i = \sigma_{i+1}.$$

We leave to the reader the task to show that the measure is invariant (see Problem 2.11).

To understand what's going on, let us consider the function $f : \Sigma \to \mathbb{Z}_n$ defined by $f(\sigma) = \sigma_0$. If we consider T^t, $t \in \mathbb{N}$, as the time evolution and f as an observation, then $f(T^t\sigma) = \sigma_t$. This can be interpreted as the observation of some phenomenon at various times. If we do not know anything concerning the state of the system, then the probability to see the value j at the time t is simply p_j. If $n = 2$ and $p_1 = p_2 = \frac{1}{2}$, it could very well be that we are observing the successive outcomes of tossing a fair coin where 1 means head and 2 tail (or vice versa); if $n = 6$ it could be the outcome of throwing a dice and so on.

[6] An outer measure has the following properties: i) $\mu^*(\emptyset) = 0$; ii) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$; iii)$\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$. Note that μ^* need not be additive on all sets.

[7] See [50] if you want a quick look at the details of the above Theorem or consult [62] if you want a more in-depth immersion in measure theory. If you think that the above construction is too cumbersome see Problem 2.13.

Dilation

–Again $X = \mathbb{T}$ and the measure is Lebesgue. T is defined by

$$Tx = 2x \mod 1.$$

This map it is not invertible (similarly to the one sided shift). Note that, in general, $\mu(TA) \neq \mu(A)$ (e.g., $A = [0, \frac{1}{2}]$).

Arnold cat

–This is an automorphism of the torus and gets its name from a picture drawn by Arnold [3]. The space X is the two dimensional torus $\mathbb{T}^2$. The measure is again Lebesgue measure and the map is

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \mod 1 =: L\begin{pmatrix} x \\ y \end{pmatrix} \mod 1.$$

Since the entries of L are integer numbers it is clear that T is well defined on the torus; in fat, it is a linear toral automorphism. The invariance of the measure follows from $\det(L) = 1$.

Hamiltonian Systems

– Up to now, we have seen only examples with discrete time. Typical examples of Dynamical Systems with continuous time are the solutions of an ODE or a PDE. Let us consider the case of a Hamiltonian system. The simplest case is when $X = \mathbb{R}^{2n}$, the σ-algebra is the Borel one and the measure μ is the Lebesgue measure m. The dynamics is defined by a smooth function $H : X \to \mathbb{R}$ via the equations

$$\frac{dx}{dt} = J\text{grad}H(x)$$

where $\text{grad}(H)_i = (\nabla H)_i = \frac{\partial H}{\partial x_i}$ and J is the block matrix

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

The fact that m is invariant with respect to the Hamiltonian flow is due to the Liouville Theorem (see [4] or Problem 2.6).

Such a dynamical system has a natural decomposition. Since H is an integral of the motion, for each $h \in \mathbb{R}$ we can consider $X_h = \{x \in X \mid H(x) =$

$h\}$. If $X_h \neq \emptyset$, then it will typically consist of a smooth manifold.[8], let us restrict ourselves to this case. Let σ be the surface measure on X_h, then $\mu_h = \frac{\sigma}{\|\text{grad} H\|}$ is an invariant measure on X_h and (X_h, ϕ_t, μ_h) is a Dynamical System (see Problem 2.6).

Geodesic flow

–Along the same lines any geodesic flow on a compact Riemannian manifold naturally defines a dynamical system.

2.2 Poincaré sections

Normally in Dynamical Systems is given a lot of emphasis to the discrete case (we have already seen an instance of this in the introduction). One reason is that there is a general device that allows to reduce the study of many properties of a continuous time Dynamical System to the study of an appropriate discrete time Dynamical System: Poincaré sections. Here we want to make few comments on this precious tool that we will largely employ in the study of billiards.

Let us consider a smooth Dynamical System (X, ϕ^t, μ) (that is a Dynamical Systems in continuous time where X is a smooth manifold and ϕ^t is a smooth flow). Then we can define the vector field $V(x) := \frac{d\phi^t(x)}{dt}|_{t=0}$.[9]

Consider a smooth compact submanifold (possibly with boundaries) Σ of codimension one such that $\mathcal{T}_x\Sigma$ (the tangent space of Σ at the point x) is transversal to $V(x)$.[10] We can then define the *return time* $\tau_\Sigma : \Sigma \to \mathbb{R}^+ \cup \{\infty\}$ by

$$\tau_\Sigma = \inf\{t \in \mathbb{R}^+ \backslash \{0\} \mid \phi^t(x) \in \Sigma\},$$

where the inf is taken to be ∞ if the set is empty. Next we define the *return map* $T_\Sigma : D(T) \subset \Sigma \to \Sigma$, where $D(T) = \{x \in \Sigma | \tau_\Sigma(x) < \infty\}$, by

$$T_\Sigma(x) = \phi^{\tau_\Sigma(x)}(x).$$

It is easy to check that there exists $c > 0$ such that $\tau_\Sigma \geq c$ (Problem 2.9).

To define the measure, the natural idea is to project the invariant measure along the flow direction: for all measurable sets $A \subset \Sigma$, define[11]

[8] By the implicit function theorem this is locally the case if $\nabla H \neq 0$.

[9] Very often it is the other way around: first is given the vector field and then the flow–as we saw in the introduction.

[10] That is $\mathcal{T}_x\Sigma \oplus V(x)$ form the full tangent space at x.

[11] We use the notation: $\phi^I(A) := \cup_{t \in I}\phi^t(A)$ for each $I \subset \mathbb{R}$.

$$\nu_\Sigma(A) := \lim_{\delta\to 0}\frac{1}{\delta}\mu(\phi^{[0,\delta]}(A)). \tag{2.2.1}$$

See Problem 2.8 for the existence of the above limit; see Problem 2.9 for the proof that τ_Σ is finite almost everywhere and Problem 2.10 for the proof that $(\Sigma, T_\Sigma, \nu_\Sigma)$ is a dynamical system. The reader is invited to meditate on the relation between this Dynamical System and the original one.

2.3 Suspension flows

A natural question is if it is possible to construct a flow with a given Poincaré section, the answer is positive and it is based on a very useful construction. Let (f, X) be a dynamical system (the Poincarè map) and $\tau : X \to \mathbb{R}_+/\{0\}$ be a positive function (the *roof* or *ceiling* function). We will construct a flow, called *supsension.* Define $\widetilde{X} = \{(x, s) \in X \times \mathbb{R} \;:\; 0 \le t < \tau(x)\}$ and the flow

$$\phi_t(x,s) = \begin{cases} (x, s+t) & \text{if } s+t < \tau(x) \\ (f(x), s-\tau(x)+t) & \text{if } \tau(x) - s < t < \tau(x) - s + \tau(f(x)) \end{cases}$$

The reader can easily check that the flow has the wanted properties.

2.4 Invariant measures

A very natural question is: given a space X and a map T there always exists an invariant measure μ? A non-exhaustive, but quite general, answer exists: Krylov-Bogoluvov Theorem.

First, we need to characterize invariance in a useful way.

Lemma 2.4.1 *Given a compact metric space X and map T continuous apart from a compact set K,*[12] *a Borel measure μ, such that $\mu(K) = 0$, is invariant if and only if $\mu(f \circ T) = \mu(f)$ for each $f \in \mathcal{C}^{(0)}(X)$.*

Proof. To prove that the invariance of the measure implies the invariance for continuous functions is obvious since each such function can be approximate uniformly by simple functions–that is, the sum of characteristic functions of measurable sets–for which the invariance is obvious.[13] The converse implication is not so obvious.

[12] This means that, if $C \subset X$ is closed, then $T^{-1}C \cup K$ is closed as well.

[13] This is essentially the definition of integral.

The first thing to remember is that the Borel measures, on a compact metric space, are regular [61]. This means that for each measurable set A the following holds[14]

$$\mu(A) = \inf_{\substack{G \supset A \\ G = \mathring{G}}} \mu(G) = \sup_{\substack{C \subset A \\ C = \overline{C}}} \mu(C). \tag{2.4.2}$$

Next, remember that for each closed set A and open set $G \supset A$, there exists $f \in \mathcal{C}^{(0)}(X)$ such that $f(X) \subset [0,1]$, $f|_{G^c} = 0$ and $f|_A = 1$ (this is Urysohn Lemma for Normal spaces [62]). Hence, setting $B_A := \{f \in \mathcal{C}^{(0)}(X) \mid f \geq \chi_A\}$,

$$\mu(A) \leq \inf_{f \in B_A} \mu(f) \leq \inf_{\substack{G \supset A \\ G = \mathring{G}}} \mu(G) = \mu(A). \tag{2.4.3}$$

Accordingly, for each A closed, we have

$$\mu(T^{-1}A) \leq \inf_{f \in B_A} \mu(f \circ T) = \inf_{f \in B_A} \mu(f) = \mu(A).$$

In addition, using again the regularity of the measure, for each A Borel holds[15]

$$\begin{aligned}
\mu(T^{-1}A) &= \inf_{\substack{U \supset K \\ U = \mathring{U}}} \mu(T^{-1}A \backslash U) \leq \inf_{\substack{U \supset K \\ U = \mathring{U}}} \sup_{\substack{C \subset T^{-1}A \backslash U \\ C = \overline{C}}} \mu(T^{-1}(TC)) \\
&\leq \inf_{\substack{U \supset K \\ U = \mathring{U}}} \sup_{\substack{C \subset A \backslash TU \\ C = \overline{C}}} \mu(T^{-1}C) \leq \sup_{\substack{C \subset A \\ C = \overline{C}}} \mu(T^{-1}C) = \sup_{\substack{C \subset A \\ C = \overline{C}}} \mu(C) = \mu(A).
\end{aligned}$$

Applying the same argument to the complement A^c of A it follow that it must be $\mu(T^{-1}A) = \mu(A)$ for each Borel set. □

Proposition 2.4.2 (Krylov–Bogoluvov) *If X is a metric compact space and $T : X \to X$ is continuous, then there exists at least one invariant (Borel) measure.*

[14] Note that this is almost obvious if one thinks of the Carathéodory construction starting from the open sets.

[15] Note that, by hypothesis, if C is compact and $C \cap K = \emptyset$, then TC is compact.

PROOF. Consider any Borel probability measure ν and define the following sequence of measures $\{\nu_n\}_{n\in\mathbb{N}}$:[16] for each Borel set A

$$\nu_n(A) = \nu(T^{-n}A).$$

The reader can easily see that $\nu_n \in \mathcal{M}^1(X)$. Indeed, since $T^{-1}X = X$, $\nu_n(X) = 1$ for each $n \in \mathbb{N}$. Next, define

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \nu_i.$$

Again $\mu_n(X) = 1$, so the sequence $\{\mu_i\}_{i=1}^{\infty}$ is contained in a weakly compact set (the unit ball) and therefore admits a weakly convergent subsequence $\{\mu_{n_i}\}_{i=1}^{\infty}$; let μ be the weak limit.[17] We claim that μ is T invariant. Since μ is a Borel measure it suffices to verify that for each $f \in \mathcal{C}^{(0)}(X)$ holds $\mu(f \circ T) = \mu(f)$ (see Lemma 2.4.1). Let f be a continuous function, then by the weak convergence we have[18]

$$\begin{aligned}\mu(f \circ T) &= \lim_{j\to\infty} \frac{1}{n_j}\sum_{i=0}^{n_j-1} \nu_i(f \circ T) = \lim_{j\to\infty} \frac{1}{n_j}\sum_{i=0}^{n_j-1} \nu(f \circ T^{i+1}) \\ &= \lim_{j\to\infty} \frac{1}{n_j}\left\{\sum_{i=0}^{n_j-1} \nu_i(f) + \nu(f \circ T^{n_j+1}) - \nu(f)\right\} = \mu(f).\end{aligned}$$

□

[16]Intuitively, if we chose a point $x \in X$ at random, according to the measure ν and we ask what is the probability that $T^n x \in A$, this is exactly $\nu(T^{-n}A)$. Hence, our procedure to produce the point $T^n x$ is equivalent to picking a point at random according to the evolved measure ν_n.

[17]This depends on the Riesz-Markov Representation Theorem [61] that states that $\mathcal{M}(X)$ is exactly the dual of the Banach space $\mathcal{C}^{(0)}(X)$. Since the weak convergence of measures in this case correspond exactly to the weak-* topology [61], the result follows from the Banach-Alaoglu theorem stating that the unit ball of the dual of a Banach space is compact in the weak-* topology. But see problem Problem 2.16 if you want a more direct proof.

[18]Note that it is essential that we can check invariance only on continuous functions: if we would have to check it with respect to all bounded measurable functions we would need that μ_n converges in a stronger sense (*strong convergence*) and this may not be true. Note as well that this is the only point where the continuity of T is used: to insure that $f \circ T$ is continuous and hence that $\mu_{n_j}(f \circ T) \to \mu(f \circ T)$.

The reason why the above theorem it is not completely satisfactory is that it is not constructive and, in particular, does not provide any information on the nature of the invariant measure. In fact, in many instances the interest is focused not just on any Borel measure but on special classes of measures, for example measures connected to the Lebesgue measure which, in some sense, can be thought as reasonably physical measures (if such measures exists).

In the following examples we will see two main techniques to study such problems: on the one hand it is possible to try to construct explicitly the measure and study its properties in the given situations (expanding maps, strange attractors, solenoid, horseshoe); on the other hand one can try to *conjugate*[19] the given problem with another, better understood, one (logistic map, circle maps). In view of this last possibility, it is very important the last example (Markov measures) that gives just a hint of the possibility of constructing a multitude of invariant measures for the shift that, as we will see briefly, is a standard system to which many others can be conjugated.

2.4.1 Examples

Contracting maps

Let $X \subset \mathbb{R}^n$ be compact and connected, $T : X \to X$ differentiable with $\|DT\| \leq \lambda^{-1} < 1$ and $T0 = 0 \in X$. In this case 0 is the unique fixed point and the delta function at zero is the only invariant measure.[20]

Expanding maps

The simplest possible case is $X = \mathbb{T}$, $T \in \mathcal{C}^{(2)}(\mathbb{T})$ with $|DT| \geq \lambda > 1$.[21] We would like to have an invariant measure absolutely continuous with respect to Lebesgue. Any such measure μ has, by definition, the Radon-Nikodym derivative $h = \frac{d\mu}{dm} \in L^1(\mathbb{T}, m)$, [62]. In Proposition 2.4.2 we saw how a measure evolves by defining the operator $T_*\mu(f) = \mu(f \circ T)$ for each $f \in \mathcal{C}^{(0)}$ and $\mu \in \mathcal{M}(X)$ (see also footnote 17 at page 31). If we want to study a smaller class of measures we must first check that T_* leaves such a class invariant. Indeed, if μ is absolutely continuous with respect to Lebesgue

[19] See Definition 5 for a precise definition and Problem 2.38 and 2.39 for some insight.

[20] The reader will hopefully excuse this physicist language, naturally we mean that the invariant measure is defined by $\delta_0(f) = f(0)$. The property that there exists only one invariant measure is called *unique ergodicity*, we will see more of it in the sequel.

[21] Note that this generalizes Examples 2.1.1–Dilations.

then $T_*\mu$ has the same property. Moreover, if $h = \frac{d\mu}{dm}$ and $h_1 = \frac{dT_*\mu}{dm}$ then (Problem 2.14)

$$h_1(x) = \mathcal{L}h(x) := \sum_{y \in T^{-1}(x)} |D_y T|^{-1} h(y).$$

The operator $\mathcal{L} : L^1(\mathbb{T}, m) \to L^1(\mathbb{T}, m)$ is called *Transfer operator* or Ruelle-Perron-Frobenius operator, and has an extremely important rôle in the study of the statistical properties of the system. Notice that $\|\mathcal{L}h\|_1 \leq \|h\|_1$. The key property of $\mathcal{L}$, in this context, is given by the following inequality (this type of inequality is commonly called of Lasota-York type) (Problem 2.15)

$$\|\frac{d}{dx}\mathcal{L}h\|_1 \leq \lambda^{-1}\|h'\|_1 + C\|h\|_1 \tag{2.4.4}$$

where $C = \frac{\|D^2T\|_\infty}{\|DT\|_\infty^2}$.

The above inequality implies immediately $\|(\mathcal{L}^n h)'\|_1 \leq \frac{C}{1-\lambda^{-1}}\|h\|_1 + \|h'\|_1$, for all $n \in \mathbb{N}$. This, in turns, implies that the $\sup_{n\in\mathbb{N}} \|\mathcal{L}^n h\|_\infty < \infty$. Consequently, the sequence $h_n := \frac{1}{n}\sum_{i=0}^{n-1} \mathcal{L}^i h$ is compact in L^1 (this is a consequence of standard Sobolev embedding theorems [33], but see Problem 2.16 for an elementary proof). In analogy with Lemma 2.4.2, we have that there exists $h_* \in L^1$ such that $\mathcal{L}h_* = h_*$. Thus $d\mu := h_* dm$ is an invariant measure of the type we are looking for.

Logistic maps

Consider $X = [0, 1]$ and

$$T(x) = 4x(1 - x).$$

This map is not an everywhere expanding map ($D_{\frac{1}{2}}T = 0$), yet it can be conjugate with one.

To see this consider the continuous change of variables $\Psi : [0, 1] \to [0, 1]$ defined by

$$\Psi(x) = \frac{2}{\pi} \arcsin \sqrt{x},$$

thus $\Psi^{-1}(x) = \left(\sin \frac{\pi}{2}x\right)^2$. Accordingly,

$$\begin{aligned} \tilde{T}(x) &:= \Psi \circ T \circ \Psi^{-1}(x) = \Psi(4 \sin^2 \tfrac{\pi}{2}x \cos^2 \tfrac{\pi}{2}x) \\ &= \Psi([\sin \pi x]^2) = \tfrac{2}{\pi} \arcsin[\sin \pi x] \end{aligned}$$

which yields[22]

$$\tilde{T}(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

The map $\tilde{T}$ is called *tent map* for its characteristic shape. What is more interesting is that the Lebesgue measure is invariant for $\tilde{T}$, as the reader can easily check. This means that, if we define $\mu(f) := m(f \circ \Psi^{-1})$, holds

$$\mu(f \circ T) = m(f \circ T \circ \Psi^{-1}) = m(f \circ \Psi^{-1} \circ \tilde{T}) = m(f \circ \Psi^{-1}) = \mu(f).$$

Hence, $([0,1], T, \mu)$ is a Dynamical Systems. In addition, a trivial computation shows

$$\mu(dx) = \frac{1}{\pi\sqrt{x(1-x)}} dx,$$

thus μ is absolutely continuous with respect to Lebesgue.

Circle maps

A circle map is an order preserving continuous map of the circle. A simple way to describe it is to start by considering its lift. Let $\hat{T} : \mathbb{R} \to \mathbb{R}$, such that $\hat{T}(0) \in [0,1]$, $\hat{T}(x+1) = \hat{T}(x) + 1$ and $\hat{T}_* \geq 0$. The circle map is then defined as $T(x) = \hat{T}(x) \mod 1$. Circle maps have a very rich theory that we do not intend to develop here, we confine ourselves to some facts (see [40] for a detailed discussion of the properties below). The first fact is that the *rotation number*

$$\rho(T) = \lim_{n \to \infty} \frac{1}{n} \hat{T}^n(x).$$

is well defined and does not depend on x.

We have already seen a concrete example of circle maps: the rotation R_ω by ω. Clearly $\rho(R_\omega) = \omega$. It is fairly easy to see that if $\rho(T) \in \mathbb{Q}$ then the map has a periodic orbit. We are more interested in the case in which the rotation number is irrational. In this case, with the extra assumption that T is twice differentiable (actually a bit less is needed) the Denjoy theorem holds stating that there exists a continuous invertible function h such that $R_{\rho(T)} \circ h = h \circ T$, that is T is *topologically conjugated* to a rigid rotation. Since we know that the Lebesgue measure is invariant for the rotations, we can obtain an invariant measure for T by pushing the Lebesgue measure by h, namely define

$$\mu(f) = m(f \circ h^{-1}).$$

[22] Remember that the domain of arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin \pi x = \sin \pi(1-x)$.

The natural question if the measure μ is absolutely continuous with respect to Lebesgue is rather subtle and depends, once again, by KAM theory. In essence the answer is positive only if T has more regularity and the rotation number in not very well approximated by rational numbers (in some sense it is 'very irrational').

Strange Attractors

We have seen the case in which all the trajectories are attracted by a point. The reader can probably imagine a case in which the attractor is a curve or some other simple set. Yet, it has been a fairly recent discovery that an attractor may have a very complex (strange) structure. The following is probably the simplest example. Let $X = Q = [0, 1]^2$ and

$$T(x,\, y) = \begin{cases} (2x,\, \frac{1}{8}y + \frac{1}{4}) & \text{if } x \in [0,\, 1/2] \\ (2x-1,\, \frac{1}{8}y + \frac{3}{4}) & \text{if } x \in]1/2,\, 1]. \end{cases}$$

We have a map of the square that stretches in one direction by a factor 2 and contract in the other by a factor 8.

Note that T it is not continuous with respect to the normal topology, so Proposition 2.4.2 cannot be applied directly. This problem can be solved in at least two ways: one is to *code* the system, and we will discuss it later, the other is to study more precisely what happens by iterating a measure in special cases.

In our situation, since T^nQ consists of a multitude of thinner and thinner strips, it is clear that there can be no invariant measure absolutely continuous with respect to Lebesgue.[23] Yet, it is very natural to ask what happens if we iterate the Lebesgue measure by the operator T_*. It is easy to see that T_*m is still absolutely continuous with respect to Lebesgue. In fact, T_* maps absolutely continuous measures in absolutely continuous measures. Once we note this, it is very tempting to define the transfer operator. An easy computation yields

$$\mathcal{L}h(x) = \chi_{TQ}(x) \sum_{y \in T^{-1}(x)} |\det(D_yT)|^{-1} h(y) = 4\chi_{TQ}(x)h(T^{-1}(x)).$$

[23] In fact, if μ is an invariant measure, $T_*\mu = \mu$, it follows

$$\mu(\chi_{T^nQ}) = T_*^n\mu(\chi_{T^nQ}) = \mu(\chi_Q) = 1,$$

so μ must be supported on $\Lambda = \cap_{n=0}^{\infty} T^nQ$.

Since the map expands in the unstable direction, it is quite natural to investigate, in analogy with the expanding case, the *unstable derivative* D^u, that is the derivative in the x direction, of the iterate of the density.

$$\|D^u \mathcal{L}h\|_1 \leq \frac{1}{2}\|D^u h\|_1 \quad \forall h \in \mathcal{C}^{(1)}(Q) \tag{2.4.5}$$

To see the consequences of the above estimate, consider $f \in \mathcal{C}^{(1)}(Q)$ with $f(0, y) = f(1, y) = 0$ for each $y \in [0, 1]$, then if ν is a measure obtained by the measure hdm ($h \in \mathcal{C}^{(1)}$) with the procedure of Proposition 2.4.2,[24] we have

$$\begin{aligned}\nu(D^u f) &= \lim_{j\to\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} (T_*)^i m(hD^u f) = \lim_{j\to\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} m(\mathcal{L}^i hD^u f)\\ &= -\lim_{j\to\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} m(fD^u \mathcal{L}^i h)\end{aligned}$$

where we have integrated by part. Remembering (2.4.5) we have

$$\nu(D^u f) = 0,$$

for all $f \in \mathcal{C}^{(1)}_{\text{per}}(Q) = \{f \in \mathcal{C}^{(1)}(Q) \mid f(0, y) = f(1, y)\}$. The enlargement of the class of functions is due to the obvious fact that, if $f \in \mathcal{C}^{(1)}_{\text{per}}(Q)$, then $\tilde{f}(x, y) = f(x, y) - f(0, y)$ is zero on the vertical (stable) boundary and $D^u\tilde{f} = D^u f$.

This means that the measure ν, when restrict to the horizontal direction, is ν-a.e. constant (see Problem 2.30). Such a strong result is clearly a consequence of the fact the map is essentially linear, one can easily imagine a non linear case (think of dilations and expanding maps) and in that case the same argument would lead to conclude that the measure, when restricted to unstable manifolds, is absolutely continuous with respect to the restriction of Lebesgue (these type of measures are commonly called *SRB* from Sinai, Bowen and Ruelle).

We can now prove that indeed the measure ν is invariant. The discontinuity line of T is $\{x = \frac{1}{2}\}$. Points close to $\{x = \frac{1}{2}\}$ are mapped close to the boundary of Q, so if $f(0, y) = f(1, y) = 0$, then $f \circ T$ is continuous. Hence, the argument of Proposition 2.4.2 proves that $\nu(f \circ T) = \mu(f)$ for all f that

[24] As we noted in the proof of Proposition 2.4.2, the only part that uses the continuity of T is the proof of the invariance. Thus, in general we can construct a measure by the averaging procedure but the invariance it is not automatic.

vanish at the stable boundary. Yet, the characterization of ν proves that $\nu(\{(x,y) \in Q \mid x \in \{0,1\}\}) = 0$, thus we can obtain $\nu(f \circ T) = \mu(f)$ for all continuous functions via the Lebesgue dominate convergence theorem and the invariance follows by Lemma 2.4.1.

Horseshoe

This very famous example consists of a map of the square $Q = [0,1]^2$, the map is obtained by stretching the square in the horizontal direction, bending it in the shape of an horseshoe and then superimposing it to the original square in such a way that the intersection consists of two horizontal strips.[25] Such a description is just topological, to make things clearer let us consider a very special case:

$$T(x,\, y) = \begin{cases} (5x \mod 1,\, \frac{1}{4}y) & \text{if } x \in [1/5,\, 2/5] \\ (5x \mod 1,\, \frac{1}{4}y + \frac{3}{4}) & \text{if } x \in [3/5,\, 4/5]. \end{cases}$$

Note that T is not explicitly defined for $x \in [0,\, 1/5[\cup[\frac{2}{3},\, \frac{3}{5}[\cup]4/5,\, 1]$ since for this values the horseshoe falls outside Q, so its actual shape is irrelevant. Since the map from Q to Q it is not defined on the all square, so we can have a Dynamical System only with respect to a measure for which the domain of definition of T, and all of its powers, has measure one. We will start by constructing such a measure.

The first step is to notice that the set

$$\Lambda = \cap_{n\in\mathbb{Z}} T^n Q \tag{2.4.6}$$

of the points which trajectory is always in Q is $\neq \emptyset$. Second, note that $\lambda = T\Lambda = T^{-1}\Lambda$, such an invariant set is called *hyperbolic set* as we will see later. We would like to construct an invariant measure on Λ. Since Λ is a compact set and T is continuous on it we know that there exist invariant measures; yet, in analogy with the previous examples, we would like to construct one *coming from Lebesgue*.

As already mentioned we must start by constructing a measure on $\Lambda_- = \cap_{n\in\mathbb{N}\cup\{0\}} T^{-n}Q$ since $T^k\Lambda_- \subset \Lambda_-$. To do so it is quite natural to construct a measure by *subtracting* the mass that leaks out of Q. namely, define the operator $\tilde{T} : \mathcal{M}(X) \to \mathcal{M}(X)$ by

$$\tilde{T}\mu(A) := \mu(TA \cap Q).$$

[25] We have already seen something very similar in the introduction.

Again we consider the evolution of measures of the type $d\mu = hdm$. For each continuous f with $\text{supp}(f) \subset Q$ holds

$$\tilde{T}\mu(f) = \mu(f \circ T^{-1}\chi_Q) = \int_{T^{-1}Q} fh \circ T|\det DT|dm.$$

We can thus define the operator $\mathcal{L}$ that evolves the densities:

$$\mathcal{L}h(x) = \frac{5}{4}\chi_{T^{-1}Q\cap Q}(x)h(Tx).$$

Clearly $\tilde{T}\mu(f) = m(f\mathcal{L}h)$.

Note that $\tilde{T}m(1) = \frac{1}{2}$, thus $\tilde{T}$ does not map probability measures into probability measures; this is clearly due to the mass leaking out of Q. Calling D^s (stable derivative) the derivative in the y direction, follows easily

$$\|D^s\mathcal{L}h\|_1 \leq \frac{1}{4}\|D^s h\|_1$$

for each h differentiable in the stable direction.

On the other hand, if $\|D^s h\|_1 \leq c$ and $\Delta = [0, 1/4] \cup [3/4, 1]$,

$$\begin{aligned}|\tilde{T}\mu(1)| &= \int_{Q\cap TQ} h = \int_\Delta dy \int_0^1 dx h(x,y) \\ &= \int_\Delta dy \int_0^1 dx \int_0^1 d\xi h(x,\xi) + \mathcal{O}(\|D^s h\|_1) \\ &= |\Delta|\|h\|_1 + \mathcal{O}(\|D^s h\|_1) = \frac{1}{2}\mu(1) + \mathcal{O}(\|D^s h\|_1).\end{aligned}$$

It is then natural to define $\hat{\mathcal{L}}h := 2\mathcal{L}h$ and $\hat{T} = 2\tilde{T}$. Thus $\|D^s\hat{\mathcal{L}}h\|_1 \leq \frac{1}{2}\|D^s h\|_1$. This means that $\{\frac{1}{n}\sum_{i=0}^{n-1}\hat{T}^i\mu\}$ are probability measures. Accordingly, there exists an accumulation point μ_* and $\mu_*(D^s f) = 0$ for each f periodic in the y direction. By the same type of arguments used in the previous examples, this means that μ_* is constant in the y direction, it is supported on Λ_- by construction and $\tilde{T}\mu_* = \frac{1}{2}\mu_*$ (*conformal invariance*) : just the measure we where looking for.

We can now conclude the argument by evolving the measure as usual:

$$T_*\mu_*(f) = \mu_*(f \circ T)$$

for all continuous f with the support in Q. Now the standard argument applies. In such a way we have obtained the invariant measure supported on Λ.

Markov Measures

Let us consider the shift (Σ_n^+, T). We would like to construct other invariant measures bedside Bernoulli. As we have seen it suffices to specify the measure on the algebra of the cylinders. Let us define

$$A(m; k_1, \ldots, k_l) = \{\sigma \in \Sigma_n^+ \mid \sigma_{i+m} = k_i \; \forall\, i \in \{1, \ldots, l\}\};$$

this are a basis for the algebra of the cylinders.

For each $n \times n$ matrix P, $P_{ij} \geq 0$, $\sum_j P_{ij} = 1$ by the Perron-Frobenius theorem there exists $\{p_i\}$ such that $pP = p$. Let us define

$$\mu(A(m; k_1, \ldots, k_l)) = p_{k_1} P_{k_1 k_2} P_{k_2 k_3} \ldots P_{k_{l-1} k_l}.$$

The reader can easily verify that μ is invariant over the algebra $\mathcal{A}$ and thus extends to an invariant measure. This is called Markov because it is nothing else than a Markov chain together with its stationary measure.[26]

2.5 Ergodicity

The examples in the previous section (strange attractor, horseshoe) show only a very dim glimpse of a much more general and extremely rich theory (the study of SRB measures) while the last (Markov measures) points toward another extremely rich theory: Gibbs (or equilibrium) measures. Although this it is not the focus here, we will see a bit more in the future.

One of the main objectives in dynamical systems is the study of the long time behavior (that is the study of the trajectories $T^n x$ for large n). There are two main cases in which it is possible to study, in some detail, such a long time behavior. The case in which the motion is rather regular[27] or close to it (the main examples of this possibility are given by the so-called KAM theory and by situations in which the motions is attracted by a simple set); and the case in which the motion is very irregular.[28] This last case may seem surprising since the irregularity of the motion should make its study very difficult. The reason why such systems can be studied is, as usual, because

[26] The probabilistic interpretation is that the probability of seeing the state k at time one, given that we saw the state l at time zero, is given by P_{lk}. So the process has a bit of memory: it remembers its state one time step before. Of course, it is possible to consider processes that have a longer–possibly infinite–memory. Proceeding in this direction one would define the so called *Gibbs measures*.

[27] Typically, quasi periodic motion, remember the small oscillation in the pendulum.

[28] Remember the example in the introduction.

we ask the right questions,[29] that is we ask questions not concerning the fine details of the motion but only concerning its statistical or qualitative properties.

The first example of such properties is the study of the invariant sets.

Definition 3 *A measurable set A is invariant for T if $T^{-1}A \subset A$.*

A dynamical system (X, T, μ) is ergodic if each invariant set has measure zero or one.

Note that if A is invariant then $\mu(A \backslash T^{-1}A) = \mu(A) - \mu(T^{-1}A) = 0$, moreover $\Lambda = \cap_{n=0}^{\infty} T^{-n}A \subset A$ is invariant as well. In addition, by definition, $\Lambda = T\Lambda$, which implies $\Lambda = T^{-1}\Lambda$ and $\mu(A\backslash\Lambda) = 0$. This means that, if A is invariant, then it always contains a set Λ invariant in the stronger (maybe more natural) sense that $T\Lambda = T^{-1}\Lambda = \Lambda$. Moreover, Λ is of full measure in A. Our definition of invariance is motivated by its greater flexibility and the fact that, from a measure theoretical point of view, zero measure sets can be discarded.

In essence, if a system is ergodic then most trajectories explore all the available space. In fact, for any A of positive measure, define $A_b = \cup_{n\in\mathbb{N}} T^{-n}A$ (this are the points that eventually end up in A), since $A_b \supset A$, $\mu(A_b) > 0$. Since $T^{-1}A_b \subset A_b$, by ergodicity follows $\mu(A_b) = 1$. Thus, the points that never enter in A (that is, the points in A_b^c) have zero measure. Actually, if the system has more structure (topology), more is true (see Problem 2.19).

2.5.1 Examples

Rotations

–The ergodicity of a rotations depends on ω. If $\omega \in \mathbb{Q}$ then the system is not ergodic. In fact, let $\omega = \frac{p}{q}$ $(p, q \in \mathbb{N})$, then, for each $x \in \mathbb{T}$ $T^q x = x + p \mod 1 = x$, so T^q is just the identity. An alternative way of saying this is to notice that all the points have a periodic trajectory of period q. It is then easy to exhibit an invariant set with measure strictly larger than 0 but strictly less than 1. Consider $[0, \varepsilon]$, then $A = \cup_{i=1}^{q-1} T^{-i}[0, \varepsilon]$ is an invariant set; clearly $\varepsilon \le \mu(A) \le q\varepsilon$, so it suffices to choose $\varepsilon < q^{-1}$.

The case $\omega \notin \mathbb{Q}$ is much more interesting. First of all, for each point $x \in \mathbb{T}$ we have that the closure of the set $\{T^n x\}_{i=0}^{\infty}$ is equal to $\mathbb{T}$, which is to say that the orbits are dense.[30] The proof is based on the fact that there

[29] Of course, the "right questions" are the ones that can be answered.

[30] A system with a dense orbits called *Topologically Transitive*.

cannot be any periodic orbit. To see this suppose that $x \in \mathbb{T}$ has a periodic orbit, that is there exists $q \in \mathbb{N}$ such that $T^q x = x$. As a consequence there must exist $p \in \mathbb{Z}$ such that $x + p = x + q\omega$ or $\omega \in \mathbb{Q}$ contrary to the hypothesis. Hence, the set $\{T^k 0\}_{k=0}^{\infty}$ must contain infinitely many points and, by compactness, must contain a convergent subsequence k_i. Hence, for each $\varepsilon > 0$, there exists $m > n \in \mathbb{N}$:

$$|T^m 0 - T^n 0| < \varepsilon.$$

Since T preserves the distances, calling $q = m - n$, holds

$$|T^q 0| < \varepsilon.$$

Accordingly, the trajectory of $T^{jq}0$ is a translation by a quantity less than ε, therefore it will get closer than ε to each point in $\mathbb{T}$ (i.e., the orbit is dense). Again by the conservation of the distance, since zero has a dense orbit the same will hold for every other point.

Intuitively, the fact that the orbits are dense implies that there cannot be a non trivial invariant set, henceforth the system is ergodic. Yet, the proof it is not trivial since it is based on the existence of Lebesgue density points [33] (see Problem 2.41). It is a fact from general measure theory that each measurable set $A \subset \mathbb{R}$ of positive Lebesgue measure contains, at least, one point x such that for each $\varepsilon \in (0, 1)$ there exists $\delta > 0$:

$$\frac{m(A \cap [x - \delta,\, x + \delta])}{2\delta} > 1 - \varepsilon.$$

Hence, given an invariant set A of positive measure and $\varepsilon > 0$, first choose an interval $I \subset A$ such that $m(I \cap A) > (1 - \varepsilon)m(I)$.Second, we know already that there exists $q, M \in \mathbb{N}$ such that $\{T^{-kq}x\}_{k=1}^{M}$ divides $[0, 1]$ into intervals of length less that $\frac{\varepsilon}{2}\delta$. Hence, given any point $x \in \mathbb{T}$ choose $k \in \mathbb{N}$ such that $m(T^{-kq}I \cap [x - \delta,\, x + \delta]) > m(I)(1 - \varepsilon)$ so,

$$\begin{aligned} m(A \cap [x - \delta,\, x + \delta]) &\geq m(A \cap T^{-kq}I) - m(I)\varepsilon \\ &\geq m(A \cap I) - m(I)\varepsilon \geq (1 - 2\varepsilon)2\delta. \end{aligned}$$

Thus, A has density everywhere larger than $1 - 2\varepsilon$, which implies $\mu(A) = 1$ since ε is arbitrary.

The above proof of ergodicity it is not so trivial but it has a definite dynamical flavor (in the sense that it is obtained by studying the evolution of the system). Its structure allows generalizations to contexts whit a less rich algebraic structure. Nevertheless, we must notice that, by taking advantage of the algebraic structure (or rather the group structure) of $\mathbb{T}$, a much simpler and powerful proof is available.

Let $\nu \in \mathcal{M}_T^1$, then define

$$F_n = \int_{\mathbb{T}} e^{2\pi i n x} \nu(dx), \quad n \in \mathbb{N}.$$

A simple computation, using the invariance of ν, yields

$$F_n = e^{2\pi i n \omega} F_n$$

and, if ω is irrational, this implies $F_n = 0$ for all $n \neq 0$, while $F_0 = 1$. Next, consider $f \in \mathcal{C}^{(2)}(\mathbb{T}^1)$ (so that we are sure that the Fourier series converges uniformly, see Problem 2.29), then

$$\nu(f) = \sum_{n=0}^{\infty} \nu(f_n e^{2\pi i n \cdot}) = \sum_{n=0}^{\infty} f_n F_n = f_0 = \int_{\mathbb{T}} f(x) dx.$$

Hence m is the unique invariant measure (unique ergodicity). This is clearly much stronger than ergodicity (see Problem 2.23)

Baker

–This transformation gets its name from the activity of bread making, it bears some resemblance with the horseshoe. The space X is the square $[0,\,1]^2$, μ is again Lebesgue, and T is a transformation obtained by squashing down the square into the rectangle $[0,\,2] \times [0,\,\frac{1}{2}]$ and then cutting the piece $[1,\,2] \times [0,\,\frac{1}{2}]$ and putting it on top of the other one. In formulas

$$T(x,\,y) = \begin{cases} (2x,\,\frac{1}{2}y) \mod 1 & \text{if } x \in [0,\,\frac{1}{2}) \\ (2x,\,\frac{1}{2}(y+1)) \mod 1 & \text{if } x \in [\frac{1}{2},\,1]. \end{cases}$$

This transformation is ergodic as well, in fact much more. We will discuss it later.

The properties of the invariant sets of a dynamical systems have very important reflections on the statistics of the system, in particular on its time averages. Before making this precise (see Theorem 2.6.8) we state few very general and far reaching results.

2.6 Some basic Theorems

Theorem 2.6.1 *(Birkhoff) Let (X, T, μ) be a dynamical system, then for each $f \in L^1(X, \mu)$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$$

exists for almost every point $x \in X$. In addition, setting

$$f^+(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x),$$

holds

$$\int_X f^+ d\mu = \int_X f d\mu.$$

Proof

Since the task at hand is mainly didactic, we will consider explicitly only the case of positive bounded functions, the completion of the proof is left to the reader. Also, this is an elementary but lengthy proof. More sophisticated and shorter proof exists [40].

Let $f \in L^\infty(X, d\mu)$, $f \geq 0$, and

$$S_n(x) \equiv \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x).$$

For each $x \in X$, there exists

$$\begin{aligned} \overline{f}^+(x) &= \limsup_{n\to\infty} S_n(x) \\ \underline{f}^+(x) &= \liminf_{n\to\infty} S_n(x). \end{aligned}$$

The first remark is that both $\overline{f}^+$ and $\underline{f}^+$ are invariant functions. In fact,

$$S_n(Tx) = S_n(x) + f(T^n x) - f(x)$$

so, tacking the limit the result follows.[31]

[31] Here we have used the boundedness, this is not necessary. If $f \in L^1(X, d\mu)$ and positive, then $S_n(Tx) \geq S_n(x) - f(x)$, so $\underline{f}^+(Tx) \geq \overline{f}^+(x)$ and it is and easy exercise to check that any such function must be invariant.

Next, for each $n \in \mathbb{N}$ and $k,\, j \in \mathbb{Z}$ we define

$$D_{n,l,j} = \left\{ x \in X \;\middle|\; \overline{f}^{+}(x) \in \left[\frac{l}{n},\, \frac{l+1}{n}\right) ;\; \underline{f}^{+}(x) \in \left[\frac{j}{n},\, \frac{j+1}{n}\right) \right\},$$

by the invariance of the functions follows the invariance of the sets $D_{n,l,j}$. Also, by the boundedness, follows that for each n exists n_0 such as

$$\bigcup_{j,l \in \{-n_0, \dots, n_0\}} D_{n,l,j} = X.$$

The key observation is the following.

Lemma 2.6.2 *For each $n \in \mathbb{N}$ and $l,\, j \in \mathbb{Z}$, setting $A = D_{n,l,j}$, holds*

$$\frac{l+1}{n}\mu(A) < \int_A f d\mu + \frac{3}{n}\mu(A)$$
$$\frac{j}{n}\mu(A) > \int_A f d\mu - \frac{3}{n}\mu(A)$$

From the Lemma follows

$$0 \le \int_X (\overline{f}^{+} - \underline{f}^{+}) d\mu = \sum_{l,\, j=-n_0}^{n_0} \int_{D_{n,l,j}} (\overline{f}^{+} - \underline{f}^{+}) d\mu$$
$$\le \sum_{l,\, j=-n_0}^{n_0} \left[\frac{l+1}{n} - \frac{j}{n}\right] \mu(D_{n,l,j}) < \frac{6}{n} \sum_{l,\, j=-n_0}^{n_0} \mu(D_{n,l,j}) = \frac{6}{n}.$$

Since n is arbitrary we have

$$\int_X (\overline{f}^{+} - \underline{f}^{+}) d\mu = 0$$

which implies $\overline{f}^{+} = \underline{f}^{+}$ almost everywhere (since $\overline{f}^{+} \ge \underline{f}^{+}$ by definition) proving that the limit exists. Analogously, we can prove

$$\int_X (f - f^{+}) d\mu = 0.$$

Proof of the Lemma 2.6.2 We will prove only the first inequality, the second being proven in exactly the same way.

For each $x \in A$ we will call $k(x)$ the first $m \in \mathbb{N}$ such that

$$S_m(x) > \frac{l-1}{n},$$

by construction $k(x)$ must be finite for each $x \in A$. Hence, setting $X_k = \{x \in A \mid k(x) = k\}$, $\cup_k X_k = A$, and for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{k=1}^{N} X_k\right) \geq \mu(A)(1-\varepsilon).$$

Let us call

$$Y = A \setminus \bigcup_{k=1}^{N} X_k.$$

Then $\mu(Y) \leq \mu(A)\varepsilon$, also set $L = \sup_{x\in A}|f(x)|$. The basic idea is to follow, for each point $x \in A$, the trajectory $\{T^i x\}_{i=0}^{M}$, where $M > N$ will be chosen sufficiently large. If the point would never visit the set Y, we could group the sum $S_M(x)$ in pieces all, in average, larger than $\frac{l-1}{n}$, so the same would hold for $S_M(x)$. The difficulties come from the visits to the set Y.

For each $n \in \{0, ..., M\}$ define

$$\widetilde{f}_n(x) = \begin{cases} f(T^n x) & \text{if } T^n x \notin Y \\ \dfrac{l}{n} & \text{if } T^n x \in Y \end{cases}$$

and

$$\widetilde{S}_M(x) = \frac{1}{M}\sum_{n=0}^{M-1} \widetilde{f}_n(x).$$

By definition $y \in Y$ implies $y \notin X_1$, i.e. $f(y) \leq \frac{l-1}{n}$. Accordingly, $\widetilde{f}(x) \geq f(T^n x)$ for each $x \in A$. Note that for each n we change the function $f \circ T^n$ only at some points belonging to the set Y and $\frac{l}{n}$ can be taken less or equal than L (otherwise $\mu(A) = 0$), consequently

$$\int_A f d\mu = \int_A S_M d\mu \geq \int_A \widetilde{S}_M d\mu - L\mu(Y) \geq \int_A \widetilde{S}_M d\mu - L\mu(A)\varepsilon.$$

We are left with the problem of computing the sum. As already mentioned the strategy consists in dividing the points according to their trajectory with respect to the sets X_n. To be more precise, let $x \in A$, then by definition it must belong to some X_n or to Y. We set $k_1(x)$ equal to j is $x \in X_j$ and $k_1(x) = 1$ if $x \in Y$. Next, $k_2(x)$ will have value j if $T^{k_1(x)}x \in X_j$ or value 1 if $T^{k_1(x)} \in Y$. If $k_1(x) + k_2(x) < M$, then we go on and define similarly $k_3(x)$. In this way, to each $x \in A$ we can associate a number $m(x) \in \{1, ..., M\}$ and indices $\{k_i(x)\}_{i=1}^{m(x)}$, $k_i(x) \in \{1, ..., N\}$, such that

$M - N \leq \sum_{i=1}^{m(x)-1} k_i(x) < M$, $\sum_{i=1}^{m(x)} k_i(x) \geq M$. Let us call $K_p(x) = \sum_{j=1}^{p} k_j(x)$. Using such a division of the orbit in segments of length $k_i(x)$ we can easily estimate

$$\begin{aligned}\widetilde{S}_M(x) =& \frac{1}{M}\left\{\sum_{i=1}^{m(x)-1} k_i(x)\left[\frac{1}{k_i(x)}\sum_{j=K_{i-1}(x)}^{K_i(x)-1}\widetilde{f}_j(x)\right] + \sum_{i=K_{m(x)-1}(x)}^{M-1}\widetilde{f}(T^i x)\right\}\\ \geq& \frac{1}{M}\sum_{i=1}^{m(x)-1} k_i(x)\frac{l-1}{n} \geq \frac{M-N}{M}\frac{l-1}{n}.\end{aligned}$$

Putting together the above inequalities we get

$$\begin{aligned}\int_A f d\mu \geq& \left\{\frac{(M-N)(l-1)}{Mn} - L\varepsilon\right\}\mu(A)\\ \geq& \frac{l+1}{n}\mu(A) - \left\{\frac{2}{n} + \frac{N(l-1)}{Mn} + L\varepsilon\right\}\mu(A).\end{aligned}$$

which, by choosing first ε sufficiently small and, after, M sufficiently large, concludes the proof. □

To prove the result for all function in $L^1(X, \mu)$ it is convenient to deal at first only with positive functions (which suffice since any function is the difference of two positive functions) and then use the usual trick to cut off a function (that is, given f define f_L by $f_L(x) = f(x)$ if $f(x) \leq L$, and $f_L(x) = L$ otherwise) and then remove the cut off. The reader can try it as an exercise. □

Birkhoff theorem has some interesting consequences.

Corollary 2.6.3 *For each $f \in L^1(X, \mu)$ the following holds*

1. *$f_+ \in L^1(X, \mu)$;*
2. *$f_+(Tx) = f_+(x)$ almost surely.*

The proof is left to the reader as an easy exercise (see Problem 2.17).

Another important consequence pertains to the case of invertible dynamics. Let T be invertible, we can then define the backward ergodic average

$$f_-(x) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^{-k}(x).$$

It is a surprising fact that the backward average equals the forward one.

Corollary 2.6.4 *Let (X, T, μ) be a dinamical systems and T be invertible. Then, for all $f \in L^1(X, \mu)$, almost surely we have $f_+ = f_-$.*

PROOF. We prove it for bounded functions, the result for L^1 function can then be obtained by approximation. By Birkhoff theorem, the set $K = \{x \in X \ : \ f_+(x) \geq f_-(x)\}$ is invariant. It follows that

$$0 \leq \int_K [f_+(x) - f_-(x)]\mu(dx) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_K [f \circ T^k(x) - f \circ T^{-k}(x)]\mu(dx) = 0$$

In the first equality, we used the Lebesgue dominated convergence theorem and then the invariance of the measure. It follows that $f_+(x) \leq f_-(x)$. Exchanging the role of f_+ and f_- the Lemma follows. □

A further interesting fact, that starts to show some connections between averages and invariant sets, emerges by considering a measurable set A and its characteristic function χ_A. A little thought shows that the ergodic average $\chi_A^+(x)$ is simply the average frequency of visit of the set A by the trajectory $\{T^n x\}$ (Problem 2.26).

One may wonder about other types of convergences that take place in the ergodic average, notably L^2 convergence. The next theorem is a consequence of Theorem 2.6.1 (see Problem 2.24). I provide an independent proof because it introduces the idea of a coboundary decomposition that turns out to be of great importance in many other situations.

Theorem 2.6.5 (Von Neumann) *Let (X, T, μ) be a Dynamical System, then for each $f \in L^2(X, \mu)$ the ergodic average converges in $L^2(X, \mu)$.*

PROOF. We have already seen that it can be useful to lift the dynamics at the level of the algebra of function or at the level of measures. This game assumes different guises according to how one plays it, here is another very interesting version.

Let us define $U : L^2(X, \mu) \to L^2(X, \mu)$ as

$$Uf := f \circ T.$$

Then, by the invariance of the measure, it follows $\|Uf\|_2 = \|f\|_2$, so U is an L^2 contraction (actually, and L^2-isometry). If T is invertible, the same argument applied to the inverse shows that U is indeed unitary, otherwise we must content ourselves with

$$\|U^* f\|_2^2 = \langle UU^* f, f\rangle \leq \|UU^* f\|_2 \|f\|_2 = \|U^* f\|_2 \|f\|_2,$$

that is $\|U^*\|_2 \leq 1$ (also U^* is and L^2 contraction).

Next, consider $V_1 = \{f \in L^2 \mid Uf = f\}$ and $V_2 = \text{Rank}(\mathbb{1} - U)$. First of all, note that if $f \in V_1$, then

$$\|U^* f - f\|_2^2 = \|U^* f\|_2^2 - \langle f, U^* f\rangle - \langle U^* f, f\rangle + \|f\|_2^2 \leq 0.$$

Thus, $f \in V_1^* := \{f \in L^2 \mid U^* f = f\}$. The same argument applied to $f \in V_1^*$ shows that $V_1 = V_1^*$. To continue, consider $f \in V_1$ and $h \in L^2$, then

$$\langle f, h - Uh\rangle = \langle f - U^* f, h\rangle = 0.$$

This implies that $V_1^\perp = \overline{V_2}$, hence $V_1 \oplus \overline{V_2} = L^2$. Finally, if $g \in V_2$, then there exists $h \in L^2$ such that $g = h - Uh$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} U^i g = \lim_{n\to\infty} \frac{1}{n}(h - U^n h) = 0.$$

On the other hand if $f \in V_1$ then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} U^i f = f$. The only function on which we do not still have control are the g belonging to the closure of V_2 but not in V_2. In such a case there exists $\{g_k\} \subset V_2$ with $\lim_{k\to\infty} g_k = g$. Thus,

$$\|\frac{1}{n} \sum_{i=0}^{\infty} U^i g\|_2 \leq \|\frac{1}{n} \sum_{i=0}^{\infty} U^i g_k\|_2 + \|g - g_k\|_2 \leq \|\frac{1}{n} \sum_{i=0}^{\infty} U^i g_k\|_2 + \frac{\varepsilon}{2},$$

provided we choose k large enough. Then, by choosing n sufficiently large we obtain

$$\|\frac{1}{n} \sum_{i=0}^{\infty} U^i g\|_2 \leq \varepsilon.$$

We have just proven that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i = P$$

where P is the orthogonal projection on V_1. □

Another very general result, of a somewhat disturbing nature, is Poincaré return theorem.

Theorem 2.6.6 (Poincaré) *Given a dynamical systems (X, T, μ) and a measurable set A, with $\mu(A) > 0$, there exists infinitely many $n \in \mathbb{N}$ such that*

$$\mu(T^{-n} A \cap A) \neq 0.$$

The proof is rather simple (by contradiction) and the reader can certainly find it out by herself (see Problem 2.18).[32]

A natural question is how long it takes, on average, to come back to a set A. Let $A \subset X$ be a measurable set, and let us define the return time

$$\tau_A(x) = \inf\{n \in \mathbb{N} \;:\; f^n(x) \in A\},$$

the reader can easily check that $\tau : A \to \mathbb{N} \cup \{\infty\}$ is a measurable function.

Lemma 2.6.7 (Kač) *Given a dynamical systems* (X, T, μ) *and a measurable set* A*, with* $\mu(A) > 0$*,*

$$\int_A \tau_A(x)\mu(dx) = 1 - \mu(Y),$$

where $Y = \{x \in X \;:\; T^n(x) \notin A \, \forall n \in \mathbb{N}\}$.

Proof. Consider the set $\tilde{X}_A = \cup_{n=1}^{\infty} T^{-n}(A)$, clearly $T^{-1}(\tilde{X}_A) \subset \tilde{X}_A$. This means that $\tilde{X}_A \setminus T^{-1}(\tilde{X}_A)$ has zero measure. We can then define $X_A = \cap_{n=0}^{\infty} T^{-n}\tilde{X}_A$, clearly $\tilde{X}_A \setminus X_A$ has zero measure and $T(X_A) = X_A = T^{-1}(X_A)$. Also, $B = A \setminus X_A$, must have zero measure, otherwise Poincarè theorem would imply that there exists $m \in \mathbb{N}$ such that $T^{-m}B \cap B \neq \emptyset$, but $T^{-m}B \subset T^m(A) \subset X_A$, which is a contradiction. The same argument shows that τ_A is almost everywhere finite on A. We can thus resctrict to the dynamical systems $(X_A, T, \bar{\mu})$, where $\bar{\mu} = \mu(X_A)^{-1}\mu$. By construction, τ_A is almost everywhere finite on X_A.

Let $E_n = \{x \in A_* \;:\; \tau_A(x) = n\}$ and $R_n = \{x \in A_* \;:\; \tau_A(x) = n\}$. Note that all this sets are disjoint, hence thier measure must tend to zero as $n \to \infty$. Then

$$T^{-1}R_n = E_{n+1} \cup R_{n+1}.$$

Consequently,

$$\mu(R_n) = \mu(E_{n+1}) + \mu(R_{n+1}) = \sum_{k=n+1}^{\infty} \mu(E_k).$$

[32] An unsettling aspect of the theorem is due to the following possibility. Consider a room full of air, the motion of the molecules can be thought to happen accordingly to Newton equations, i.e. it is an Hamiltonian systems, hence a dynamical system to which Poincaré theorem applies. Let A be the set of configurations in which all the air is in the left side of the room. Since we ignore, in general, the past history of the room, it could very well be that at some point in the past the systems was in a configuration belonging to A–maybe some silly experiment was performed. So there is a positive probability for the system to return in the same state. Therefore the disturbing possibility of sudden death by decompression.

It follows

$$1 - \mu(Y) = \mu(X_A) = \sum_{n=1}^{\infty} \mu(E_n) + \mu(R_n) = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \mu(E_k)$$
$$= \sum_{k=1}^{\infty} k\mu(E_k) = \int_A \tau_A(x)\mu(dx).$$

□

Let us go back to the relation between ergodicity and averages. From an intuitive point of view a function from X to $\mathbb{R}$ can be thought as an "observable," since to each configuration it associates a value that can represent some relevant property of the configuration (the property that we observe). So, if we observe the system for a long time via the function f, what we see should be well represented by the function f^+. Furthermore, notice that there is a simple relations between invariant functions and invariant sets. More precisely, if a measurable set A is invariant, then its characteristic function χ_A is a measurable invariant function; if f is an invariant function then for each measurable set $I \in \mathbb{R}$ the set $f^{-1}(I)$ is a measurable invariant set (if the implications of the above discussions are not clear to you, see Problem 2.25).

As a byproduct of the previous discussion it follows that if a system is ergodic then for each function $f \in L^1(X, \mu)$ the function f_+ is almost everywhere constant and equal to $\int_X f$. We have just proven an interesting characterization of the ergodic systems:

Theorem 2.6.8 *A Dynamical System (X, T, μ) is ergodic if and only if for each $f \in L^1(X, \mu)$ the ergodic average f^+ is constant; in fact, $f^+ = \mu(f)$ a.e..*

In other words, if we observe the time average of some observable for a sufficiently long time then we obtain a value close to its space average. The previous observation is very important especially because the space average of a function does not depend on the dynamics. This is exactly what we where mentioning previously: the fact that the dynamics is sufficiently 'complex' allows us to ignore it completely, provided we are interested only in knowing some average behavior. The relevance of ergodic theory for physical systems is largely connected to this fact.

2.7 Mixing

We have argued the importance of ergodicity, yet from a physical point of view ergodicity may be relevant only if it takes places at a sufficiently fast rate (i.e., if the time average converges to the space average on a physically meaningful time scale). This has prompted the study of stronger statistical properties of which we will give a brief, and by no mean complete, account in the following.

Definition 4 *A Dynamical System (X, T, μ) is called mixing if for every pairs of measurable sets A, B we have*

$$\lim_{n\to\infty} \mu(T^{-n}(A)\cap B) = \mu(A)\mu(B).$$

Obviously, if a system is mixing, then it is ergodic. In fact, if A is an invariant set for T, then $T^{-n}A \subset A$, so, calling A^c the complement of A, we have

$$\mu(A)\mu(A^c) = \lim_{n\to\infty} \mu(T^{-n}A\cap A^c) = 0,$$

and the measure of A is either one or zero.

An equivalent characterization of mixing is the following:

Proposition 2.7.1 *A Dynamical System (X, T, μ) is mixing if and only if*

$$\lim_{n\to\infty} \int_X f\circ T^n g d\mu = \int_X f d\mu \int_X g d\mu$$

for every f, $g \in L^2(X, \mu)$ or for every $f \in L^\infty(X,\mu)$ and $g \in L^1(X,\mu)$.[33]

The proof is rather straightforward and it is left as an exercise to the reader (see Problem 2.27) together with the proof of the next statement.

Proposition 2.7.2 *A Dynamical System (X, T, μ), with X a compact metric space, T continuous and μ Borel, is mixing if and only if for each probability measure λ absolutely continuous with respect to μ*

$$\lim_{n\to\infty} \lambda(f\circ T^n) = \mu(f)$$

for each $f \in \mathcal{C}^{(0)}(\mathbb{T}^2)$.

[33] The quantity $\int_X f\circ Tg - \int_X f\int_X g$ is called "correlation," and its tending to zero–which takes places always in mixing systems–it is called "decay of correlation."

This last characterization is interesting from a mathematical point of view. Define, as usual, the evolution of a measure via the equation

$$(T_*\lambda)(f) \equiv \lambda(f \circ T)$$

for each continuous function f. If for each measure, absolutely continuous with respect to the invariant one, the evolved measure converges weakly to the invariant measure, then the system is mixing (and thus the evolved measures converge strongly). This has also a very important physical meaning: if the initial configuration is known only in probability, the probability distribution is absolutely continuous with respect to the invariant measure, and the system is mixing, then, after some time, the configurations are distributed according to the invariant measure. Again the details of the evolution are not important to describe relevant properties of the system.

2.7.1 Examples

Rotations

–We have seen that the translations by an irrational angle are ergodic. They are not mixing. The reader can easily see why.

Bernoulli shift

–The key observation is that, given a measurable set A, for each $\varepsilon > 0$ there exists a set $A_\varepsilon \in \mathcal{A}$, thus depending only on a finite subset of indices,[34] with the property[35]

$$\mu(A_\varepsilon \backslash A) \leq \varepsilon.$$

This is not immediately obvious, but it is a general measure theoretic consequence of our definition of the σ-algebra (be more precise refers to previous discussion). Then, given A, B measurable, and for each $\varepsilon > 0$, let A_ε, B_ε be such an approximation, and I_A, I_B the defining sets of indices, then

$$\left|\mu(T^{-m}A \cap B) - \mu(A)\mu(B)\right| \leq 4\varepsilon + \left|\mu(T^{-m}A_\varepsilon \cap B_\varepsilon) - \mu(A_\varepsilon)\mu(B_\varepsilon)\right|.$$

If we choose m so large that $(I_A + m) \cap I_B = \emptyset$, then by the definition of Bernoulli measure we have

$$\mu(T^{-m}A_\varepsilon \cap B_\varepsilon) = \mu(T^{-m}A_\varepsilon)\mu(B_\varepsilon) = \mu(A_\varepsilon)\mu(B_\varepsilon),$$

[34] Remember, this means that there exists a finite set $I \subset \mathbb{Z}$ such that it is possible to decide if $\sigma \in \Sigma_n$ belongs or not to A_ε only by looking at $\{\sigma_i\}_{i \in I}$.

[35] This follows from our construction of the σ-algebra and by the definition of outer measure.

which proves

$$\lim_{m\to\infty} \mu(T^{-m}A \cap B) = \mu(A)\mu(B).$$

Dilation

–This system is mixing. In fact, let $f,\, g \in \mathcal{C}^{(1)}(\mathbb{T})$, then we can represent them via their Fourier series $f(x) = \sum_{k\in\mathbb{Z}} e^{2\pi ikx} f_k,\ f_{-k} = \overline{f}_k$. It is well known that $\sum_{k\in\mathbb{Z}} |f_k| < \infty$ and $|f_k| \le \frac{c}{|k|}$, for some constant c depending on f. Therefore,

$$f(T^n x) = \sum_{k\in\mathbb{Z}} e^{2\pi i 2^n kx} f_k,$$

which implies that the only Fourier coefficients of $f \circ T^n$ different from zero are the $\{2^n k\}_{k\in\mathbb{Z}}$. Hence,

$$\left|\int_{\mathbb{T}} f \circ T^n g - \int_{\mathbb{T}} f \int_{\mathbb{T}} g\right| = \left|\sum_{k\in\mathbb{Z}} f_k g_{2^n k} - f_0 g_0\right| \le c 2^{-n} \sum_{k\in\mathbb{Z}} |f_k|.$$

The previous inequalities imply the exponential decay of correlations for each smooth function. The proof is concluded by a standard approximation argument: given $f,\, g \in L^2(X,\, d\mu)$, for each $\varepsilon > 0$ exists $f_\varepsilon,\, g_\varepsilon \in \mathcal{C}^{(1)}(X)$: $\|f - f_\varepsilon\|_2 < \varepsilon$ and $\|g - g_\varepsilon\|_2 < \varepsilon$. Thus,

$$\left|\int_{\mathbb{T}} f \circ T^n g - \int_{\mathbb{T}} f \int_{\mathbb{T}} g\right| \le \left|\int_{\mathbb{T}} f_\varepsilon \circ T^n g_\varepsilon - \int_{\mathbb{T}} f_\varepsilon \int_{\mathbb{T}} g_\varepsilon\right| + 2(\|f\|_2 + \|g\|_2)\varepsilon,$$

which yields the result by choosing first ε small and then n sufficiently large.

2.8 Stronger statistical properties

Even more extreme form statistical behaviors are possible, to present them we need to introduce the idea of equivalent systems. This is done via the concept of conjugation that we have already seen informally in Example 2.4.1 (logistic map, circle map).

Definition 5 *Two Dynamical Systems (X_1, T_1, μ_1), (X_2, T_2, μ_2) are (measurably) conjugate if there exists a measurable map $\phi : X_1 \to X_2$ almost everywhere invertible*[36] *such that $\mu_1(A) = \mu(\phi(A))$ and $T_2 \circ \phi = \phi \circ T_1$.*

[36] This means that there exists a measurable function $\phi^{-1} : X_2 \to X_1$ such that $\phi \circ \phi^{-1} =$ id μ_2-a.e. and $\phi^{-1} \circ \phi =$ id μ_1-a.e..

Clearly, the conjugation is an equivalence relation. Its relevance for the present discussion is that conjugate systems have the same ergodic properties (Problem2.39).[37]

We can now introduce the most extreme form of stochasticity.

Definition 6 *A dynamical system* (X, T, μ) *is called Bernoulli if there exists a Bernoulli shift* (M, ν, σ) *and a measurable isomorphism* $\phi : X \to M$ *(i.e., a measurable map one one and onto apart from a set of zero measure and with measurable inverse) such that, for each* $A \in X$*,*

$$\nu(\phi(A)) = \mu(A)$$

and

$$T = \phi^{-1} \circ \sigma \circ \phi.$$

That is a system is Bernoulli if it is isomorphic to a Bernoulli shift. Since we have seen that Bernoulli systems are very stochastic (remind that they can be seen as describing a random event like coin tossing) this is certainly a very strong condition on the systems. In particular it is immediate to see that Bernoulli systems are mixing (Problem2.39).

2.8.1 Examples

Dilation

–We will show that such a system is indeed Bernoulli. The map ϕ is obtained by dividing $[0, 1)$ in $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$. Then, given $x \in \mathbb{T}$, we define $\phi : \mathbb{T} \to \Sigma_2^+$ by

$$\phi(x)_i = \begin{cases} 1 & \text{if } T^i x \in [0, \frac{1}{2}) \\ 2 & \text{if } T^i x \in [\frac{1}{2}, 1) \end{cases}$$

the reader can check that the map is measurable and that it satisfy the required properties. Note that the above shows that the Bernoulli measure with $p_1 = p_2 = \frac{1}{2}$ is nothing else than Lebesgue measure viewed on the numbers written in basis two. This may explain why we had to be so careful in the construction of the Bernoulli measure.

[37] Of course the reader can easily imagine other forms of conjugacy, e.g. topological or differential conjugation.

Baker

–Let us define ϕ^{-1}; for each $\sigma \in \Sigma_2$

$$x = \sum_{i=0}^{\infty} \frac{\sigma_{-i}}{2^{i+1}},$$
$$y = \sum_{i=1}^{\infty} \frac{\sigma_i}{2^i}.$$

Again the rest is left to the reader.

Problems

2.1 Given an invariant set A prove that, if $T(A)$ is measurable, then $\mu(TA) \geq (A)$.

2.2 Set $\mathcal{M}^1(X) = \{\mu \in \mathcal{M} \mid \mu(X) = 1\}$ and $\mathcal{M}^1_T(X) = \mathcal{M}^1(X) \cap \mathcal{M}_T(X)$. Prove that $\mathcal{M}^1_T(X)$ and $\mathcal{M}^1(X)$ are convex sets in $\mathcal{M}(x)$.

2.3 Call $\mathcal{M}^e(X) \subset \mathcal{M}^1(X)$ the set of ergodic probability measures. Show that $\mathcal{M}^e(X)$ consists of the extremal points of $\mathcal{M}_T(X)$. (Hint: Krein-Milman Theorem [63]).

2.4 Prove that the Lebesgue measure is invariant for the rotations on $\mathbb{T}$.

2.5 Consider a rotation by $\omega \in \mathbb{Q}$, find invariant measures different from Lebesgue.

2.6 Prove that the measure μ_h defined in Examples 2.1.1 (Hamiltonian systems) is invariant for the Hamiltonian flow. (Hint: Use the properties of H to deduce $\langle \nabla_{\phi^t x} H, \, d_x\phi^t \nabla_x H \rangle = \|\nabla_x H\|^2$, and thus $d_x\phi^t \nabla_x H = \frac{\|\nabla_x H\|^2}{\|\nabla_{\phi^t x} H\|^2} \nabla_{\phi^t x} H + v$ where $\langle \nabla_{\phi^t x} H, \, v \rangle = 0$. Then study the evolution of an arbitrarily small parallelepiped with one side parallel to $\nabla_x H$– or look at the volume form if you are more mathematically incline– remembering the invariance of the volume with respect to the flow.)

2.7 Given a Poincaré section prove that there exists $c > 0$ such that $\inf \tau_\Sigma \geq c > 0$.

2.8 Show that ν_Σ, defined in (2.2.1) is well defined.(Hint: use the invariance of μ and the fact that, by Problem 2.7, if $A \subset \Sigma$ then $\mu(\phi^{[0,\delta]}(A) \cap \phi^{[n\delta, \, (n+1)\delta]}) = 0$ provided $(n+1)\delta \leq c$.)

2.9 Show that the return time τ_Σ is finite ν_Σ-a.e. .(Hint: let $\delta < c$ and $\Sigma_\delta := \phi^{[0,\delta]}\Sigma$, apply Poincaré return theorem to Σ_δ.)

2.10 Show that ν_Σ is T_Σ invariant. Verify that, collecting the results of the last exercises, $(\Sigma, T_\Sigma, \nu_\Sigma)$ is a Dynamical Systems.

2.11 Prove that the Bernoulli measure is invariant with respect to the shift. (Hint: check it on the algebra $\mathcal{A}$ first.)

2.12 Let Σ_p be the set of periodic configurations of Σ. If μ is the Bernoulli measure prove that $\mu(\Sigma_p) = 0$ (Hint: Σ_p is the countable union of zero measure sets.)

2.13 Consider the Bernoulli shift on $\mathbb{Z}$ and define the following equivalence relation: $\sigma \sim \sigma'$ iff there exists $n \in \mathbb{Z}$ such that $T^n\sigma = \sigma'$ (this means that two sequences are equivalent if they belong to the same orbit). Consider now the equivalence classes (the space of orbits) and choose[38] a representative from each class, call the set so obtained K. Show that K cannot be a measurable set. (Hint: show that $K \cap T^n K \subset \Sigma_p$, then by using Problem 2.12 show that if K is measurable $\sum_{i=1}^{\infty} \mu(T^n K) = 1$ which, by the invariance of μ, is impossible).

2.14 Compute the transfer operator for maps of $\mathbb{T}$. (Hint: Use the equivalent definition $\int g\mathcal{L}f dm = \int fg \circ T dm$.) Prove that $\|\mathcal{L}h\|_1 \leq \|h\|_1$.

2.15 Prove the Lasota-York inequality (2.4.4).

2.16 Prove that for each sequence $\{h_n\} \subset \mathcal{C}^{(1)}(\mathbb{T})$, with the property

$$\sup_{n\in\mathbb{N}} \|h_n'\|_1 + \|h_n\|_1 < \infty,$$

it is possible to extract a subsequence converging in L^1. (Hint: Consider partitions $\mathcal{P}_n$ of $\mathbb{T}$ in intervals of size $\frac{1}{n}$. Define the conditional expectation $\mathbb{E}(h|\mathcal{P}_n)(x) = \frac{1}{m(I(x)} \int_{I(x)} hdm$, where $x \in I(x) \in \mathcal{P}_n$. Prove that $\|\mathbb{E}(h|\mathcal{P}_n) - h\|_1 \leq \frac{1}{n}\|h'\|_1$. Notice that the functions $\mathbb{E}(h_n|\mathcal{P}_m)$ have only m distinct values and, by using the standard diagonal trick, construct an subsequence h_{n_j} such that all the $\mathbb{E}(h_{n_j}|\mathcal{P}_m)$ are converging. Prove that h_{n_j} converges in L^1.)

2.17 Prove Corollary 2.6.3.

[38] Attention !!!: here we are using the *Axiom of choice*.

2.18 Prove Theorem 2.6.6 (Hint: Note that $\mu(T^{-n}A \cap T^{-m}A) \neq 0$ then, supposing without loss of generality $n < m$, $\mu(A \cap T^{-m+n}A) \neq 0$. Then prove the theorem by absurd remembering that $\mu(X) < \infty$.)

2.19 A topological Dynamical System (X, T) is called *Topologically transitive*, if it has a dense orbit. Show that if $(\mathbb{T}^d, T, m)$ is ergodic and T is continuous, then the system is topologically transitive. (Hint: For each $n \in \mathbb{N}$, $x \in \mathbb{T}^d$ consider $B_{\frac{1}{m}}(x)$–the ball of radius $\frac{1}{m}$ centered at x. By compactness, there are $\{x_i\}$ such that $\cup_i B_{\frac{1}{m}}(x_i) = \mathbb{T}^d$. Let

$$A_{m,i} = \{y \in \mathbb{T}^d \mid T^k y \cap B_{\frac{1}{M}}(X_I) = \emptyset \ \ \forall k \in \mathbb{N}\},$$

clearly $A_{m,i} = \cap_{k\in\mathbb{N}} T^{-k} B_{\frac{1}{m}}(x_i)^c$ has the property $T^{-1}A_{m,i} \supset A_{m,i}$. It follows that $\tilde{A}_{m,i} = \cup_{n\in\mathbb{N}} T^{-n} A_{m,i} \supset A_{m,i}$ is an invariant set and it holds $\mu(\tilde{A}_{m,i} \backslash A_{m,i}) = 0$. Since $A_{m,i}$ it is not of full measure, $\tilde{A}_{m,i}$, and thus $A_{m,i}$, must have zero measure. Hence, $\bar{A}_m = \cap_i A_{m,i}$ has zero measure. This means that $\cup_{m\in\mathbb{N}} \bar{A}_m$ has zero measure. Prove now that, for each $y \in \mathbb{T}^d$, the trajectories that never get closer than $\frac{2}{m}$ to y are contained in $\bar{A}_m$, and thus have measure zero. Hence, almost every point has a dense orbit.)
Extend the result to the case in which X is a compact metric space and μ charges the open sets (that is: if $U \subset X$ is open, then $\mu(U) > 0$).

2.20 Give an example of a system with a dense orbit which it is not ergodic.

2.21 Give an example of an ergodic system with no dense orbit.

2.22 Give an example of a Dynamical Systems which does not have any invariant probability measure. (Hint: $X = \mathbb{R}^d$, $Tx = x + v$, $v \neq 0$.)

2.23 Show that a Dynamical Systems (X, T, μ) is ergodic if and only if there does not exists any invariant probability measure absolutely continuous with respect to μ, beside μ itself.

2.24 Prove that Birkhoff theorem implies Von Neumann theorem. (Hint: Note that the ergodic average is an isometry in L^2. Use Lebesgue dominate convergence theorem to prove convergence in L^2 for bounded functions. Use Fatou to show that if $f \in L^2$ then $f^+ \in L^2$ and a $3 - \varepsilon$ argument to conclude).

2.25 Prove that if (X, T, μ) is ergodic, then all $f \in L^1(X, \mu)$ and $f \circ T = f$ are a.e. constant. Prove also the converse.

2.26 For each measurable set A, let

$$F_{A,n}(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \chi_A(T^i x).$$

be the average number of times x visits A in the time n. Show that there exists $F_A = \lim_{n\to\infty} F_{A,n}$ a.e. and prove that, if the system is ergodic, $F_A = \mu(A)$. (Hint: Birkhoff theorem and Theorem 2.6.8).

2.27 Prove Proposition 2.7.1 and Proposition 2.7.2. (Hint: Note that for each measurable set A and $\varepsilon > 0$ there exists $f \in \mathcal{C}^{(0)}(X)$ such that $\mu(|f - \chi_A|) < \varepsilon$ –by Uryshon Lemma and by the regularity of Borel measures. To prove that $\mu(T^{-n}A \cap B) \to \mu(A)\mu(B)$ choose $d\lambda = \mu(B)^{-1}\chi_B d\mu$ and use the invariance of μ to obtain the uniform estimate $\lambda(|f \circ T^n - \chi_A \circ T^n|) \leq \mu(B)^{-1}\mu(|f - \chi_A|)$.)

2.28 Show that the irrational rotations are not mixing.

2.29 Prove that if $f \in \mathcal{C}^{(2)}(\mathbb{T})$, then its Fourier series converges uniformly.[39] (Hint: Remember that $f_n = \frac{1}{2\pi}\int_{\mathbb{T}} e^{2\pi i n x} f(x)dx$. Thus $f_n = \frac{1}{(2\pi i n)^2 2\pi}\int_{\mathbb{T}} e^{2\pi i n x} f^{(2)}(x)dx$.)

2.30 Let ν be a Borel measure on $Q = [0,1]^2$ such that $\nu(\partial_x f) = 0$ for all $f \in \mathcal{C}^{(1)}_{\text{per}}(Q) = \{f \in \mathcal{C}^{(1)}(Q) \mid f(0,y) = f(1,y) \ \forall\ y \in [0,1]\}$. Prove that there exists a Borel measure ν_1 on $[0,1]$ such that $\nu = m \times \nu_1$. (Hint: The measure ν_1 is nothing else then the marginal with respect to x, that is: for each continuous function $f : [0,1] \to \mathbb{R}$ define $\tilde{f} : Q \to \mathbb{R}$ by $\tilde{f}(x,y) = f(y)$, then $\nu_1(f) = \nu(\tilde{f})$. To prove the statement use Fourier series. If f is smooth enough $f(x,y) = \sum_{k\in\mathbb{Z}} \hat{f}_k(y)e^{2\pi i k x}$ where the Fourier series for f and $\partial_x f$ converge uniformly. Then notice that $0 = \nu(\partial_x e^{2\pi i k \cdot}) = 2\pi i k \nu(e^{2\pi i k \cdot})$ implies $\nu(f) = \nu(\hat{f}_0) = m \times \nu_1(f)$.)

2.31 Prove that is a flow is ergodic (mixing) so is each Poincarè section. Prove that is a map is ergodic so is any suspension on the map. Give an example of a mixing map with a non-mixing suspension (constant ceiling).

2.32 Consider $([0,1], T)$ where

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$$

[39] This result is far from optimal, see [71] if you want to get deeper into the theory of Fourier series

($[a]$ is the integer part of a), and

$$\mu(f) = \frac{1}{\ln 2}\int_0^1 f(x)\frac{1}{1+x}dx.$$

Prove that $([0,1], T, \mu)$ is a Dynamical System.[40] (Hint: write $\mu(f \circ T) = \sum_{i=1}^{\infty}\int_{\frac{1}{i+1}}^{\frac{1}{i}} f \circ T(x)\mu(dx)$, change variable and use the identity $\frac{1}{a^2+a} = \frac{1}{a} - \frac{1}{a+1}$ to obtain a series with alternating signs.)

2.33 Prove that for each $x \in \mathbb{Q} \cap [0,1]$ holds $\lim_{n\to\infty} T^n(x) = 0$. (Hint: if $x = \frac{p_0}{q_0}$, $p_0 \leq q_0$, then $q_0 = k_1 p_0 + p_1$, with $p_1 < p_0$, and $T(x) = \frac{p_1}{p_0}$. Let $q_1 = p_0$ and go on noticing that $p_{i+1} < p_i$.)[41]

2.34 In view of the two previous exercises explain why it is problematic to study the statistical properties of the Gauss map on a computer.(Hint: The computer uses only rational numbers. It is quite amazing that tese type of pathologies arises rather rarely in the nu,erical studies carried out by so many theoretical physicist.)

2.35 Prove that any infinite continuous fraction of the form

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

with $a_i \in \mathbb{N}$ defines a real number. (Hint: Note that if you fix the first n $\{a_i\}$, this corresponds to specifying which elements of the partition $\{[\frac{1}{i+1}, \frac{1}{i}]\}$ are visited by the trajectory of $\{T^i x\}$. By the expansivity of the map readily follows that x must belong to an interval of size λ^{-n} for some $\lambda > 1$.)

---

[40] The above map is often called *Gauss map* since to him is due the discovery of the above invariant measure

[41] This is nothing else that the *Euclidean algorithm* to find the greatest common divisor of two integers [32] Elements, Book VII, Proposition 1 and 2. The greatest common divisor is clearly the last non-zero p_i. This provides also a remarkable way of writing rational numbers: *continuous fractions*

$$\frac{p_0}{q_0} = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \ddots + \frac{1}{k_n}}}.$$

2.36 Prove that, for each $a \in \mathbb{N}$,

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \ddots}}} = \frac{-a + \sqrt{a^2 + 4}}{2}.$$

(Hint: Note that $T(x) = x$.) Study other periodic continuous fractions.

2.37 Choose a number in $[0, 1]$ at random according to Lebesgue distribution. Assuming that the Gauss map is mixing (which is), compute the average percentage of numbers larger than n in the associated continuous fraction. (Hint: Define $f(x) = [x^{-1}]$, then the entries of the continuous fraction of x are $\{f \circ T^i\}$. The quantity one must compute is then $m(\lim_{k\to\infty} \frac{i}{k} \sum_{i=0}^{k-1} \chi_{[n,\infty)} \circ f \circ T^i) = \mu([n, \infty))$.)

2.38 Let (X_0, T_0, μ_0) be a Dynamical System and $\phi : X_0 \to X_1$ an homeomorphism. Define $T_1 := \phi \circ T_0 \circ \phi^{-1}$ and $\mu_1(f) = \mu_0(f \circ \phi^{-1})$. Prove that (X_1, T_1, μ_1) is a Dynamical System.

2.39 Let (X_0, T_0, μ_0) be measurably conjugate to (X_1, T_1, μ_1), then show that one of the two is ergodic if and only if the other is ergodic. Prove the same for mixing.

2.40 Show that the systems described in Examples 2.4.1–strange attractor and horseshoe, are Bernoulli.

2.41 Prove Lebesgue density theorem: for each measurable set A, $m(A) > 0$, there exists $x \in A$ such that for each $\varepsilon > 0$ exists $\delta > 0$ such that $m(A \cap [x - \delta, x + \delta]) > (1 - \varepsilon)2\delta$. (Hint: we have seen in Examples 2.8.1-Dilations that Lebesgue measure is equivalent to Bernoulli measure and that the cylinder correspond to intervals. It the suffices to prove the theorem for the latter. Let $A \subset \Sigma^+$ such that $\mu(A) > 0$, then, for each $\varepsilon > 0$,there exists $A_\varepsilon \in \mathcal{A}$ such that $A_\varepsilon \supset A$ and $\mu(A_\varepsilon) - \mu(A) < \varepsilon\mu(A)$. Since $A_\varepsilon \in \mathcal{A}$, it exists $n_\varepsilon \in \mathbb{N}$ such that it is possible to decide if $\sigma \in A_\varepsilon$ only by looking at $\{\sigma_1, \dots, \sigma_{n_\varepsilon}\}$. Consider all the cylinders $\mathcal{I}\{A(0; k_1, \dots, k_{n_\varepsilon})\}$, clearly if $I \in \mathcal{I}$ then $I \cap A_\varepsilon$ is either I or $\emptyset$. Let $\mathcal{I}_+ = \{I \in \mathcal{I} \mid I \cap A_\varepsilon = I\}$ and $\mathcal{I}_+ = \{I \in \mathcal{I} \mid I \cap A_\varepsilon = \emptyset\}$. Now suppose that for each $I \in \mathcal{I}_+$ holds $\mu(I \cap A) \le (1 - \varepsilon)\mu(I)$ then

$$\mu(A) = \sum_{I \in \mathcal{I}_+} \mu(A \cap I) \le (1 - \varepsilon)\mu(A_\varepsilon) < \mu(A),$$

which is absurd. Thus there must exists $I \in \mathcal{I}_+$: $\mu(A \cap I) > (1 - \varepsilon)\mu(I)$.)

Chapter 3

Uniformly hyperbolic systems

The concept of ergodicity is a very important one in dynamical systems, yet it turns out to be surprisingly difficult to establish if a system is or not ergodic and very few examples have been fully analyzed. Nonetheless, in this chapter we will see that a very simple idea introduced by Hopf [45, 46] allows to discuss the ergodicity in some special cases. The relevance of Hopf's idea is that, properly generalized, it allows to prove ergodicity in a vast class of systems. Much in the following chapters will deal with such a generalization.

3.1 A Basic Example

To explain the Hopf approach we will study a very simple case: a slight generalization of Arnold's cat, see Examples 2.1.1. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ (here by $\mathbb{T}^2$ we mean $\mathbb{R}^2 \mod 1$) be defined by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a & 1+a^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mod 1 \tag{3.1.1}$$

It is obvious that if $a \in \mathbb{Z}$, then T is well defined and it is a linear automorphism of $\mathbb{T}^2$. Moreover, for all $x \in \mathbb{T}^2$

$$D_x T = \begin{pmatrix} 1 & a \\ a & 1+a^2 \end{pmatrix} \equiv L.$$

Since $\det L = 1$, Lebesgue measure is preserved. It is immediate to see that there exists $\lambda > 1$; $v_+, v_- \in \mathbb{R}^2$:

$$\begin{aligned} Lv_+ &= \lambda v_+ \\ Lv_- &= \lambda^{-1} v_-. \end{aligned}$$

We will call v_+ the unstable eigenvector (direction) and v_- the stable eigenvector (direction). Remark that, since $L^* = L$, $\langle v_+, v_- \rangle = 0$.

The dynamical system just described is a basic model of hyperbolic systems (see next chapter) and will appear in various disguises in this book.

Proposition 3.1.1 *The Arnold cat is ergodic.*

Sections 3.1.1 and 3.2.1 contain two different proofs of the above proposition.

3.1.1 An algebraic proof

A first idea to studying the ergodic properties of this system is to imitate what we have done for the Rotations (Examples 2.5.1) and the Dilations: use Fourier series. Let us see how such an approach would work.

Let $f,\ g \in C^{(m)}(\mathbb{T}^2)$, then[1]

$$f \circ T^n(x) = \sum_{k \in \mathbb{Z}^2} e^{2\pi i \langle k,\, L^n x \rangle} f_k = \sum_{k \in \mathbb{Z}^2} e^{2\pi i \langle k,\, x \rangle} f_{L^{-n}k},$$

so

$$\begin{aligned} \int_{\mathbb{T}^2} f \circ T^{2n} g = \int_{\mathbb{T}^2} f \circ T^n g \circ T^{-n} &= \sum_{k \in \mathbb{Z}^2} f_{L^{-n}k} g_{L^n k} \\ &= f_0 g_0 + \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} f_{L^{-n}k} g_{L^n k}. \end{aligned}$$

It is well known that $f \in C^{(m)}(\mathbb{T}^2)$ implies[2]

$$|f_k| \le \frac{\|f^{(m)}\|_1}{\|k\|^m} \text{ for } k \neq 0$$

hence

$$\left| \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} f_{L^{-n}k} g_{L^n k} \right| \le \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{\|f^{(m)}\|_1 \|g^{(m)}\|_1}{\|L^{-n}k\|^m \|L^n k\|^m}.$$

---

[1]Note that $e^{2\pi i \langle k, T^n x \rangle} = e^{2\pi i \langle k, L^n x \rangle}$.

[2]Here for $\|f^{(m)}\|_1$ we mean $\sup_{\substack{i+j=m \\ i,j \ge 0}} \frac{1}{(2\pi)^m} \int_{\mathbb{T}^2} |\partial_{x_1}^i \partial_{x_2}^j f| dx_1 dx_2$; and $\|k\| = \sqrt{k_1^2 + k_2^2}$.

For each $k \in \mathbb{Z}^2$ holds $\|k\|^2 = \langle k,\, v^+\rangle^2 + \langle k,\, v^-\rangle^2$ hence one of the two terms must be larger than $\|k\|^2/2$.[3] Moreover if $k \neq 0$ $\|L^n k\| \geq 1$ for each $n \in \mathbb{Z}$. Using the above facts it yields

$$
\begin{aligned}
\left|\sum_{k\in\mathbb{Z}^2\setminus\{0\}} f_{L^{-n}k}\, g_{L^n k}\right| &\leq \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{\|f^{(m)}\|_1 \|g^{(m)}\|_1 2^{m/2}}{\lambda^{nm}\|k\|^m} \\
&\leq \text{const.} \|f^{(m)}\|_1 \|g^{(m)}\|_1 \lambda^{-nm},
\end{aligned}
$$

where the constant does not depend on f or g and we have assumed $m \geq 3$ to insure the convergence of the series.

Accordingly, for each $f,\ g \in C^{(3)}(\mathbb{T}^2)$ we have

$$
\left|\int_{\mathbb{T}^2} f\circ T^n g - \int_{\mathbb{T}^2} f \int_{\mathbb{T}^2} g\right| \leq \text{const.}\|f^{(3)}\|_1 \|g^{(3)}\|_1 \lambda^{-3n/2}.
$$

To obtain the final result we need an approximation argument. If $f,\, g \in L^2(\mathbb{T}^2)$ we can choose $f_n,\, g_n \in C^{(3)}(\mathbb{T}^2)$ such that they converge to f and g, respectively, in L^2.

Then, for each $\varepsilon \geq 0$, choose $m \in \mathbb{N}$ such that

$$
\|f - f_m\|_2 + \|g - g_m\|_2 \leq \varepsilon.
$$

Accordingly,

$$
\begin{aligned}
\left|\int_{\mathbb{T}^2} f\circ T^n g - \int_{\mathbb{T}^2} f \int_{\mathbb{T}^2} g\right| \leq& \left|\int_{\mathbb{T}^2} f_m\circ T^n g_m - \int_{\mathbb{T}^2} f_m \int_{\mathbb{T}^2} g_m\right| \\
&+ 2\|f - f_m\|_2\|g\|_2 + 2\|f_m\|_2\|g - g_m\|_2 \\
\leq& 2(\|g\|_2 + \|f\|_2)\varepsilon + \varepsilon,
\end{aligned}
$$

where we have chosen n large enough depending on m and ε. We have just proven mixing.

The above result is certainly rather satisfactory: non only it proves the mixing–hence the ergodicity–of the map but gives an explicit estimate on the rate of decay and shows how such a rate depends on the regularity of the functions.[4] Therefore, an eventual critique can not concern the type of result but only the method; indeed the method does have a shortcoming.

[3] Here we have normalized the eigenvalues so that $\|v^{\pm}\| = 1$.

[4] In fact, the obtained estimate it is not optimal: using the Diofantine properties of the stable and unstable directions a better estimate can be obtained.

The use of Fourier series is strictly related to the group structure of $\mathbb{T}^2$ and the linearity of the map. Clearly in more general systems, where both such properties may fail, such a technique has no hope whatsoever of being applied.[5] In some sense, much of the theory of hyperbolic systems may be view as an attempt to find an alternative proof of the above facts. Such a proof must be *dynamical* meaning that it must use properties of the dynamics and as little as possible of the structure of the space.

The best way to gain a real feeling of what is meant by *dynamical* is to see such type of arguments in action.

3.2 An Idea by Hopf

The following argument, due to Hopf [45, 46] is exactly such a dynamical proof of ergodicity. Let $f : \mathbb{T}^2 \to \mathbb{R}$ be a continuous function. We want to prove that for almost every $x \in \mathbb{T}^2$ the time averages converge as $n \to +\infty$ to the average value of f, i.e., $\int_{\mathbb{T}^2} f d\mu$. Once this is established, one can obtain the same property for all integrable functions by an approximation argument, which proves ergodicity due to the characterization provided by Theorem 2.6.8 (see also Problem 2.25). From Birkhoff Ergodic Theorem (BET) we know that the time averages converge almost everywhere to a function $f^+ \in L^1(\mathbb{T}^2, \mu)$ which is invariant on the orbits of T, i.e., $f^+ \circ T = f^+$, and has the same average value as f, i.e., $\int f^+ d\mu = \int f d\mu$. Further, applying BET to f and T^{-1} we obtain that the time averages in the past

$$\frac{f(x) + f(T^{-1}x) + \cdots + f(T^{-n+1}x)}{n}$$

converge almost everywhere as $n \to +\infty$ to $f^- \in L^1(\mathbb{T}^2, \mu)$, $f^- \circ T = f^-$ and $\int f^- d\mu = \int f d\mu$.

The next Lemma is part of the usual magic of the ergodic theory.

Lemma 3.2.1 *The functions f^+ and f^- coincide almost everywhere.*

Proof. Let

$$\mathcal{A}_+ = \{x \in \mathbb{T}^2 \mid f_+(x) > f_-(x)\};$$

by definition $\mathcal{A}_+$ is an invariant set, hence

$$\int_{\mathcal{A}_+} [f_+(x) - f_-(x)]\, d\mu(x) = \int_{\mathcal{A}_+} f(x) d\mu(x) - \int_{\mathcal{A}_+} f(x) d\mu(x) = 0$$

[5] In fact, there are very few cases in which this type of ideas has produced relevant results, noticeably the case of geodesic flows on surfaces of constant negative curvature.

which implies $\mu(\mathcal{A}_+) = 0$ and $f_+ \leq f_-$ μ-almost everywhere. The same argument, this time applied to the set $\mathcal{A}_- = \{x \in \mathbb{T}^2 \mid f_-(x) > f_+(x)\}$, implies the converse inequality. □

3.2.1 A dynamical proof

For $x \in \mathbb{T}^2$ let us denote by $W^u(x)$ ($W^s(x)$) the line in $\mathbb{T}^2$ passing through x and having the direction of the unstable eigenvector (the stable eigenvector), i.e., the eigenvector with eigenvalue λ (λ^{-1}). We call $W^u(x)$ ($W^s(x)$) the unstable (stable) leaf (or manifold) of x. The leaves of x have the following property. If $y \in W^u(x)$ ($y \in W^s(x)$) then the distance (computed along the leaf)

$$d(T^n y,\, T^n x) = \lambda^{-|n|} d(y,\, x) \to 0 \quad \text{as } n \to -\infty(+\infty).$$

Hence for $y, z \in W^{u(s)}(x)$

$$|f(T^n y) - f(T^n z)| \to 0 \text{ as } n \to -\infty(+\infty).$$

It follows that for $y, z \in W^{u)}(x)$ either $f^-(y)$ and $f^-(z)$ are both defined and equal or they are both undefined; the same can be said for $f^+(y)$ and $f^+(z)$ if $y, z \in W^s(x)$.

It is interesting to notice that $W^u(x)$ is an infinitely long line in the direction v_+ that fills densely the torus This implies that the collection (foliation) $\{W^u(x)\}_{x \in \mathbb{T}^2}$ of these global manifolds has a quite complex structure. For this reason, it turns out to be much more convenient to deal only with *local manifolds*.

A local manifold of size δ is simply a piece of $W^u(x)$ of size δ centered at x. In the following, by $W^u(x)$ and $W^s(x)$ we will always mean local manifolds (lines) of some length. The exact length is, most of the time, irrelevant and often will not be specified (in the following, it will be frequently chosen so that the lines do not wrap around the torus more than once).

Up to now, we have seen that f^+ is constant along a.e. stable lines while f^- is constant along a.e. unstable line, since they are equal a.e. it seems obvious that they must be equal and constant. Yet, in the last sentence there are a lot of almost everywhere and, being measure theory a rather subtle subject, it is better to spell out the reasoning in full detail.[6]

Let us choose any point $x \in \mathbb{T}^2$ and prove that there is a neighborhood of x in which f^+ is a.e. constant. Since x is arbitrary this implies that

[6] We have already seen in Examples 2.5.1–Rotations that this type of argument must employ measure theory in a non-trivial way.

f^+ is a.e. constant.[7] Chose a square Q_δ of size $2\delta < \frac{1}{4}$ centered at x with sides parallel to v_+ and v_- respectively. Let $\phi : [-\delta, \delta]^2 \to Q_\delta$ be defined by $\phi(\alpha, \beta) = x + \alpha v_+ + \beta v_-$, where we have chosen $\|v_\pm\| = 1$. It is then convenient to transport the problem in $[-\delta, \delta]^2$ by ϕ: doing so the Lebesgue measure is sent in the Lebesgue measure, and that $f^+ \circ \phi$ is a.e. constant in the vertical direction (α constant), while $f^- \circ \phi$ is a.e. constant in the horizontal direction. This corresponds simply to a change of variables and from now on we will confuse Q_δ and $[-\delta, \delta]^2$ since this does not create any ambiguity.

There are three full measure sets to consider:
$\widetilde{\mathcal{B}}_+ = \{\xi \in Q_\delta \mid f^+(\xi) \text{ is defined}\}$; $\widetilde{\mathcal{B}}_- = \{\xi \in Q_\delta \mid f^-(\xi) \text{ is defined}\}$ and $G = \{\xi \in \widetilde{\mathcal{B}}_+ \cap \widetilde{\mathcal{B}}_- \mid f^+(\xi) = f^-(\xi)\}$.

Let us call $W^s_\alpha := \{(a, b) \in Q_\delta \mid a = \alpha\}$ the segment in Q_δ parallel to the stable direction passing through the point $(\alpha, 0)$, and $W^u_\beta := \{(a, b) \in Q_\delta \mid b = \beta\}$ the segment in Q_δ parallel to the unstable direction passing through the point $(0, \beta)$. The previous discussion proves that there exist $\mathcal{B}_\pm \in [-\delta, \delta]$ such that $\widetilde{\mathcal{B}}_+ = \cup_{\alpha \in \mathcal{B}_+} W^s_\alpha$ and $\widetilde{\mathcal{B}}_- = \cup_{\beta \in \mathcal{B}_-} W^u_\beta$.

Since m is the product of two one dimensional Lebesgue measures[8] Fubini theorem [62] implies that $\mathcal{B}_\pm$ are measurable sets of full measure. Again by Fubini Theorem, it follows

$$4\delta^2 = m(Q_\delta) = m(\widetilde{\mathcal{B}}_+ \cap G) = \int_{\mathcal{B}_+} d\alpha \int_{-\delta}^{\delta} d\beta \chi_{W^s_\alpha \cap G}(\alpha, \beta).$$

This implies immediately that there exists a set $I \subset \mathcal{B}_+$, of full measure, such that, for each $\alpha \in I$ the set $J_\alpha = \{\beta \in \mathcal{B}_- \mid (\alpha, \beta) \in G\}$ is measurable and has full measure as well; the same holds for $E = \cup_{\alpha \in I} W^s_\alpha$.

Finally, let $z, y \in E$, $z = (a, b)$ and $y = (c, d)$. If $a = c$, then $z, y \in W^s_a$ and $f^+(z) = f^+(y)$. On the other hand, if $a \neq c$ then by choosing $\beta \in J_a \cap J_c$ it follows

$$\begin{aligned} f^+(z) &= f^+(W^s_a) = f^+(a, \beta) = f^-(a, \beta) \\ &= f^-(W^u_\beta) = f^-(c, \beta) = f^+(c, \beta) = f^+(y). \end{aligned}$$

That is, f^+ is constant on E, hence f^+ (and f^-) is a.e. constant on Q_δ. By the arbitrariness of Q_δ follows that $f^+ = f^-$ =constant a.e..

[7]Please, note this apparently naïve idea to look at the problem first locally and then globally, we will see much more of it in the following.

[8]Here, to have an unambiguous notation, we should use m_n for the Lebesgue measure in $\mathbb{R}^n$, then we just said $m_2 = m_1 \times m_1$. For simplicity, I have suppressed all the subscript hoping not to confuse the reader too much.

Up to now we have proved that f^+ is a.e. constant only if $f \in \mathcal{C}^{(0)}(\mathbb{T}^2)$, to prove ergodicity we need the same result for each $f \in L^1(\mathbb{T}^2)$. This can be easily obtained by an approximation argument; yet, it is probably more interesting to prove directly that all invariant sets have measure zero or one.

Let us consider a T-invariant measurable subset A. Let

$$f_n \to \chi_A \text{ in } L^1(\mathbb{T}^2, \mu)$$

be a sequence of uniformly bounded continuous approximations to the indicator function.[9] We will use the fact that the time average is continuous with respect to the L^1 norm to establish that the time average of χ_A must be constant on $\mathbb{T}^2$. Indeed, if we denote by $\|\cdot\|_1$ the $L^1(\mathbb{T}^2, m)$ norm, then

$$\begin{aligned}\|f_n^+ - \chi_A^+\|_1 &= \left\| \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left(f_n \circ T^i - \chi_A \circ T^i\right) \right\|_1 \\ &= \lim_{N\to\infty} \frac{1}{N} \left\| \sum_{i=1}^{N} \left(f_n \circ T^i - \chi_A \circ T^i\right) \right\|_1\end{aligned}$$

by the Lebesgue Dominated Convergence Theorem.

Using the invariance of the measure we obtain

$$\|f_n^+ - \chi_A^+\|_1 \le \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \|f_n - \chi_A\|_1 = \|f_n - \chi_A\|_1 .$$

Since the time averages $f_n^+ = m(f_n)$ a.e. in $\mathbb{T}^2$ and $\lim_{n\to\infty} m(f_n) = m(A)$, the Lebesgue dominated convergence theorem implies $\|m(A) - \chi_A^+\|_1 = 0$, that is $\chi_A^+ = m(A)$ a.e.. In addition, the invariance of A forces $\chi_A^+ = \chi_A$ so that either A or A^c has measure zero. In view of the arbitrariness of the invariant set A it follows that T must be ergodic.

3.2.2 What have we done?

The question remains of how and if such an argument can be extended to more general systems. The answer must lie in the possibility to generalize the main ingredients of the previous proof. Such ingredients are essentially

[9] If the existence of such a sequence $\{f_n\}$ it is not obvious, consider the following: for each $\varepsilon > 0$, by the regularity of the Lebesgue measure, there exists $C_\varepsilon \subset A \subset G_\varepsilon$ (C_ε closed and G_ε open) such that $m(G_\varepsilon) - m(C_\varepsilon) \le \varepsilon$. Then Uryshon lemma implies that there exists $f_\varepsilon \in \mathcal{C}^{(0)}(\mathbb{T}^2)$ such that $f_\varepsilon(\mathbb{T}^2) \subset [0,1]$, $f_\varepsilon|_{C_\varepsilon} = 1$ and $f_\varepsilon|_{G_\varepsilon^c} = 0$. Thus $\|f_\varepsilon - \chi_A\|_1 \le m(G_\varepsilon \backslash C_\varepsilon) \le \varepsilon$.

two: a) the *existence* of two foliations on which f^+ (f^- respectively) are constant; b) some *regularity* property of such foliations.

In general the foliations will be provided by the stable and unstable manifolds (the existence of which is the content of the next chapter). A careful look at the previous proof should convince the reader that the needed regularity is a property of the type: consider two manifolds W_1^s, W_2^s and define a map $\phi : W_1^s \to W_2^s$ by $\phi(x) = W^u(x) \cap W_2^s$ (this is often called *holonomy map* or *Poincaré transformation*[10], we will use the first name), then ϕ is measurable and *absolutely continuous* that is : if $A \subset W_2^s$ has positive measure so has $\phi^{-1}A$.

3.3 About mixing

We continue our investigations with a discussion of an other dynamical proofs in which we will see the role of hyperbolicity and some basic ideas associated to it at work. The final goal will be to obtain a dynamical proof of the following.

Proposition 3.3.1 *The Arnold cat is mixing.*

We will start by proving Topological Mixing.

Definition 7 *A smooth Dynamical System is topologically mixing if for each two open sets U and V there exists an integer $n \in \mathbb{N}$ such that*

$$T^{-m}U \cap V \neq \emptyset \quad \forall m \geq n.$$

Note that the all point in the above definition is that it holds for <u>all</u> n large enough.

Remark that it suffices to have the above property for any class of sets that can be used as a basis for the topology. The most convenient choice is given by the so called "rectangles." Such sets are an extremely important tool in hyperbolic theory and we have already met them several times–although I will not insist on them in the present book–here they appear in the simplest possible form.

Definition 8 *By rectangle we mean a quadrilater (i.e. a region with boundaries consisting of four segments) with sides parallel to the stable or unstable directions.*

[10]Note that if one could define a flow along the unstable direction–and in our case it is possible–then the above map would indeed be a Poincaré map with respect to such a flow.

Proposition 3.3.2 *The Arnold cat is topologically mixing.*

PROOF. Let us consider two rectangles A and B. A first key observation is that, for each $m \in \mathbb{N}$, $T^m A$ and $T^m B$ are rectangles as well. The second key observation is that they have a very special shape: in the stable direction their size has contracted by a factor λ^m while in the unstable direction the size has expanded by the same factor. Hence, provided m is chosen large enough, $T^m A$ and $T^m B$ are very thin in the stable direction and very elongated in the unstable direction. This property of stretching and squeezing, that we are witnessing here, is the cornerstone of almost all arguments in hyperbolic theory. Of course, similar, but symmetrical, arguments hold for $T^{-m}A$ and $T^{-m}B$. We can then choose $m \in \mathbb{N}$ so large that the length of the unstable sides of $T^m B$ is larger than 2 and, at the same time, the same is true for the stable side of $T^{-m}A$. It is then a trivial geometric observation, best seen on the covering of $\mathbb{T}^2$, that $T^n A \cap T^{-n} B \neq \emptyset$, for each $n \geq m$, thus $T^{-2n}A \cap B \neq \emptyset$, which suffices to prove the topological mixing. □

The reader who starts to appreciate the spirit of the game may be unhappy about the previous proof. The problem is that we have used a bit too heavily the structure of the foliation (straight lines) and of $\mathbb{T}^2$ (the covering).

It is then quite natural to wonder if a more flexible and dynamical proof is available. Here it is.

ANOTHER PROOF OF PROPOSITION 3.3.2. Let us start by a preliminary result.

Given any rectangle A let us call A_c a rectangle of half the size and situated at its center.[11]

Lemma 3.3.3 *If $T^{-n}A_c \cap A_c \neq \emptyset$ for some $n \in \mathbb{N}$ such that $\lambda^n > 4$, then $T^{-mn}A \cap A \neq \emptyset$ for all $m \in \mathbb{N}$.*

PROOF. By construction $T^{-n}A$ intersects A completely from one unstable side to the other (see figure 3.1)

This means that $T^{-2n}A \supset T^{-n}(T^{-n}A \cap A)$, which is a very thin rectangle contained in $T^{-n}A$ and that crosses it from one unstable side to the other. Accordingly $T^{-2n}A$ will intersect A completely (from one unstable side to the other). By induction the result follows. □

[11] This may seem a silly construction but it is a rather general trick used to exploit topological mixing and we will see it again under the name of *core* of a rectangle in chapter 3.

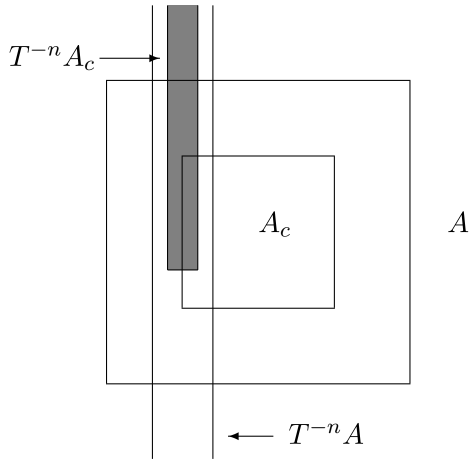

Figure 3.1: Intersection between A and $T^{-n}A$

Note that the $n \in \mathbb{N}$ required by the above statement always exists.

Next, let $A, B \subset \mathbb{T}^2$ be two rectangles and let $n_B \in \mathbb{N}$ such that Lemma 3.3.3 applies to B. We then consider the Dynamical Systems $(\mathbb{T}^2, T^{n_B}, m)$, this is ergodic as well .[12] Consequently, for each integer $i \in \{1, \dots, n_B - 1\}$ there exists $k_i \in \mathbb{N}$ such that

$$T^{-k_i n_B}(T^{-i}A_c) \cap B_c \neq \emptyset,$$

and the unstable size of A times $\lambda^{-k_i n_B}$ is smaller than one quarter of the unstable size of B. This implies immediately that

$$T^{-kn_B}(T^{-i}A) \cap B \neq \emptyset \quad \forall k \geq k_i. \tag{3.3.2}$$

In fact, $T^{-k_i n_B}(T^{-i}A)$ crosses B from one unstable side to the other and touches B_c, thus (3.3.2) can be proved by the same type of arguments used in Lemma 3.3.3.

Finally, set $k_m := \max\{k_i \mid i \in \{1, \dots, n_B\}\}$. For each $n > k_m n_B$ we can write $n = kn_B + i$ where $0 < i < n_B$, thus

$$T^{-n}A \cap B = T^{kn_B}(T^{-i}A) \cap B \neq \emptyset,$$

by (3.3.2). □

[12] This is the crucial property always needed to obtain mixing in hyperbolic systems: ergodicity of all the powers of the map.

By the same arguments, one can prove the following.

Lemma 3.3.4 *Given any stable segment W^s of length δ, and any unstable segment W^u of length $L > \lambda\delta^{-1}$, then it holds $W^s \cap W^u \neq \emptyset$.*

To start discussing the problem of mixing we need to adopt a point of view among the many possible. We will take the one that looks at the measures (see Proposition 2.7.2 and Problem 2.27) which, by now, should be rather familiar to the reader. Calling μ_0 a measure absolutely continuous with respect to Lebesgue we would like to study the asymptotic behavior of $\mu_n := {T_*}^n\mu_0$. Thanks to Proposition 2.7.2 we need to study only the weak convergence. The first observation is that such a set of measures is compact hence we can study the set of its limit points Γ (of course with the goal of showing that it consists of only one point).[13] Such a set is simply the set of limits of convergent subsequences. Since the measure μ_0 is absolutely continuous with respect to m there exists a function $h \in L^1(\mathbb{T}^2)$, $h \geq 0$, such that

$$\mu_0(f) = m(hf).$$

A lesson that we have learned from the computation in Fourier transform and from the Hopf argument is that the regularity of the functions do matter considerably and that it may be useful to consider, at first, regular functions and then obtain the wanted result by an approximation argument. Accordingly, we will restrict ourself to the case $h \in C^{(1)}(\mathbb{T}^2)$ and establish two fundamental facts.[14]

Lemma 3.3.5 *If $\bar{\mu} \in \Gamma$ then $\bar{\mu}$ is absolutely continuous with respect to Lebesgue. In addition, $\bar{h} = \frac{d\bar{\mu}}{dm} \in L^\infty(\mathbb{T}^2, m)$.*

Proof. We notice that the sequence μ_n is uniformly absolutely continuous with respect to Lebesgue, that is $\forall f \in \mathcal{C}^{(0)}(\mathbb{T}^2)$ such that $f \geq 0$

$$\mu_n(f) = \int_{\mathbb{T}^2} h \circ T^{-n} f \leq \|h\|_\infty \|f\|_1.$$

This implies $\bar{\mu}(f) \leq \|h\|_\infty m(f)$ and

$$\bar{\mu}(A) = \sup_{\substack{C \subset A \\ C = \overline{C}}} \bar{\mu}(C) = \sup_{\substack{C \subset A \\ C = \overline{C}}} \inf_{\{f \in \mathcal{C}^{(0)} \mid f > \chi_C\}} \bar{\mu}(f) \leq \|h\|_\infty m(A), \tag{3.3.3}$$

[13] Note that such accumulation points are not necessarily invariant measures, this is why we considered accumulation points of averages in section 2.4.

[14] Actually, this regularity condition on h will be needed only in Lemma 3.3.6.

where we have used (2.4.2) and (2.4.3). Clearly (3.3.3) implies the absolute continuity. Hence, by the Radon-Nikodym theorem [62], there exists $\bar{h} \in L^1(\mathbb{T}^2, m)$ such that $d\bar{\mu} = \bar{h}dm$.

Next, let $A = \{x \in \mathbb{T}^2 \mid \bar{h}(x) > \|h\|_\infty\}$. If $m(A) \neq 0$, then

$$\|h\|_\infty m(A) < \int_A \bar{h}dm = \bar{\mu}(A) \leq \|h\|_\infty m(A)$$

which is a contradiction, thus $\bar{h} \leq \|h\|_\infty$ a.e.. □

The next argument is very similar to what we have already seen in Examples 2.4.1–Strange Attractors. Let us call D^u the derivative along the unstable direction (if v^+ is the normal vector in the unstable direction then $D^u f := \langle \nabla f, v^+ \rangle$).

Lemma 3.3.6 *There exists $c > 0$: for each $f \in \mathcal{C}^{(1)}(\mathbb{T}^2)$*

$$|\mu_n(D^u f)| \leq \lambda^{-n} c \|f\|_\infty.$$

PROOF.

$$\begin{aligned}
\mu_n(D^u f) &= \int_{\mathbb{T}^2} h(D^u f) \circ T^n = \int_{\mathbb{T}^2} h\langle (\nabla f) \circ T^n, v^+ \rangle \\
&= \int_{\mathbb{T}^2} h\langle L^{-n} \nabla (f \circ T^n), v^+ \rangle = \lambda^{-n} \sum_{i=1}^{2} \int_{\mathbb{T}^2} h \partial_{x_i} (f \circ T^n) v_i^+ \\
&= -\lambda^{-n} \int_{\mathbb{T}^2} D^u h f \circ T^n,
\end{aligned}$$

where the last equality is obtained by integrating by parts with respect to both coordinates. Accordingly,

$$|\mu_n(D^u f)| \leq \lambda^{-n} \|\nabla h\|_1 \|f\|_\infty.$$

□

From the above results it follows that if $\bar{\mu} \in \Gamma$ then there exists $\bar{h} \in L^\infty(\mathbb{T}^2)$ such that, for each $f \in L^1(\mathbb{T}^2, m)$,

$$\bar{\mu}(f) = \int f \bar{h} dm$$

and for each $f \in \mathcal{C}^{(1)}(\mathbb{T}^2)$, $\bar{\mu}(D^u f) = 0$. This two facts together imply that $\bar{h}$ is constant almost everywhere.

To see this we start by a **local** argument showing that $\bar{h}$ is constant along the unstable direction. We have already done a similar argument, in Examples 2.4.1–Strange Attractors, by using Fourier series, let us see here a more measure theoretical argument to convince the reader that the global structure of $\mathbb{T}^2$ has nothing to do with the result.

Let us consider an arbitrary rectangle R of size smaller than 1/4. Consider an arbitrary $f \in \mathcal{C}^{(1)}(\mathbb{T}^2)$ with support contained in $\overset{\circ}{R}$. Then consider coordinates in R parallel to its sides (since this is achieved by rotations and rigid translations it leaves invariant the Lebesgue measure). As before, the unstable sides are horizontal. Let us call x the coordinate along the stable direction and y the one along the unstable direction. In such coordinates $R = [0, a] \times [0, b]$ (we have translates the origin at the bottom left corner of R). Given $f \in \mathcal{C}^{(1)}$, we define

$$\begin{aligned}\tilde{f}(x, y) &= f(x, y) - \frac{1}{a}\int_0^a f(\xi, y)d\xi,\\ F(x, y) &= \int_0^x \tilde{f}(\xi, y)d\xi.\end{aligned}$$

Then $F|_{\partial R} = 0$ so F can be extended to a function on $\mathbb{T}^2$ by setting $F = 0$ outside R. Note, that F is continuous and differentiable everywhere apart from the boundary ∂R where the derivative can be discontinuous. In the new coordinates D^u becomes simply the derivative with respect to x.

$$\int_{\mathbb{T}^2} \bar{h} f = \int_R \bar{h} f = \int_0^a dx \int_0^b dy \bar{h}\tilde{f} + \frac{1}{a}\int_0^b dy \int_0^a dx \bar{h}(x, y)\int_0^a d\xi f(\xi, y),$$

and, setting $\tilde{h}(y) = \frac{1}{a}\int_0^a d\xi \bar{h}(\xi, y)$, $\bar{f}(y) = \int_0^a d\xi f(\xi, y)$,

$$\int_{\mathbb{T}^2} \bar{h} f = \int_0^a dy \int_0^b dx \bar{h}\partial_x F + \int_0^b dy \tilde{h}(y)\bar{f}(y) = \int_{\mathbb{T}^2} \bar{h} D^u F + \int_0^b dy \tilde{h}(y)\bar{f}(y).$$

At this point a small obstacle appears, due to the fact that F is not $\mathcal{C}^{(1)}$. The problem is easily solved by approximating F by $\mathcal{C}^{(1)}$ functions F_ε such that $\|D^u F - D^u F_\varepsilon\|_1 \leq \varepsilon$. Then

$$\left|\int_{\mathbb{T}^2} \bar{h} D^u F\right| = \left|\int_{\mathbb{T}^2} \bar{h} D^u F - \int_{\mathbb{T}^2} \bar{h} D^u F_\varepsilon\right| \leq \|\bar{h}\|_\infty \varepsilon.$$

Hence, $\int_{\mathbb{T}^2} \bar{h} D^u F = 0$ also if the derivative is not continuous, consequently

$$\int_{\mathbb{T}^2} \bar{h} f = \int_{\mathbb{T}^2} \tilde{h} f. \tag{3.3.4}$$

By the arbitrariness of f (3.3.4) implies that $\bar{h} = \tilde{h}$ almost everywhere in $\overset{\circ}{R}$. Since R is arbitrary it follows that $\bar{h}$ is constant a.e. along the unstable direction.

A **global** argument is now needed to show that $\bar{h}$ must be constant.[15]

Proof of Proposition 3.3.1–a shortcut. Consider a line $\ell_a = \{x = a\}$. Clearly for each point $p = (a, y) \in \ell_a$ W^u_p intersects again ℓ_a at the point $(a, y+\omega_+ \mod 1)$ where $(1, \omega_+)$ is the unstable direction. Then we can consider the Dynamical Systems $(\ell_a, R_{\omega_+}, m)$, and the function $h_a = \bar{h}(a, y)$. By the previous discussion (and Fubini Theorem) it follows that, for almost every a, the function h_a is an $L^1(\ell_a, m)$ invariant function for the rotation R_{ω_+}; but we know that the irrational rotations are ergodic (see Examples 2.5.1), thus h_a =constant which implies immediately $\bar{h}$ constant. □

The above proof is simple but uses quite heavily the global properties of the foliation and of $\mathbb{T}^2$ to reduce the problem to one already studied (the irrational rotations). Clearly it is not clear how such a trick could work in more general situations. Again we would like a more flexible and dynamical argument.

Proof of Proposition 3.3.1–dynamical. We will use a strategy already employed to prove the ergodicity of irrational rotations based on the existence of density points. Morally, this allows us to consider only rectangles. By topological mixing we can ensure that any two rectangle are crossed by the same unstable line (although it is more convenient to take preimmages of the rectangle and show that they must intersect a given unstable segment), so it is not possible that $\bar{h}$ has values different in the two rectangles. This very naïve argument can be made precise as follows.

If $\bar{h}$ it is not a.e. constant then there exists two sets A and B of positive measure such that $\bar{h}|_A > \bar{h}|_B$ a.e.. Let x_A and x_B be density points, of A and B respectively, and choose two rectangle R_A and R_B of the same size, smaller than $\frac{1}{4}$, and such that

$$\begin{aligned} m(A \cap R_A) &\geq \alpha m(R_A) \\ m(B \cap R_B) &\geq \alpha m(R_B) \end{aligned} \tag{3.3.5}$$

where $\alpha \in [0, 1)$ will be chosen later.

[15]The fact that the argument is global, i.e. uses some properties of $\mathbb{T}^2$, reflects the fact that it is not as general as the Hopf argument which, instead, is of a completely local nature, as we will see better later.

Let us consider $h\circ T^n$, clearly $h\circ T^n|_{T^{-n}A} > h\circ T^n|_{T^{-n}B}$ and the relations (3.3.5) hold for $T^{-n}A$, $T^{-n}R_A$ and $T^{-n}B$, $T^{-n}R_B$.

Next, let $\hat{R}_A \subset R_A$ and $\hat{R}_B \subset R_B$ be two shorter rectangles obtained by the original ones by chopping off a quarter of the length in the stable direction from each side. Let n_0 be so large that the stable length of the rectangles time λ^{n_0} is larger than one. Now chose another rectangle R, of size $\rho \leq \frac{1}{4}$, as you please. By topological mixing it follows that there exists $n > n_0$ such that $T^{-n}\hat{R}_A \cap R \neq \emptyset$ and $T^{-n}\hat{R}_B \cap R \neq \emptyset$. In addition, by the construction of $\hat{R}_A$ and $\hat{R}_B$ and the choice of n_0, it follows that $T^{-n}R_A$ and $T^{-n}R_B$ cross $\tilde{R}$ completely from one unstable side to the other, where $\tilde{R}$ is a rectangle containing R at its center and of double size. Moreover, the same quantitative argument of Lemma 3.3.4 shows that it is possible to choose n such that the stable length of $T^{-n}R_A$, $T^{-n}R_B$ is shorter than $8\lambda^2$.

Let L_A, L_B the two rectangles contained in $T^{-n}R_A \cap \tilde{R}$ and $T^{-n}R_B \cap \tilde{R}$, respectively, that cross $\tilde{R}$ from an unstable side to the other. Chose

$$\alpha = 1 - \frac{m(L_B)}{4m(R_B)} = 1 - \frac{m(L_A)}{4m(R_A)}.$$

The all point is that, on almost all the unstable lines in $\tilde{R}$, $\bar{h}\circ T^n$ is constant, so if one of this unstable lines intersects both $T^{-n}A$ and $T^{-n}B$ we have a contradiction. Thus, it must be

$$m\left(\left[\bigcup_{x\in T^{-n}A} W^u_x \cap L_B\right] \bigcap \left[\bigcup_{x\in T^{-n}B} W^u_x \cap L_B\right]\right) = 0.$$

Fubini theorem implies

$$m\left(\bigcup_{x\in T^{-n}A} W^u_x \cap L_B\right) = m\left(\bigcup_{x\in T^{-n}A} W^u_x \cap L_A\right) \geq m(T^{-n}A \cap L_A),$$

and

$$m\left(\bigcup_{x\in T^{-n}B} W^u_x \cap L_B\right) \geq m(T^{-n}B \cap L_B),$$

This yields:

$$\begin{aligned} m(L_B) &\geq m(T^{-n}A \cap L_A) + m(T^{-n}B \cap L_B) \\ &\geq m(T^{-n}A \cap T^{-n}R_A) - m(T^{-n}R_A \backslash L_A) \\ &\quad + m(T^{-n}B \cap T^{-n}R_B) - m(T^{-n}R_B \backslash L_B) \\ &\geq 2\{\alpha m(T^{-n}R_B) - m(T^{-n}R_B) + m(L_B)\} \\ &\geq \frac{3}{2} m(L_B) \end{aligned}$$

which is a contradiction. This shows that is not possible that the unstable manifolds starting at $T^{-n}A$ systematically avoid $T^{-n}B$.

Hence, $\bar{h}$ is constant, but then $\bar{h} = \int_{\mathbb{T}^2} \bar{h} = \bar{\mu}(1) = \mu_0(1)$. We have just proved that Γ consists of only one measure: the Lebesgue measure. Thus

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} hf \circ T^n dm = \int_{\mathbb{T}^2} hdm \int_{\mathbb{T}^2} fdm,$$

for each g, $f \in C^{(1)}(\mathbb{T}^2)$. The mixing follows by the same approximation argument used in the Fourier series analyses. □

3.4 Shadowing

In this section we explore the topological complexity of the dynamics of our model systems. I have already remarked that when such a strong instability with respect to the initial condition is present it is impossible to follow exactly an orbit of the system. In fact if we compute (e.g. with a computer) the orbit of the initial point $x \in \mathbb{T}^2$, due to round off errors we do not get an orbit but rather a *pseudo-orbit*.

Definition 9 *Give an systems (X, T), X Riemannian manifold, an infinite sequence $\{x_i\}_{i\in\mathbb{Z}} \subset \mathbb{T}^2$ is called an ε-pseudo orbit if, for all $i \in \mathbb{Z}$,*

$$d(x_{i+1}, Tx_i) \leq \varepsilon.$$

Which means exactly that at each step an error of order ε is allowed.

The following result, beside being very useful, is a partial replay to the argument that it is not possible to follow orbits on a computer. Although the result is quite general, we state, and prove, it in our special context.

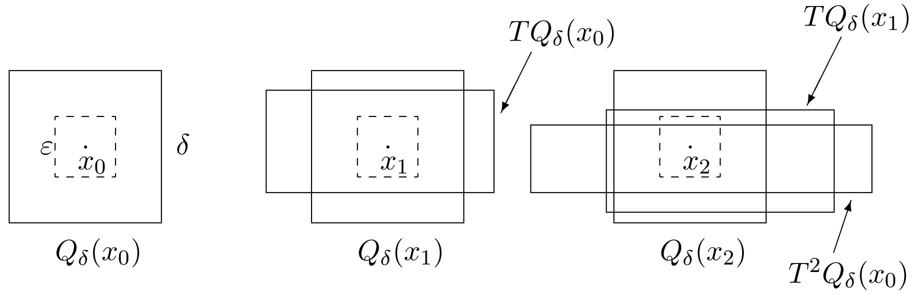

Figure 3.2: Intersection between $T^nQ_\delta(x_0)$ and $Q_\delta(x_n)$

Proposition 3.4.1 *For each $\delta > 0$ there exists and $\varepsilon > 0$ such that, if $\{x_i\}$ is a ε-pseudo-orbit for the Arnold cat, then there exists $\xi \in \mathbb{T}^2$ such that*

$$d(x_i, T^i\xi) \leq \delta \quad \forall i \in \mathbb{Z}.$$

That is, there exists an orbit that δ-shadows the pseudo-orbit, moreover such an orbit is unique.

Proof. As usual we consider rectangular (better yet: square) neighborhood of points. So, let $Q_\varepsilon(x)$ be a square of size ε centered at x with sides parallel to the stable and unstable direction, respectively.

Next, let us consider $TQ_\delta(x_0)$, since $d(Tx_0, x) \leq \varepsilon$, if $\frac{\delta}{2\lambda} + \varepsilon < \frac{\delta}{2}$ and $\frac{\lambda\delta}{2} - \varepsilon > \frac{\delta}{2}$, then $TQ_\delta(x_0)$ crosses $Q_\delta(x_1)$ completely from the stable side to the other stable side. Thus, provided we choose $\delta \geq \frac{2\lambda}{\lambda-1}\varepsilon$, we have the picture of the intersection between rectangle that we have already learned to like.

Of course the same transversal intersection takes place for each $TQ_\delta(x_i)$ and $Q_\delta(x_{i+1})$. This immediately implies that $T^nQ_\delta(x_0)$ crosses $Q_\delta(x_n)$ from one stable side to the other (see figure 3.2)

Thus $K_n = T^{-n}(T^nQ_\delta(x_0) \cap Q_\delta(x_n))$ is a sequence of nested ($K_{n+1} \subset K_n$) vertical rectangles. The unstable side of K_n is of size $\lambda^{-n}\delta$ while the stable side is of size δ.

Clearly, if $\xi \in K_n$, then

$$d(T^i\xi, x_i) < \delta \quad \forall i \in \{0, \ldots, n\}.$$

We can then consider the vertical line $K_\infty = \cap_{n\in\mathbb{N}} K_n$, by construction K_∞ consists of points whose orbit δ shadows $\{x_i\}_{i\in\mathbb{N}}$. By doing the same exact construction in the past we obtain an horizontal line $\tilde{K}_\infty$ of points that δ shadows $\{x_{-i}\}_{i\in\mathbb{N}}$. The theorem is then proven by choosing $\{\xi\} = \tilde{K}_\infty \cap K_\infty$.

the uniqueness should be obvious from the construction. In alternative the reader can prove it by contradiction. □

The above theorem is not so helpful from the measure-theoretical point of view, since it could happen that the set of trajectories that shadow pseudo-orbits is of measure zero. (*say more*)

Nevertheless, it is very useful from the topological point of view.

3.5 Markov partitions

In all the above constructions, the concept of rectangle has played a key rôle. In this section, we present a construction that is the glorification of such a point of view.

Consider the stable and unstable manifolds of zero and prolong them until they meet (of course when they meet we meet an old friend: an homoclinic intersection) few times.

Clearly in such a way we have obtained a partition of $\mathbb{T}^2$. Such a partition consists of rectangles with sides that are either stable or unstable manifolds. We call them respectively the stable and the unstable sides of the rectangles. A partition is Markov if the preimage of each unstable side of a rectangle is contained in the unstable side of a rectangle and the image of every stable side is contained in the stable side of a rectangle. The reader can check that it is possible to use the above construction to have a Markov partition with (for example) three rectangles.

Problems

3.1 Use the Diofantine properties of the stable and unstable direction to obtain better estimates of the decay of correlations. The Diofantine property refers to the following fact: if we normalize the eigenvectors in such a way that $v_\pm = (1, \omega_\pm)$, then $\omega_\pm$ are irrational numbers that are badly approximated by rationals: there exists $c \geq 0$ such that $|\omega_\pm - \frac{p}{q}| \geq \frac{c}{q^2}$ for each $p, q \in \mathbb{N}$.

3.2 Prove that the dynamical System $(\mathbb{T}^2, T^n, m)$ (where T is the Arnold cat map) is ergodic for each $n \in \mathbb{N}$. (Hint: the same proof as for $n = 1$.)

3.3 Let (X, T, μ) be a Dynamical Systems where X is a compact metric space, T is continuous, and μ charges the open sets (i.e. if $U \subset X$ is open, then $\mu(U) > 0$). Prove that for each $U \subset X$ open, there exist infinitely many $n \in \mathbb{N}$ such that $T^{-n}U \cap U \neq \emptyset$. (Hint: Poincaré Theorem.)

3.4 Let (X, T, μ) be an ergodic Dynamical Systems where X is a compact metric space, T is continuous, and μ charges the open sets. Prove that for each $U, V \subset X$ open, there exist infinitely many $n \in N$ such that $T^{-n}U \cap V \neq \emptyset$. (Hint:For each $k \in \mathbb{N}$, $A = \cup_{n \leq k} T^{-n}U$ is an invariant open set, if it does not intersect V, then $m(A) < 1$, thus, by ergodicity, $m(A) = 0$ which implies $U = \emptyset$.)

3.5 Prove Lemma 3.3.4. (Hint: As in the proof of Topologically mixing consider $T^{-n}W^s$, T^nW^u and chose n so large that $\lambda\delta > 2$ while the length L of W^u must satisfy $\lambda^{-n}L > 2$.)

3.6 Show that for each $x \in \mathbb{T}^2$ the global unstable manifold $W^u(x)$ is dense in $\mathbb{T}^2$. (Hint: *An algebraic proof*–Let us normalize $v_+ = (1, \omega)$, then ω is irrational. Clearly $W^u(x) = \{x + tv_+ \bmod 1\}_{t\in\mathbb{R}}$. Consider a point $y = (y_1, y_2)$ and chose $t_0 = y_1 - x_1$, then, for each $n \in \mathbb{Z}$, $x + (t_0 + n)v_1^+ \bmod 1 = (y_1, R_\omega^n \xi \bmod 1)$ where $\xi = x_2 + (y_1 - x_1)\omega \bmod 1$. Now, we know that R_ω has dense orbits (see Examples 2.5.1–Rotations), thus the result.
A dynamical proof–It follows Lemma 3.3.4 plus the fact that $T^{-n}W^u$ is shorter than W^u.)

3.7 Consider the global unstable foliation $\{W^u(x)\}$ and choose an interval of length (in the horizontal direction) one from each fiber.[16] Let K be the set obtained by the union of all such segments. Prove that K it is not measurable. (Hint: Define $R : \mathbb{T}^2 \to \mathbb{T}^2$ by $R(x, y) = (x, R_\omega y)$. Then, remember Problem 2.13.)

3.8 Let $W^u(x)$, $W^s(x) \subset U \subset \mathbb{R}^2$, $U = \overset{\circ}{U}$ and $\bar{U}$ compact, smooth manifolds ($\mathcal{C}^{(1)}$ curves) such that, the $\{W^{u(s)}(x)\}_{x\in U}$ are pairwise disjoint, $\partial W^{u(s)}(x) \subset \partial U$, if $z \in W^u(x) \cap W^s(y)$, then the angle between $W^u(x)$ and $W^s(y)$ at z is larger than some $\theta > 0$. In addition, assume that, calling $v^{u(s)}(x)$ the unit tangent vector to $W^{u(s)}(x)$ at x, $v^{u(s)} \in \mathcal{C}^{(1)}$. We will call such two foliation " $\mathcal{C}^{(1)}$ uniformly transversal foliations." Show that to each such a foliation it is associated a change of variable

[16] The Axiom of choice again.

(a diffeomorphism $\Psi : U \to U$) and that to each change of variables is associated such a foliation. (Hint: ...)

3.9 Consider two $\mathcal{C}^{(1)}$ uniformly transversal foliations. Prove that if $f \in L^\infty$ is constant along almost every fiber of the two foliations, then it is constant almost everywhere. (Hint: Do the argument locally and change variables so that the foliations becomes straight.)

3.10 Consider the Bernoulli measures μ_p^B defined on Σ_2^+ (the one sided sequences with two symbols) by choosing $p_0 = p$ and $p_1 = 1 - p$ (see Examples 2.1.1–Bernoulli shift). Show that, if $p \neq p'$ then μ_p^B and $\mu_{p'}^B$ are mutually singular. (Hint: All the dynamical systems $(\Sigma_2^+, \tau, \mu_p^B)$ are ergodic.)

3.11 Let μ_p be the measure on $[0,1]$ obtained from μ_p^B by the binary representation of the real numbers, let

$$F_p(x) := \mu_p([0,x]).$$

Show that, for each $p \in (0,1)$, $F_p : [0,1] \to [0,1]$ is one one, onto, continuous. In addition, show that there exists $c \in \mathbb{R}^+$ such that, for each $p, q \in [\frac{1}{4}, \frac{3}{4}]$, holds

$$|F_p(x) - F_q(x)| \leq c|p - q|.$$

(Hint: Note that the cylinder correspond to intervals with end points made of binary rationals. It is then immediately clear that all the measures μ_p give positive measures to the open sets. To prove the last inequality prove the representation

$$F_p(x) = \sum_{n=0}^{\infty} \sigma_n \prod_{i=0}^{n} p^{\sigma_i}(1-p)^{1-\sigma_i}$$

where σ is the binary representation of x.)

3.12 Construct $\phi : [0,1] \to [0,1]$, invertible and continuous, such that there exists $A \subset [0,1]$ with $m(A) = 0$ while $m(\phi(A)) = 1$. (Hint: Any of the above F_p will do.)

3.13 Construct a continuous foliation Ψ in $[0,1]^2$made of $\mathcal{C}^\infty$ leaves (that is Ψ is a isomomorphism of $[0,1]^2$ and $\Psi(\cdot, y) \in \mathcal{C}^\infty$). In addition, the foliation must be made of straight lines in $\{(x,y) \in [0,1]^2 \mid x \in$

$[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]\}$ but is should not be absolutely continuous in the region $\{(x, y) \in [0, 1]^2 \mid x \in [\frac{1}{4}, \frac{3}{4}]\}$. (Hint: Let $\varphi \in \mathcal{C}^{(\infty)}(\mathbb{R})$, $\varphi(\mathbb{R}) = [0, 1]$, $\varphi(x) = 0$ for $x < 0$ and $\varphi(x) = 1$ for $x > \frac{1}{2}$. Then, using and approriate ϕ, define

$$\Psi(x, y) = \begin{cases} (x, y) & \text{if} \quad x \in [0, \frac{1}{4}] \\ (x, [1 - \varphi(x - \frac{1}{4})]y + \varphi(x - \frac{1}{4})\phi(y)) & \text{if} \quad x \in [\frac{1}{4}, \frac{3}{4}] \\ (x, \phi(y)) & \text{if} \quad x \in [\frac{3}{4}, 1]. \end{cases}$$

Clearly the leaves $\Psi(\cdot, y)$ are $\mathcal{C}^{(\infty)}$, yet the foliation it is not absolutely continuous.)

3.14 Find two $\mathcal{C}^{(0)}$ uniformly transversal foliations in $[0, 1]^2$, with $\mathcal{C}^{(\infty)}$ leaves, such that the Hopf argument does not apply. (Hint: Call Ψ_p, $p \in [\frac{1}{4}, \frac{3}{4}]$ the foliation constructed in the Problem 13 starting from the function F_p defined in the Problem 11. Choose a sequence p_n converging to one quarter, e.g. $p_n = \frac{1}{4} + \frac{1}{4^n}$, then let $x_n = \frac{1}{2} - \frac{1}{2n}$. Finally define the foliation

$$\Psi(x, y) = \begin{cases} \Psi_{p_n}(x_n + (x_{n+1} - x_n)x, y) & \text{for } x \in [x_n, x_{n+1}] \\ (x, F_{\frac{1}{4}}(y)) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

Further define the function $g : [0, 1] \to [0, 1]$ to be one on a set of full measure for $\mu_{\frac{1}{4}}$ and of zero measure for μ_{p_n} and zero otherwise. The functions f^+, f^- defined by

$$f^-(x, y) = \begin{cases} 0 & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$f^+(x, \Psi(x, y)) = g(x),$$

are then constant on the vertical and the Ψ foliation respectively. Moreover they clearly are equal Lebesgue almost everywhere, nevertheless they are certainly not constant.)

3.15 Show (first without using Markov Partitions and then by using Markov partitions) that the Arnold cat has at least e^{cn} periodic orbits of period n, for some $c > 0$. (Hint: If we have a rectangle R of size ε, then $T^{-n}R \cap R \neq \emptyset$ for some $n \leq c \ln \varepsilon^{-1}$. Then, if $x \in T^{-n}R \cap R$ we consider the pseudo orbit $x_k = T^i x$ where $i = k \mod n$. Then Proposition 3.4.1 implies the existence of a periodic orbit in an ε-neighborhood

R_ε of R. On the other hand the boxed $T^{-k}R_\varepsilon$, $k \in \{0, \ldots, n\}$ invade a part of $\mathbb{T}^2$ of measure $c\varepsilon^2 \ln \varepsilon^{-1}$. The argument is then concluded taking boxes in the remaining space and continuing until all the available space is exhausted. On the other hand, if one takes in account Markov partions, then the number of periodic orbits is given–appart from the non-invertibility of the coding–by the number of periodic simbolic sequences of period n.)

Chapter 4

Hyperbolic Systems–general facts

This chapter is designed to give a general idea of hyperbolic theory. Since such a theory covers a rather vast landscape, and it contains very technical results our exposition is bound to be quite sketchy.

4.1 Hyperbolicity

Our goal in this section is to introduce and discuss a class of systems for which we can hope to investigate the properties introduced in the previous section. As we have seen, the chief property that we used in the study of the Arnold cat were the expanding and contracting properties of the map. These are generalized in the following definition.

Definition 10 *By Hyperbolic System (with discrete time) we mean a Dynamical System* (X, f, μ) *such that* X *is a smooth compact Riemannian manifold (possibly with boundary),* f *is* μ*-almost everywhere differentiable and there exists two measurable families of invariant*[1] *subspaces* $E^u(x)$, $E^s(x) \in \mathcal{T}_x X$ *almost surely transversal,*[2] *and measurable functions* $\nu(x) > 1$, $c(x) > 0$ *such that for almost all* $x \in X$

$$\begin{aligned} \|D_x f^n v\| &\geq c(x)^{-1} \nu(x)^n \|v\| \quad \forall v \in E^u(x) \\ \|D_x f^n v\| &\leq c(x) \nu(x)^{-n} \|v\| \quad \forall v \in E^s(x). \end{aligned}$$

[1]That is $D_x f E^{s(u)}(x) = E^{s(u)}(fx)$.

[2]That is, $E^u(x) \cap E^s(x) = \{0\}$ and $E^u(x) \oplus E^s(x) = \mathcal{T}_x X$ a.e.

If the functions c, ν can be chosen constant and the distributions are transversal at each point, then the system is called Uniformly Hyperbolic*. In addition, if f is a diffeomorphism and E^u, E^s vary with continuity, then the system is called* Anosov *(or sometimes C or U systems).*

The condition in Definition 10 is essentially equivalent to saying that two very close initial conditions almost certainly will grow apart at an exponential rate. This corresponds to a strong instability with respect to the initial conditions and characterizes the sense in which the dynamics of hyperbolic systems is a very complex one. Such complex behaviour has captured the popular fantasy under the ambiguous name of chaos.

4.1.1 Examples

Rotations

Clearly the rotations are not hyperbolic since $Df = 1$.

Dilation

One can easily see that such a system is expanding, hence $E^u = \mathbb{R}$ and $E^s = \emptyset$.

Arnold cat

We have seen it in detail in the previous chapter.

Baker

In this case one direction is expanding and one is contracting, $\dim E^u = \dim E^s = 1$

A more general notion of hyperbolicity is the one of *hyperbolic set*.

Definition 11 *Given a diffeomorphism f of a manifold X, we say that $\Lambda \subset X$ is hyperbolic if Λ is compact, $f(\Lambda) = \Lambda$ and there exists two measurable families of invariant subspaces $E^u(x)$, $E^s(x) \in \mathcal{T}_x X$ transversal at each point and measurable functions $\nu(x) > 1$, $c(x) > 0$ such that for all $x \in \Lambda$*

$$\|D_x f^n v\| \geq c(x)^{-1} \nu(x)^n \|v\| \quad \forall v \in E^u(x)$$
$$\|D_x f^n v\| \leq c(x) \nu(x)^{-n} \|v\| \quad \forall v \in E^s(x).$$

If the constants c, ν can be chosen independently of $x \in \Lambda$ then Λ is called Uniformly Hyperbolic*.*

4.1.2 Examples

Smale Horseshoe

In this case the set Λ is the one constructed in Examples 2.4.1 and $\dim E^s = \dim E^u = 1$.

Forced pendulum

Same situations as for the horseshoe, see Examples 2.8.1.

Definition 10 it is not particularly helpful in concrete cases since, in general, it is not clear how to verify if a systems is hyperbolic or not.

4.2 Lyapunov exponents and invariant distributions

We start by a different and very helpful characterization of hyperbolicity obtained by introducing the so called Lyapunov Exponents (LE).

Definition 12 *For each $x \in X$, $v \in \mathcal{T}_x X$ we define*

$$\lambda(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\|.$$

If $\lambda(x, v)$ exists it is called "Lyapunov exponent" (LE).

It is interesting to notice that $\lambda(f(x), D_x f v) = \lambda(x, v)$ (see Problem 4.1). Moreover, it should be clear that, if the system is ergodic and the map invertible, then $\lambda(x, v)$, if it exists, can assume only finitely many values (see Problem 4.3).

The existence and properties of the LE have been intensively studied and have given rise to a multitude of results. Here, we content ourselves with the following theorem, which is by far not the most general version, but it suffices for our needs. See [76] for a more extensive presentation of the theorem and its proof. I will provide some ideas related to the proof at the end of the section.

Theorem 4.2.1 (Oseledets [57]) *Let (X, μ) be a probability space and $f : X \to X$ a measure-preserving transformation. Let $L : X \to GL(d, \mathbb{R})$ be a measurable mapping from X to the invertible $n \times n$ matrices such that $\ln \|L(\cdot)^{\pm 1}\| \in L^1(X, \mu)$. Then for μ-almost all $x \in X$ there are subspaces*

$\{0\} = V_x^0 \subset V_x^1 \subset \cdots \subset V_x^d = \mathbb{R}^d$ and numbers $\lambda_1(x) \le \cdots \le \lambda^d(x)$ such that, for all $i \in \{1, \ldots, d\}$,[3]

$$\lim_{n\to\infty} \frac{1}{n} \ln \|L(f^{n-1}(x)) \cdots L(f(x))L(x)v\| = \lambda_i(x)$$

if $v \in V_x^i \setminus V_x^{i-1}$. In addition, if μ is ergodic, the spaces and exponents are almost surely constant and denoting by d_k the difference between the dimension of $\mathbb{V}^k$ and $\mathbb{V}^{k-1}$ we have

$$\sum_k d_k \lambda_k = \int \log|\det(L(x))|\mu(dx).$$

The above theorem is taylored to study *cocylces*, that is the dynamical systems $(X \times \mathbb{R}^d, F)$ where $F(x, v) = (f(x), L(x)v)$. Indeed one can easily check that $F^n(x, v) = (f^n(x), L(f^{n-1}(x)) \cdots L(x)v))$.
Given a measurable dynamical system (X, f, μ), where X is a d dimensional Riemannian manifold (possibly with boundary) and T is almost surely differentiable we have the natural cocycle $(X \times \mathbb{R}^d, F)$ where $F(x, v) = (f(x), D_x f(v))$. Note that, by the chain rule, $F^n(x, v) = (f^n(x), D_x f^n v)$. So, the Lyapunov exponents of f are exactly the numbers given by Oseledets' Theorem for the associated cocycle. The connection between Lyapunov exponents and hyperbolicity is illustrated by the following.

Theorem 4.2.2 *A system $(X,\ f,\ \mu,)$, where X is a Riemannian manifold and f is a diffeomorphism. Then, f is hyperbolic iff for almost all $x \in X$*

$$\lambda(x,\ v) \ne 0 \quad \forall v \in \mathcal{T}_x X, \quad v \ne 0.$$

PROOF. Clearly, if the system is hyperbolic, then all the LE are different from zero. The other implication is almost as trivial. Define $E^s(x) = \{v \in \mathcal{T}_x \mid \lambda(x, v) < 0\}$; then consider the Dynamical system (X, f^{-1}, μ) and its LE $\lambda^-(x, v)$ and define $E^u(x) = \{v \in \mathcal{T}_x \mid \lambda^- < 0\}$. Next, let

$$\rho(x) = \sup\{\lambda(x, v),\ \lambda_-(x, w) \mid v \in E^s(x),\ w \in E^u(x)\}$$

clearly $\rho(x) < 0$ a.e.. Then setting $\nu(x) = e^{-\rho(x)/2}$ and

$$c(x) = \sup_n\{\nu(x)^n \|D_x f^n v\|,\ \nu(x)^n \|D_x f^{-n} w\| \mid v \in E^s(x);\ w \in E^u(x)\}_{n\in\mathbb{N}},$$

which is almost surely finite by construction, hence proving the theorem. □

[3] Note that the $\mathbb{V}_i$ and the λ_i are not necessarily distinct.

To conclude the section, let me provide a few ideas related to the proof of Theorem 4.2.1 to give a feeling of what is involved. Note that, thanks to the ergodic decomposition, we can assume w.l.o.g. that μ is ergodic. Let us define

$$\bar{\lambda}(x,v) = \limsup_{n\to\infty} \frac{1}{n} \ln \|L(f^{n-1}(x)) \cdots L(f(x))L(x)v\|. \tag{4.2.1}$$

Note that, for each $\alpha \in \mathbb{R}$,

$$\bar{\lambda}(x,\alpha v) = \limsup_{n\to\infty} \frac{1}{n} \ln \left\{\|L(f^{n-1}(x)) \cdots L(f(x))L(x)v\||\alpha|\right\} = \lambda(x,v).$$

In addition,

$$\begin{aligned}\bar{\lambda}(x,v) &\leq \limsup_{n\to\infty} \frac{1}{n} \ln \|L(f^{n-1}(x)) \cdots L(f(x))L(x)\| \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \|L(f^{k-1}(x))\| = \int_X \ln \|L(x)\| \mu(dx) = \lambda_+(x).\end{aligned}$$

where the last inequality follows by Birkhoff's theorem. In addition, for $\|v\| = 1$,

$$\begin{aligned}0 &= \ln \|L(x)^{-1}L(f(x))^{-1} \cdots L(f^{n-1}(x))^{-1}L(f^{n-1}(x)) \cdots L(x)v\| \\ &\leq \ln \|L(x)^{-1}L(f(x))^{-1} \cdots L(f^{n-1}(x))^{-1}\| + \ln \|L(f^{n-1}(x)) \cdots L(x)v\|.\end{aligned}$$

Which, arguing as before, yields

$$\bar{\lambda}(x,v) \geq \int_X \ln \|L(x)^{-1}\|^{-1} \mu(dx).$$

Hence, the numbers $\bar{\lambda}(x,v)$ are almost surely bounded. Next, for $v, w \in \mathbb{R}^d$ let

$$\begin{aligned}v_n &= \|L(f^{n-1}(x)) \cdots L(f(x))L(x)(v+w)\| \\ w_n &= \|L(f^{n-1}(x)) \cdots L(f(x))L(x)(v+w)\|\end{aligned}$$

then

$$\bar{\lambda}(x,v+w) \leq \limsup_{n\to\infty} \frac{1}{n} \ln \left[2\max\{v_n,w_n\}\right] \leq \max\{\bar{\lambda}(x,v), \bar{\lambda}(x,w)\}.$$

Finally, by definition,

$$\bar{\lambda}(x,L(x)v) = \bar{\lambda}(f(x),v).$$

It follows that if we define $\mathbb{V}(x,\alpha) = \{v \in \mathbb{R}^d \ : \ \bar\lambda(x,v) \leq \alpha\}$, then the $\mathbb{V}(x,\alpha)$ are vector spaces and $L(x)\mathbb{V}(x,\alpha) = \mathbb{V}(f(x),\alpha)$. A technical issue that I will not discuss is the proof that the functions $\mathbb{V}(\cdot,\alpha)$ are measurable;[4] see [76] for a proof. Note that the dimension $d(x,\alpha)$ of $\mathbb{V}(x,\alpha)$ is an invariant function with respect to x, hence it must be constant almost surely. Additionally, it is clearly increasing with α. It follows that there must be at most d values $\{\alpha_i\}$ of α for which the dimension of the space changes. We have so obtained the flag $\{0\} \subset \mathbb{V}_1 \subset \cdots \subset \mathbb{V}_d = \mathbb{R}^d$, where some of the spaces can be repeated.

It remains to show that the limsup in (4.2.1) is indeed a limit. I refer to [76] for a complete proof; here, I provide an outline of the idea, looking at the maximal exponent. Assume, for simplicity, that $\ln \|L\| \in L^\infty$.

Let

$$\begin{aligned} a_n(x) &= \ln \sup_{\{\|v\|=1\}} \|L(f^{n-1}(x)) \cdots L(f(x))L(x)v\| \\ \lambda_+(x) &= \limsup_{n\to\infty} \frac{1}{n} a_n(x). \end{aligned}$$

We have that, for all $n, m \in \mathbb{N}$, $a_{n+m}(x) \leq a_n(f^m(x)) + a_m(x)$. This implies that

$$\lambda_+(x) \leq \lambda_+(f(x)) + \limsup_{n\to\infty} \frac{1}{n} \ln \|L(x)\| = \lambda_+(f(x)).$$

Accordingly, by the invariance of the measure,

$$0 \leq \int [\lambda_+(f(x)) - \lambda_+(x)]\mu(dx) = 0.$$

Which implies that $\lambda_+(x) = \lambda_+(f(x))$ almost surely, and hence λ_+ almost surely constant, by ergodicity.

....

For each $\varepsilon > 0$, thre exists $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{p} a_p(x) \leq \lambda_+(x) + \varepsilon$ for all $p \geq n_\varepsilon$. We can then write

$$\frac{1}{km} a_{km}(x) \leq \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{m} a_m(f^{jm}(x)).$$

[4] Here we see $\mathbb{V}(\cdot,\alpha)$ as elements of the union of Grassmanian, which is a topological space. If you do not want to be very fancy, given $\mathbb{V}_1, \mathbb{V}_2 \subset \mathbb{R}^d$, let their distance be the Hausdorff distance between their intersection with the unit sphere. This gives a topology, and we consider the associated Borel σ-algebra.

Taking the limsup for $k \to \infty$ Birkhoff's theorem implies that, μ-almost surely,

$$\lambda_+(x) \leq \int \frac{1}{m} a_m(y)\mu(dy).$$

If we set $\lambda_-(x) = \liminf_{n\to\infty} \frac{1}{m} a_m(x)$ and we take the linsup of the above expression we obtain

$$\begin{aligned}\lambda_+(x) &\leq \limsup_{n\to\infty} \int \frac{1}{m} a_m(y)\mu(dy) = \liminf_{n\to\infty} \int \frac{1}{m} a_m(y)\mu(dy)\\ &\leq \int \lambda_-(y)\mu(dy).\end{aligned}$$

where, in the first equality, we have used the fact that the sequence $\bar{a}_m := \int a_m(y)\mu(dy)$ is subattive, hence $\frac{1}{m}\bar{a}_m$ has a limit. Integrating, we have

$$\int [\lambda_+(y) - \lambda_-(y)]\, \mu(dy) \leq 0$$

which implies $\lambda_+ - \lambda_-$ almost surely. We can now take the limsup in m and obtain $\lambda_+(x) \leq \int \lambda_+(x)\mu(dx)$, which implies $\lambda_+(x) = \int \lambda_+(x)\mu(dx) =: \lambda_+$ almost surely.

4.3 Comments on the non-smooth case

The results of the last section can be applied to non-smooth systems, however to develop a useful theory the singularities of the system cannot be arbitrary. As we will see in the following, systems that are quite natural both from the mathematical point of view and from the physical one are not smooth–typically they have discontinuities. In this section we will discuss a class of systems called *smooth systems with singularities*. Although the theory of such systems has been done in great generality, here we will give a restrictive definition, just sufficient for our later purposes. See the notes at the end of the chapter for information on more general settings.

Definition 13 *By Smooth Dynamical System with singularities we mean a Dynamical Systems (X, T, μ), where*

- *X is the union of finitely many compact pieces X_i of $\mathbb{R}^n$, ∂X_i is the union of finitely many $n-1$ dimensional smooth manifolds.*

- *T is smooth outside a compact set $\mathcal{S}$. The singularity set $\mathcal{S}$ is the finite union of smooth $n-1$ dimensional manifolds with boundary $\mathcal{S}_i$, $\mathcal{S}_i \cap \mathcal{S}_j \not\subset \partial\mathcal{S}_i \cap \partial\mathcal{S}_j$ implies $i = j$. In addition, the boundary $\partial\mathcal{S}_i$ is the finite union of smooth $n-2$ dimensional manifolds.*

- *There exists $c_1, c_2 > 0$ such that*

$$\|D_x T\| + \|D_x^2 T\| \leq c_1 \, dist\,(x, \mathcal{S})^{-c_2}.$$

- *The measure μ is absolutely continuous with respect to Lebesgue.*

Remark 4.3.1 *Note that the fact that (X, T, μ) is a Smooth Dynamical System with singularities does not implies immediately that the same holds for (X, T^k, μ). The problem is that the map T can be very wild near the set $\mathcal{S}$, so it is not clear that the singularity set of T^k will satisfy our requirements. Nevertheless, in the examples we will consider, all the Dynamical System (X, T^k, μ) will always be Smooth Dynamical System with singularities.*

Remark 4.3.2 *We will call a smooth Dynamical System with singularities invertible if T^{-1} is densely defined and (X, T^{-1}, μ) is itself a smooth Dynamical System with singularities.*

Note that the above conditions imply the applicability of Oseledets Theorem.

4.3.1 Examples

Backer map

It is easy to check that the Backer map is a Smooth Dynamical System with singularities.

Discontinuous Arnold cat

If we consider $(\mathbb{R}^2, L, m)$ where

$$L \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a & 1 + a^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{4.3.2}$$

with $a \notin \mathbb{Z}$, then it is not possible to project the system down to a torus preserving the continuity of the map. Yet, we can construct a discontinuous version of the Arnold cat.

Consider $\mathcal{M}_+ = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x + ay < 1;\ 0 \le y < 1\}$ and $\mathcal{M}_- = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x < 1;\ 0 \le ax - y < 1\}$. It is easy to see that, if Π is the projection from the universal cover $\mathbb{R}^2$ to the torus $\mathbb{T}^2$ ($\Pi\xi = \xi \mod 1$), then Π, restricted to $\mathcal{M}_\pm$, is one-one and onto. Moreover, $L\mathcal{M}_+ = \mathcal{M}_-$. This means that we can define $T : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T = \Pi L(\Pi|_{\mathcal{M}_+})^{-1}.$$

Of course T is discontinuous on $\mathcal{S}_+ := \partial\mathcal{M}_+$ and T^{-1} is discontinuous on $\mathcal{S}_- := \partial\mathcal{M}_-$. In addition, the Lebesgue measure is invariant and the map is hyperbolic since $DT = L$.

The question arises if there exists stable and unstable manifolds. A moment of thought shows that this is equivalent to the following question: there exist segments in the stable (unstable) direction such that their images in the future (past) never meet the discontinuity set $\mathcal{S}_+$ ($\mathcal{S}_-$)?

Let us analyze the unstable manifolds. Call $\mathcal{S}_\delta$ the δ neighborhood of $\mathcal{S}_+$. Consider a segment J centered at x, in the unstable direction, and suppose that $T^{-n}J \cap \mathcal{S}_+ \neq \emptyset$, then J cannot be the unstable manifold since its points do not have the same asymptotic trajectory in the past. Let $\lambda > 1$ the eigenvalue of L, then $T^{-n}J$ has total length $\lambda^{-n}|J|$, so the trajectory of x can be fairly close to $\mathcal{S}_+$ without having a problem. This discussion leads naturally to considering the set

$$G_\delta = \{x \in \mathbb{T}^2 \mid \text{dist } (T^{-n}x, \mathcal{S}_+) \ge \lambda^{-n}\delta\}.$$

On the one hand, it is clear that if $x \in G_\delta$, a segment in the unstable direction of size δ is indeed an unstable manifold. On the other hand, $m(G_\delta) \le c\delta$. Thus almost all the points do have an unstable manifold of some positive size. This it is encouraging, yet it is clearly not sufficient to perform the Hopf argument. For the time being it suffices to notice that what we have seen so far implies that the discontinuous Arnold cat has, at most, countably many ergodic components.

4.4 Flows

All what we have described so far has a rather straightforward generalization in the case of flows, yet some natural changes are called for.

To appreciate the problem let us consider a flow, on a compact Riemannian manifold, generated by a smooth non-zero vector field V. By definition $\frac{d}{dt}\phi^t|_{t=0} = V(x)$ and $d\phi^t V(x) = V(\phi^t x)$, thus $\lambda(x, V(x)) = 0$. This is a rather general fact: the Lyapunov exponent in a flow, with a nonvanishing vector field, is zero in the flow direction. The only relevant exception is constituted by hyperbolic fixed points (think of the unstable equilibrium point of the pendulum) that, in the previous example, was ruled out by the assumption that the vector field be non zero. We will consider only such case.

Consequently a flow is hyperbolic if the tangent space is split in three transversal subspaces E^s, E^u, E^0, where E^0 is the flow direction and corresponds to a zero Lyapunov exponents.

Oseledec Theorem (Theorem 4.2.1) holds unchanged with the L^1 condition on the cocycle obviously replaced by

$$\int_X \| \log d\phi^t \| d\mu < \infty.$$

For a smooth flow coming from a non vanishing vector field Theorem 4.2.2 holds unchanged as well.

4.4.1 Examples

Smooth flows with collisions

Let M be a smooth manifold with piecewise smooth boundary ∂M. We assume that the manifold M is equipped with a symplectic structure ω.[5] Given a smooth function H on M with *non vanishing* differential we obtain the non vanishing Hamiltonian vector field $F = \nabla_\omega H$ on M by $\omega(\nabla_\omega H, v) = dH(v)$. The vector field F is tangent to the level sets of the Hamiltonian $M^c = \{z \in M | H(z) = c\}$.

We distinguish in the boundary ∂M the regular part, ∂M_r, consisting of the points which do not belong to more than one smooth piece of the boundary and where the vector field F is transversal to the boundary. The regular part of the boundary is further split into "outgoing" part, ∂M_-, where the vector field F points outside the manifold M and the "incoming" part, ∂M_+, where the vector field is directed inside the manifold. Suppose that additionally we have a piecewise smooth mapping $\Gamma : \partial M_- \to \partial M_+$, called the collision map. We assume that the mapping Γ preserves the Hamiltonian, $H \circ \Gamma = H$, and so it can be restricted to each level set of the Hamiltonian.

We assume that all the integral curves of the vector field F that end (or begin) in the singular part of the boundary lie in a codimension 1 submanifold of M.

[5] That is a non–degenerate closed antisymmetric two form.

We can now define a flow $\Psi^t : M \to M$, called a flow with collisions, which is a concatenation of the continuous time dynamics Φ^t given by the vector field F, and the collision map Γ. More precisely a trajectory of the flow with collisions, $\Psi^t(x),\ x \in M$, coincides with the trajectory of the flow Φ^t until it gets to the boundary of M at time $t_c(x)$, the collision time. If the point on the boundary lies in the singular part then the flow is not defined for times $t > t_c(x)$ (the trajectory "dies" there). Otherwise the trajectory is continued at the point $\Gamma(\Psi^{t_c}x)$ until the next collision time, i.e., for $0 \le t \le t_c\left(\Gamma(\Psi^{t_c(x)}x)\right)$

$$\Psi^{t_c+t}x = \Phi^t\Gamma\Psi^{t_c}x.$$

We define a flow with collisions to be symplectic, if for the collision map Γ restricted to any level set M^c of the Hamiltonian we have

$$\Gamma^*\omega = \omega.$$

More explicitly we assume that for every vectors ξ and η from the tangent space $T_z\partial M^c$ to the boundary of the level set M^c we have

$$\omega(D_z\Gamma\xi, D_z\Gamma\eta) = \omega(\xi, \eta).$$

We restrict the flow with collisions to one level set M^c of the Hamiltonian and we denote the resulting flow by Ψ_c^t. This flow is very likely to be badly discontinuous but we can expect that for a fixed time t the mapping Ψ_c^t is piecewise smooth, so that the derivative $D\Psi_c^t$ is well defined except for a finite union of codimension one submanifolds of M^c. We will consider only such cases.

The symplectic volume $\wedge^d\omega$ is clearly invariant for the flow, so will be the measure μ_c obtained by restricting the symplectic volume to the manifold M^c. Clearly for such an invariant measure all the trajectories that begin (or end) in the singular part of the boundary have measure zero. With respect to the measure μ_c the flow Ψ_c^t is a measurable flow in the sense of Definition 2 and we obtain a measurable derivative cocycle $D\Psi_c^t : T_xM^c \to T_{\Psi_c^t x}M^c$. We can define Lyapunov exponents of the flow Ψ_c^t with respect to the measure μ_c, if we assume that[6]

$$\begin{aligned} &\int_{M^c} \log_+ \|D_x\Psi_c^t\| d\mu_c(x) < +\infty \\ &\int_{\partial M_-^c} \log_+ \|D_y\Gamma\| d\mu_{cb}(y) < +\infty \end{aligned} \tag{4.4.3}$$

(cf.[57]).

[6] Here μ_{cb} is the restriction of the volume to ∂M_-^c.

Problems

4.1 Prove that $\lambda(Tx, D_xTv) = \lambda(x, v)$.

4.2 Prove that $\lambda(x, v+w) \leq \max\{\lambda(x,v), \lambda(x,w)\}$ and $\lambda(x, \alpha v) = \lambda(x, v)$ for each $\alpha \in \mathbb{R}$, if they all exist. (Hint: Just apply the definition of LE and note that

$$\lambda(x, v+w) \leq \lim_{n\to\infty} \max\{\frac{1}{n}\log\|D_xT^n v\|, \frac{1}{n}\log\|D_xT^n w\|\}. \quad)$$

4.3 Assuming only that the LE are well defined a.e., prove that, if (X, T, μ) is ergodic, X is a d dimensional manifold and T a diffeomorphism, then there exists d numbers $\{\lambda_i\}$ such that the Lyapunov exponents $\lambda(x, v) \in \{\lambda_i\}$ a.e.. (Hint: For each $\alpha \in \mathbb{R}$ define $V_\alpha(x) := \{v \in \mathcal{T}_xX \mid \lambda(x, v) \leq \alpha\}$. By Problem 4.2 $V_\alpha(x)$ is a linear vector space and, by Problem 4.1 the distribution V_α is invariant. Then $d_\alpha(x) := \dim V_\alpha(x)$ is an invariant function, thus a.e. constant for each α. In addition, d_α is an increasing function of α and can assume only the values $\{0, \dots, d\}$. Thus there are at most $s \leq d$ $\{\alpha_j\}$ where d_α jumps. But this means that the LE are discrete. In fact, let $v \in V_\alpha(x)\backslash V_\beta(x)$, $\alpha > \beta$, then for each $w \in \text{span}\{v, V_\beta(x)\}$ it is easy to compute that $\lambda(x, w) = \lambda(x, v) > \beta$, which means: the LE is constant over $V_\alpha(x)$ apart for lower dimensional subspaces. In addition, we have a flag of subspaces $\{V_i\}_{i=0}^s$, $s \leq d$, such that $V_\alpha \in \{V_i\}_{i=0}^s$ for each $\alpha \in \mathbb{R}$. Hence, if $V_\alpha \supset V_i$ but $V_\alpha \not\supset V_{i+1}$ it must be $V_\alpha = V_i$, thus if $v \in V_\alpha$ but $v \notin V_{i-1}$ $\lambda(x, v) = \alpha_i$ where $\alpha_i = \inf\{\alpha \in \mathbb{R} \mid V_\alpha \supset V_i\}$.)

4.4 Show that, if T is invertible, $\{\lambda_i(x)\}$ is equal a.e. to $\{-\lambda_i^-(x)\}$ where $\{\lambda_i^-(x)\}$ are the LE of (X, T^{-1}, μ).

4.5 Show that

$$\lim_{n\to\infty} \frac{1}{n}\log|\det(D_xT^n)|$$

exists almost everywhere. (Hint: Apply BET.)

4.6 Let (X, T, μ) be a Dynamical Systems, X a compact Riemannian manifold and T a.e. differentiable. Suppose that there exists a one-dimensional distribution $E(x)$ such that $D_xTE(x) = E(Tx)$. Prove, without using Oseledets theorem, that for each $v \in E(x)$ the LE $\lambda(x, v)$ is well defined. (Hint: Let $v(x) \in E(x)$, $\|v(x)\| = 1$, then $D_xTv(x) = \alpha(x)v(Tx)$ and thus $D_xT^nv(x) = \prod_{i=1}^n \alpha(T^ix)v(T^ix)$. Then the result follows by the BET.)

4.7 Define a cocycle associated with a flow with collision, which yields all the Lyapunov exponents, but the one in the flow direction. (Hint: The derivative of the flow with collisions can also be naturally factored onto the quotient of the tangent bundle TM^c of M^c by the vector field F, which we denote by $\widehat{T}M^c$. Note that for a point $z \in \partial M^c$ the tangent to the boundary at z can be naturally identified with the quotient space. We will again denote the factor of the derivative cocycle by

$$A^t(x) : \widehat{T}_x M^c \to \widehat{T}_{\Psi_c^t x} M^c.$$

We will call it the transversal derivative cocycle. If the derivative cocycle has well defined Lyapunov exponents then the transversal derivative cocycle has also well defined Lyapunov exponents which coincide with the former ones except that one zero Lyapunov exponent is skipped.)

The theory of foliations for piecewise continue maps is developed in great generality in [41].

Chapter 5

Hyperbolicity: how to establish it

Here, we discuss how to establish hyperbolicity for symplectic maps and flows. The ideas put forward can also be used for more general systems, but simplecitiy provides an extra structure that allows the development of a much richer theory. Since Billiards are Hamiltonian systems, and hence give rise to symplectic flows and maps, this theory is relevant for Billiards.

The material of this chapter is taken from [74, 52], and the reader is referred to such articles for the full details. Here I just try to present the ideas in the simplest possible form.

5.1 Hamiltonian flows and Symplectic structure

Given the matrix $2d \times 2d$ defined by

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

Hamilton's equations can be written as[1]

$$\dot{x} = J\nabla H(x) \tag{5.1.1}$$

where $x = (q, p)$. Note that $J^2 = -\mathbb{1}$ e $J^T = -J$.[2] The matrix J plays a fundamental role in the Hamiltonian structure. In particular, one can define

[1]The gradient of a function $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ is given by the vector $\nabla f := (\partial_{x_i} f)$.

[2]Note the similarity with the imaginary number i, where the transpose takes the place of the complex conjugation; this is no accident!

the bilinear form on $\mathbb{R}^{2d}$

$$\omega(v, w) := \langle v, Jw\rangle. \tag{5.1.2}$$

The form ω is called the *symplectic form*. A matrix A with the property $\omega(Av, Aw) = \omega(v, w)$, for every $v, w \in \mathbb{R}^{2d}$, is called *symplectic*. A transformation $F \in \mathcal{C}^1(\mathbb{R}^{2d}, \mathbb{R}^{2d})$ such that $DF(x)$ is symplectic for every $x \in \mathbb{R}^{2d}$ is said to be *symplectic transformation*.

Lemma 5.1.1 *For each Hamiltonian H the Hamiltonian flow ϕ_t is a symplectic transformation.*

PROOF. Let $\Xi(x, t) = D\phi_t$, then

$$\dot{\Xi}(x, t) = JD^2H \circ \phi_t(x) \cdot \Xi(x, t)$$

hence, for each $v, w \in \mathbb{R}^{2d}$,

$$\frac{d}{dt}\omega(\Xi v, \Xi w) = \omega(\dot{\Xi}v, \Xi w)+\omega(\Xi v, \dot{\Xi}w) = \langle JD^2H\Xi v, J\Xi w\rangle-\langle \Xi v, D^2H\Xi w\rangle = 0,$$

where we used the fact that D^2H is a symmetric matrix.[3] □

Lemma 5.1.2 *The set of symplectic matrices form a group (called $Sp(2d, \mathbb{R})$). Furthermore, if $L \in Sp(2d, \mathbb{R})$, then $L^T \in Sp(2d, \mathbb{R})$.*

PROOF. First note that a matrix is symplectic if and only if $L^TJL = J$. Then it is trivial to verify that $\mathbb{1} \in Sp(2d, \mathbb{R})$. Furthermore, if $L, B \in Sp(2d, \mathbb{R})$, then

$$(LB)^TJLB = B^TL^TJLB = J,$$

therefore $LB \in Sp(2d, \mathbb{R})$. Moreover, $L[-JL^TJ] = \mathbb{1}$ shows that L is invertible and $L^{-1} = -JL^TJ$, furthermore

$$(L^{-1})^TJL^{-1} = (-JL^TJ)^TJL^{-1} = JLL^{-1} = J.$$

Hence $L^{-1} \in Sp(2d, \mathbb{R})$. Finally, if $L \in Sp(2d, \mathbb{R})$, then $L^{-1}J(L^T)^{-1} = J$ which impleis $(L^T)^{-1} \in Sp(2d, \mathbb{R})$ and $L^T \in Sp(2d, \mathbb{R})$. □

Next, we provide a useful decomposition.

[3]Obviously we are assuming that $H \in \mathcal{C}^2$ and symmetry follows from Schwartz's Lemma.

Lemma 5.1.3 *If* $L := \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \in Sp(2d, \mathbb{R})$*, where* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ *are* $d \times d$ *matrices, and* $\det(\boldsymbol{a}) \neq 0$*, then there exist symmetric* $d \times d$ *matrices* R, P *such that*

$$L = \begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ P & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & R \\ 0 & \mathbb{1} \end{pmatrix}, \tag{5.1.3}$$

Proof. A direct computation shows that $L \in Sp(2d, \mathbb{R})$ if and only if

$$\boldsymbol{c}^T \boldsymbol{a} = (\boldsymbol{a}^T \boldsymbol{c})^T = \boldsymbol{a}^T \boldsymbol{c}\,; \quad \boldsymbol{d}^T \boldsymbol{b} = (\boldsymbol{b}^T \boldsymbol{d})^T = \boldsymbol{b}^T \boldsymbol{d}\,; \quad \boldsymbol{a}^T \boldsymbol{d} - \boldsymbol{c}^T \boldsymbol{b} = \mathbb{1}. \tag{5.1.4}$$

Since $\boldsymbol{a}$ is invertible, we can write

$$L = \begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbb{1} & R \\ P & H \end{pmatrix}, \tag{5.1.5}$$

where $R = \boldsymbol{a}^{-1}\boldsymbol{b}$, $P = \boldsymbol{a}^T \boldsymbol{c}$ and $H = \boldsymbol{a}^T \boldsymbol{d}$. Condition (5.1.4) implies that $\begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix}$ is symplectic. Then, by Lemma 5.1.2, also the matrix $\begin{pmatrix} \mathbb{1} & R \\ P & H \end{pmatrix}$ must be symplectic. Accordingly, (5.1.4) implies

$$P^T = P\,; \quad H = \mathbb{1} + P^T R = \mathbb{1} + PR.$$

On the other hand, by Lemma 5.1.2, also the matrix $\begin{pmatrix} \mathbb{1} & P \\ R^T & H^T \end{pmatrix}$ is symplectic, hence (5.1.4) implies

$$R^T = R$$

from which the Lemma follows. □

Note that $L^T J L = J$ implies $\det(L)^2 = 1$. In fact, since the symplectic group is connected, the above decomposition implies that $\det(L) = 1$ by continuity (see Problem 5.8 for a more direct proof of this latter fact).

5.2 Symplectic Poincarè sections and time one maps

Let $\tau : \mathbb{R}^{2d} \to \mathbb{R}_+$ be a piecewise differentiable function and define the map $f(x) = \phi_{\tau(x)}(x)$. Where f is differentiable, we have

$$D_x f = D_x \phi_\tau + J\nabla H(\phi_\tau(x)) \otimes \nabla\tau.$$

We restrict the map f to a constant energy surface $M_E = \{x \in \mathbb{R}^{2d} : H(x) = E\}$. Then, for $v \in TM_E$ we have $\langle \nabla H, v\rangle = 0$. It follows that, for $v, w \in TM_E$,

$$\begin{aligned}\omega(Dfv, Dfw) =&\langle D\phi_\tau v + J\nabla H(\phi_\tau(x))\langle\nabla\tau, v\rangle, J(D\phi_\tau w + J\nabla H(\phi_\tau(x))\langle\nabla\tau, w\rangle)\rangle\\ =&\omega(v,w) + \langle\nabla H(\phi_\tau(x)), D\phi_\tau w\rangle\langle\nabla\tau, v\rangle\\ &- \langle D\phi_\tau v, \nabla H(\phi_\tau(x))\rangle\langle\nabla\tau, w\rangle\\ &+ \langle\nabla H(\phi_\tau(x)), J\nabla H(\phi_\tau(x))\rangle\langle\nabla\tau, v\rangle\langle\nabla\tau, w\rangle = \omega(v,w).\end{aligned}$$

It is then natural to introduce the equivalence relation $v \sim w$ is $v - w = \lambda J\nabla H$ for some $\lambda \in \mathbb{R}$. Let $\mathbb{V}_x = T_xM_E/\sim$ be the vector space formed by the equivalence classes. Note that

$$\begin{aligned}D_xf(v + \lambda J\nabla H(x)) &= D_xfv + \lambda D_x\phi_\tau J\nabla H(x) + \lambda J\nabla H(\phi_\tau(x))\langle\nabla\tau, J\nabla H(x)\rangle\\ &= D_xfv + \lambda J\nabla H(f(x))\left[1 + \langle\nabla\tau, J\nabla H(x)\rangle\right].\end{aligned}$$

Hence, the action of Df from T_xM_E to $T_{f(x)}M_E$ quotients naturally in an action between $\mathbb{V}_x$ and $\mathbb{V}_{f(x)}$. On the other hand, for $v \in \mathbb{V}_x$ we have

$$\omega(J\nabla H, v) = \langle\nabla H, v\rangle = 0.$$

Thus $\omega(v + \lambda J\nabla H, w + \mu J\nabla H) = \omega(v, w)$, that is we can quotient ω as well on $\mathbb{V}_x$. It follows that ω induces canonically a symplectic form, which we still call ω, on each $\mathbb{V}_x$. By the above discussion the d dimensional spaces $W_1^+ = \{(v, 0) : v \in \mathbb{R}^d\}$ and $W_2^+ = \{(0, v) : v \in \mathbb{R}^d\}$ quotient to $d-1$ dimensional spaces W_i in each $\mathbb{V}_x$, moreover $\omega(w, w') = 0$ for each $w, w' \in W_1$ or $w, w' \in W_2$ (such subspaces, as we will see briefly, are called Lagrangian). Next, one can check that it is possible to choose basis $\{e_i\}$ in W_1 and $\{f_i\}$ in W_2 such that $\omega(e_i, f_j) = \delta_{ij}$. Then we can write any vector $a \in \mathbb{V}_x$ as $a = \sum_{i=1}^{d-1}\xi_ie_i + \sum_{i=1}^{d-1}\eta_if_i$ and

$$\omega(a, a') = \sum_{i,j}\xi_i\eta_j'\omega(e_i, f_j) + \eta_i\xi_j'\omega(f_i, e_j) = \sum_i \xi_i\eta_i' - \xi'\eta_i = \langle(\xi, \eta), J(\xi', \eta').\rangle$$

That is, in such coordinates, the symplectic form has the standard form (5.1.2). We can thus identify all the spaces $\mathbb{V}_x$ and, in such coordinates, $Df|_{\mathbb{V}}$ is symplectic.

By choosing $\tau \equiv 1$, the map ϕ_1 can be seen as a $2d-2$ symplectic map. Moreover, if Σ is a Poncarè section for the flow, then we can choose τ to be the first return time and since $\mathbb{V}$ is naturally isomorphic to $T\Sigma$, again we have that the Poincarè map $f(x) = \phi_{\tau(x)}(x)$ is symplectic.

5.3 Two dimensions

We are interested in the case $L(x) = D_x\phi_1$, where ϕ_t is the billiard flow. Of course, the flow will have a zero Lyapunov exponent (the flow direction).

Definition 14 *A symplectic flow is hyperbolic if the only zero Lyapunov exponent is the one associated with the flow direction. Equivalently, a symplectic flow is hyperbolic if the Poincarè map has no zero Lyapunov exponent.*

The problem is to have a tool to establish hyperbolicity. The following theorem provides a very efficient tool (we do not provide the proof as it is a special case of Theorem 5.4.2).

Theorem 5.3.1 (Wojtkowski [74]) *Let X be a Riemannian manifold, possibly with boundaries, $\{\mathcal{C}(x) \subset \mathcal{T}_x X \ : \ x \in X\}$ a family of closed cones in the tangent space. Let $f : X \to X$ and $L : X \to SL(n, \mathbb{R})$ as in Theorem 4.2.1. If for μ almost $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $L(f^{n(x)-1}) \cdots L(x)\mathcal{C}(x) \subset \operatorname{int}(\mathcal{C}(f^{n(x)}(x)))$, then the maximal Lyapunov exponent is strictly positive.*

The above theorem suffices for planar billiards, where there are two Lyapunov exponents λ_i and, by volume conservation $\lambda_1 = -\lambda_2$. For higher dimensional billiard, it does not control all the Lyapunov exponents. To achieve this, we have to use more heavily the fact that the Billiards flows are Hamiltonian, and hence symplectic. In addition, while a two-dimensional cone is simply a sector, a higher-dimensional cone can have many different shapes, and it is not obvious what is a natural cone shape.

5.4 Higher dimensions: the symplectic structure

Given a symplectic form ω, which is left invariant by map $f : M \to M$, we have a symplectic flow. If $\mathcal{T}M = \mathbb{R}^{2d}$, then a d-dimensional subspace $V \subset \mathbb{R}^{2d}$ is called *Lagrangian* if $\omega|_V \equiv 0$. Given two transversal Lagrangian subspaces V_1, V_2, we can write uniquely $v \in \mathbb{R}^{2d}$ ad $v = v_1 + v_2$, with $v_i \in V_i$. we can then define the quadratic function

$$Q(v) = \omega(v_1, v_2).$$

This allows us to define special cones with remarkable properties:

$$\mathcal{C} = \{v \in \mathbb{R}^{2n} \ : \ Q(v) > 0\}. \tag{5.4.6}$$

Accordingly, if we specify a field of transversal Lagrangian subspace, we have the quadratic functions Q_x and the cone field $\mathcal{C}_x$.

Obviously, if $Q_{f(x)}(d_x f v) \geq Q_x(v)$, then $d_x f \mathcal{C}_x \subset \mathcal{C}_{f(x)}$, hence we have cone invarince. Such maps are called *monotone*.

If $Q_{f(x)}(d_x f v) > Q_x(v)$ for all $v \neq 0$, then $d_x f(\overline{\mathcal{C}_x} \setminus \{0\}) \subset \mathcal{C}_{f(x)}$, such maps are called *strictly monotone*.

Lemma 5.4.1 ([52], Sections 6) *A map is monotone if and only if the cone field is invariant. The same is true for strict monotonicity.*

Theorem 5.4.2 ([52] Sections 5, 6, or [51]) *If a map is eventually strictly monotone, then all its Lyapunov exponents are non-zero.*

This is proven exactly as Theorem 5.5.1, so we refer to the proof of the latter.
The above also has a continuous version: a Hamiltonian flow in a $2d+2$ dimensional manifold, is determined by a Hamiltonian

5.4.1 Lagrangian subspaces

By a symplectic change of variables, we can assume that the space is $\mathbb{R}^{2d}$, the vectors are written as (ξ, η), $\xi, \eta \in \mathbb{R}^d$ and the symplecit form is given by

$$\omega((\xi, \eta), (\eta', \eta') = \langle \xi, \eta' \rangle - \langle \eta, \xi' \rangle.$$

Then, $A \in GL(2d, \mathbb{R})$ is sympletc if and only if $\omega(Av, Aw) = \omega(v, w)$ for all $v, w \in \mathbb{R}^{2d}$. That is if

$$A^T J A = J$$
$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

To introduce an appropriate higher dimensional formalism, it is convenient to discuss briefly *Lagrangian subspaces.*

Definition 15 *A d-dimensional subspace $\mathbb{V}$ of $\mathbb{R}^{2d}$ is Lagrangian iff*

$$\omega(v, w) = 0$$

for all $v, w \in \mathbb{V}$.

Lemma 5.4.3 *For each $d \times d$ matrix U, the space $\mathbb{V} = \{(v, Uv) \, : \, v \in \mathbb{R}^d\}$ is Lagrangian iff U is symmetric.*

PROOF. Clearly $\mathbb{V}$ is d-dimensional. To conclude, it suffices to compute

$$\omega((v, Uv), (w, Uw)) = \langle v, Uw\rangle - \langle w, Uv\rangle$$

which is zero only if U is symmetric. □

Let $V_1, V_2 \in \mathbb{R}^{2d}$ two transversal Lagrangian subspaces, then, for each $v \in \mathbb{R}^{2d}$ we can write uniquely $v = v_1 + v_2$ with $v_i \in V_i$. We then write

$$Q(v) := \omega(v_1, v_2)$$

By a symplectic change of variable, we can always reduce the general case to the case $V_1 = \{(v_1, 0) \,:\, v_1 \in \mathbb{R}^d\}$ $V_1 = \{(0, v_2) \,:\, v_2 \in \mathbb{R}^d\}$. In this case

$$Q((v_1, v_2)) = \langle v_1, v_2\rangle.$$

We say that a symplectic matrix L is monotone if $Q(Lv) \geq Q(v)$ for each $v \in \mathbb{R}^{2d}$, and we say that a symplectic matrix L is strictly monotone if $Q(Lv) > Q(v)$ for each $v \in \mathbb{R}^{2d} \setminus \{0\}$.

To measure precisely how much the quadric form increases, it is convenient to introduce the cones

$$\mathcal{C} = \{v \in \mathbb{R}^{2d} \,:\, Q(v) > 0\}\,; \quad \overline{\mathcal{C}} = \{v \in \mathbb{R}^{2d} \,:\, Q(v) \geq 0\}.$$

Lemma 5.4.4 *A Lagrangian space $\mathbb{V}$ belongs to $\mathcal{C} \cup \{0\}$ iff it is of the form (v, Uv), with U strictly positive.*

PROOF. If $\pi_i(v_1, v_2) = v_i$, then $\pi_1 : \mathbb{V} \to \mathbb{R}^n$ is injective. If not, there exists $(v_1, v_2) \in \mathbb{V} \setminus \{0\}$ such that $v_1 = 0$. But then $Q((v_1, v_2)) = 0$ contrary to the hypothesis. We can then define $U := \pi_2 \circ \pi_1^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbb{V} = \{(v, Uv) \,:\, v \in \mathbb{R}^d\}$. Then, by Lemma 5.4.3 U must be symmetric. Finally, for $v \neq 0$,

$$0 < Q((v, Uv)) = \langle v, Uv\rangle$$

hence U is strictly positive. The opposite implication is trivial. □

Lemma 5.4.5 *A symplectic matrix $L = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$ is strictly monotone if and only if $\det \boldsymbol{a} \neq 0$ and the matrices R, P in the factorization (5.1.3) are strictly positive.*

PROOF. Indeed, if $\det \boldsymbol{a} = 0$, then there exists $\xi \in \mathbb{R}^d \setminus \{0\}$ such that $\boldsymbol{a}\xi = 0$, but then

$$Q(L(\xi, 0)) = \langle \boldsymbol{a}\xi, \boldsymbol{c}\xi\rangle = 0 = Q((\xi, 0))$$

contrary to the hypothesis. We can then apply Lemma 5.1.3 to write

$$\begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ P & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & R \\ 0 & \mathbb{1} \end{pmatrix} (v_1, v_2) = (\boldsymbol{a}(v_1 + Rv_2), (\boldsymbol{a}^{-1})^T(Pv_1 + (\mathbb{1} + PR)v_2)).$$

Thus,

$$Q(L(v_1, v_2)) = \langle v_1 + Rv_2, Pv_1 + (\mathbb{1} + PR)v_2\rangle \tag{5.4.7}$$

If $v_2 = 0$, then we have

$$0 < Q(L(v_1, 0)) = \langle v_1, Pv_1\rangle$$

hence P is a strictly positive matrix. On the other and, for each $\mu > 0$ and $\|v\| = 1$, we have that

$$\mu < Q(L(v, \mu v)) = \langle v + \mu Rv, Pv + \mu(\mathbb{1} + PR)v\rangle.$$

We can then chose v to be an eigenvector of R, so $Rv = \lambda v$. Then we obtain

$$\mu < \langle (1 + \mu\lambda)v, Pv + \mu(\mathbb{1} + \lambda P)v\rangle = (1 + \lambda)\mu + (1 + \lambda\mu)^2\langle v, Pv\rangle$$

that is

$$\lambda\mu + (1 + \lambda\mu)^2\langle v, Pv\rangle > 0.$$

It follows that it must be $\lambda \geq 0$ otherwise we can choose $\mu = -\lambda^{-1}$ and obtain the contradiction $-1 > 0$. On the other hand, if $\lambda = 0$, then

$$0 < Q(L(0, v)) = \langle 0, v\rangle = 0$$

which is also impossible. Finally, if $\det(\boldsymbol{a}) \neq 0$ and the matrices P, R are strictly positive, then

$$Q(L(v_1, v_2)) = \langle v_1, v_2\rangle + \langle v_2, Rv_2\rangle + \langle v_1 + Rv_2, P(v_1 + Rv_2)\rangle > Q((v_1, v_2)).$$

□

The above implies that if L is strictly monotone, then $LV_i \subset \mathcal{C} \cup \{0\}$. There is a useful partial converse of this fact.[4]

[4] Note that [52, Proposition 4.8] is false as the example $L = \begin{pmatrix} \mathbb{1} & 0 \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & -2\mathbb{1} \\ 0 & \mathbb{1} \end{pmatrix}$ shows.

Lemma 5.4.6 *If $LV_i \subset \mathcal{C} \cup \{0\}$ and, for all $v \in \mathbb{R}^d$,*

$$\omega(L(0,v), ((\boldsymbol{a}^T)^{-1}v, 0)) \geq 0,$$

then L is strictly monotone.

Proof. First of all, note that

$$0 < Q(L(v,0)) = \langle Q((\boldsymbol{a}v, \boldsymbol{c}v)) = \langle v, \boldsymbol{c}^T \boldsymbol{a}v\rangle.$$

Since (5.1.4) implies that $\boldsymbol{c}^T\boldsymbol{a}$ is a symmetric matrix, it follows that $\boldsymbol{c}^T\boldsymbol{a}$ is strictly positive, hence $\det(\boldsymbol{a}) \neq 0$. We can then use the decomposition (5.1.3) which yields the expression (5.4.7) which implies

$$0 < Q(L(v,0)) = \langle v, Pv\rangle$$

which implies that P is a strictly positive matrix. This implies that

$$Q\left(\begin{pmatrix} \mathbb{1} & 0 \\ P & \mathbb{1}\end{pmatrix}\begin{pmatrix} v_1 \\ v_2\end{pmatrix}\right) = Q((v_1, Pv_1 + v_2)) = Q((v_1,v_2)) + \langle v_1, Pv_1\rangle \geq Q((v_1,v_2)).$$

On the other hand

$$0 < Q(L(0,v)) = \langle Rv, (\mathbb{1} + PR)v\rangle = \langle v, (R + RPR)v\rangle,$$

that is $R + RPR$ is strictly positive matrix. Since R is symmetric it has d eigenvectors, let w, $\|w\| = 1$, and eigenvector and λ the corresponding eigenvalue, then

$$0 < \langle w, (R + RPR)w\rangle = \lambda + \lambda^2\langle w, Pw\rangle = \lambda(1 + \lambda\langle w, Pw\rangle).$$

which implies $\lambda \neq 0$. Finally,

$$0 \leq \omega(L(0,v), ((\boldsymbol{a}^T)^{-1}w, 0)) = \langle \boldsymbol{a}Rw, (\boldsymbol{a}^T)^{-1}w\rangle = \langle Rw, w\rangle$$

implies that R is positive and hence strictly positive. The Lemma follows then from Lemma 5.4.5. □

Let us define

$$\sigma(L) = \inf_{v \in \mathcal{C}} \sqrt{\frac{Q(Lv)}{Q(v)}}.$$

Lemma 5.4.7 *If a symplectic matrix* $L = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$ *is strictly monotone, then the eaigenvalues of* $\boldsymbol{c}^T \boldsymbol{b}$ *are all strictly positive and, calling* t *the minimal such eigenvalue, we have*

$$\sigma(L) \geq \sqrt{t} + \sqrt{1+t} > 1.$$

Proof. We use the decomposition (5.1.3) and note that the matrix

$$\mathcal{R} = \begin{pmatrix} R^{-\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{pmatrix}$$

is a Q-isometry, that is $Q(\mathcal{R}v) = Q(v)$ for all $v \in \mathbb{R}^{2d}$. In particular, this implies that $\mathcal{R}\mathcal{C} = \mathcal{C}$. Hence, setting $\mathcal{L} = \begin{pmatrix} \mathbb{1} & R \\ P & \mathbb{1} + PR \end{pmatrix}$,

$$\inf_{v \in \mathcal{C}} \sqrt{\frac{Q(Lv)}{Q(v)}} = \inf_{v \in \mathcal{C}} \sqrt{\frac{Q(\mathcal{L}v)}{Q(v)}} = \inf_{v \in \mathcal{C}} \sqrt{\frac{Q(\mathcal{R}\mathcal{L}\mathcal{R}^{-1}(\mathcal{R}v))}{Q(\mathcal{R}v)}} = \inf_{v \in \mathcal{C}} \sqrt{\frac{Q(\mathcal{R}\mathcal{L}\mathcal{R}^{-1}v)}{Q(v)}}.$$

Setting $T = R^{\frac{1}{2}} P R^{\frac{1}{2}}$, we have

$$\mathcal{R}\mathcal{L}\mathcal{R}^{-1} = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ T & \mathbb{1} + T \end{pmatrix} =: \mathcal{T}$$

Note that T is a strictly positive matrix; hence, calling t_i its eigenvalues and w_i the associated eigenvector, we have $t_i > 0$. In addition, we have

$$PR(R^{-\frac{1}{2}} w_i) = R^{-\frac{1}{2}} T w_i = t_i R^{-\frac{1}{2}} w_i.$$

That is, the eigenvalues of T are also the eigenvalues of $PR = \boldsymbol{c}^T \boldsymbol{a} \boldsymbol{a}^{-1} \boldsymbol{b} = \boldsymbol{c}^T \boldsymbol{b}$, where we have used (5.1.5) and the fact that $P^T = P$. To conclude, we note that, setting $v = (v_1, v_2)$ and calling t the minimal eigenvalue ot T,

$$\begin{aligned}
\frac{Q(\mathcal{T}v)}{Q(v)} &= \frac{\langle v_1, v_2 \rangle + \langle v_2, v_2 \rangle + \langle (v_1 + v_2), T(v_1 + v_2) \rangle}{\langle v_1, v_2 \rangle} \geq 1 + \frac{\|v_2\|^2 + t\|v_1 + v_2\|^2}{\langle v_1, v_2 \rangle} \\
&= 1 + \frac{(1+t)\|v_1\|^2 + 2t\langle v_1, v_2 \rangle + t\|v_2\|^2}{\langle v_1, v_2 \rangle} \\
&= 1 + \frac{2t\langle v_1, v_2 \rangle + (1+t)^{\frac{1}{2}} t^{\frac{1}{2}} \left[(1+t)^{\frac{1}{2}} t^{-\frac{1}{2}} \|v_1\|^2 + (1+t)^{-\frac{1}{2}} t^{\frac{1}{2}} \|v_2\|^2 \right]}{\langle v_1, v_2 \rangle} \\
&\geq 1 + \frac{2\left[t + (1+t)^{\frac{1}{2}} t^{\frac{1}{2}} \right] \langle v_1, v_2 \rangle}{\langle v_1, v_2 \rangle} = \left[\sqrt{t} + \sqrt{1+t} \right]^2.
\end{aligned}$$

□

5.5 Higher dimensions: hyperbolicity

We say that a Hamiltonian flow (M, ϕ_t) is hyperbolic on a constant energy surface M_E if, when restricted to such a surface, all his Lyponov exponents, but one (the one in the flow direction), are non-zero. For simplicity, we restrict to the case $M \subset \mathbb{R}^{2d}$, but the result holds for general symplectic manifolds. Also, we require that M_E is compact. Let μ be the Liouville measure normalized so that $\mu(M) = 1$. The goal of this section is to prove the following theorem:

Theorem 5.5.1 ([74], or see [52] Sections 5, 6, or [51]) *If a flow on M_E is* eventually strictly monotone*, then all its Lyapunov exponents, apart from the one in the flow direction, are non-zero.*

By the results of section 5.2, we can restrict ourselves to a discrete-time analysis. We will consider the time one map $f = \phi_1$ with the differential acting on the quotient space there described; the study of the Poincarè map being similar. For $x \in M$, le $s(x) = \min\{k \ : \ Df^k$ is strictly monotone$\}$.

By *eventually strictly monotone*, we mean that, for almost all $x \in M$, $D_x f$ is monotone and $s(x) < \infty$.

Proof of Theorem 5.5.1. Let $A_m = \{x \in M \ : \ s(x) = m\}$. For such m we define the first hyperbolic return time to A_m as

$$n_m(x) = \begin{cases} 0 & \text{if } x \notin A_m \\ \min\{k \geq m \ : \ f^k(x) \in A_m\} & \text{otherwise.} \end{cases}$$

If f is eventually strictly monotone, then $\sum_{m=1}^{\infty} \mu(A_m) = 1$. Hence, there exists $m > 0$ such that $\mu(A_m) > 0$. Define the return map $F(x) = f^{n_m(x)}(x)$, $x \in A_m$. For each $n \in \mathbb{N}$ let $k(x) = \min\{k \in \mathbb{N} \ : \ \sum_{j=0}^{k} n_m(F^j(x)) \geq n\}$. Then

$$\begin{aligned} \sigma(Df^n) &\geq \sigma(D_{F^{k(x)-1}(x)} f^{n_m(F^{k(x)-1}(x))} \cdots D_x f^{n_m(x)}) \\ &\geq \prod_{j=0}^{k(x)-1} \sigma(D_{F^j(x)} f^{n_m(F^j(x))}). \end{aligned}$$

Also, note that, by definition, it must be $n_m(s) \geq m$. So, by Lemma 5.4.7, we have, for each $y \in A_m$, $\sigma(D_y f^{n_m(y)}) \geq \sqrt{t}(x) + \sqrt{1 + t(x)} =: e^{\alpha(x)}$

where $\alpha(x) > 0$. Since α could be unbounded it is convenient to set $\bar{\alpha}(x) = \min\{1, \alpha(x)\}$ and again $\sigma(D_y f^{n_m(y)}) \geq e^{\bar{\alpha}(x)}$. Accordingly,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \ln \sigma(D_x f^n) &\geq \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{k(x)-1} \ln \sigma(D_{F^j(x)} f^{n_m(F^j(x))}) \\ &\geq \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{k(x)-1} \bar{\alpha}(F^j(x)) \geq \frac{\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \bar{\alpha}(F^j(x))}{\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k} n_m(F^j(x))}.\end{aligned}$$

By Birkhoff's ergodic theorem, the limits exist almost surely and are L^1 functions. Hence, the limit can be zero on a positive measure set only if the numerator is. Also, the points for which the numerator is zero form an invariant set $B \subset A_m$. But if $\mu(B) > 0$, then we can restrict the above argument to B and we obtain, for almost al $x \in B$, the contradiction

$$0 = \int_B \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \bar{\alpha}(F^j(x)) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \int_B \bar{\alpha}(F^j(x)) = \int_B \bar{\alpha}(x) > 0.$$

Accordingly, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \ln \sigma(D_x f^n) > 0. \tag{5.5.8}$$

The above implies that for each $v \in \mathcal{C}$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \ln \|Df^n v\| &= \lim_{n\to\infty} \frac{1}{2n} \ln \|Df^n v\|^2 \geq \lim_{n\to\infty} \frac{1}{2n} \ln Q(Df^n v) \\ &\geq \lim_{n\to\infty} \frac{1}{2n} \ln \sigma(Df^n) > 0.\end{aligned}$$

Since the Lagrangian space $\mathbb{W} = \{(w, w)\} \subset \mathcal{C} \cup \{0\}$, we have a d-dimensional subspace with strictly positive Lyapunov exponents. Hence, we have d strictly positive Lyapunov exponents. Let $\mathbb{V}_i$ the spaces in Oseldets' Theorem, so that $\dim(\mathbb{V}_i) = i$ (note that if the spectrum is not simple, there are many possible choices). Note that for each $i \in \{d+1, \ldots 2d\}$, there must be $v \in \mathbb{V}_i$, and $w \in \mathbb{V}_{2d-i+1}$ such that $\omega(v, w) \neq 0$; otherwise the two spaces would be skew orthogonal which is impossible since the sum of their dimensions is $2d+1$. Then, by continuity, we can find $v_i \in \mathbb{V}_i \setminus \mathbb{V}_{i-1}$ and $w_i \in \mathbb{V}_{2d-i+1} \setminus \mathbb{V}_{2d-i}$ such that $\omega(v_i, w_i) \neq 0$. By construction, λ_i is the Lyapunov exponent associated to v_i ad λ_{2d-i+1} the Lyapunov exponents associated with $\hat{w}_i$. Then

$$\begin{aligned}0 &= \lim_{n\to\infty}\frac{1}{n}\ln\omega(v_i,w_i) = \lim_{n\to\infty}\frac{1}{n}\ln\omega(Df^nv_i,Df^nw_i)\\ &\leq \lim_{n\to\infty}\frac{1}{n}\ln\|Df^nv_i\|\|Df^nw_i\| = \lambda_i+\lambda_{2d-i+1}.\end{aligned}$$

On the other hand, by Oseledets Theorem 4.2.1,

$$\begin{aligned}0 &= \lim_{n\to\infty}\frac{1}{n}\ln|\wedge^d\omega(v_{d+1},\dots v_{2d},w_{d+1},\dots,w_{2d})|\\ &= \lim_{n\to\infty}\frac{1}{n}\ln|\wedge^d\omega(Df^nv_{d+1},\dots Df^nv_{2d},Df^nw_{d+1},\dots,Df^nw_{2d})|\\ &= \lim_{n\to\infty}\frac{1}{n}\ln|\det(Df^n)||\wedge^d\omega(v_{d+1},\dots v_{2d},w_{d+1},\dots,w_{2d})|\\ &= \sum_{i=1}^{2d}\lambda_i = \sum_{i=1}^{d}[\lambda_i+\lambda_{2d-i+1}]\geq 0\end{aligned}$$

which implies $\lambda_{2d-i+1} = -\lambda_i$, hence all the Lyapunov exponents are non zero. □

Problems

5.1 Construct a strictly invariant cone family for the irrational translation on $\mathbb{T}^2$ (see Examples 2.1.1) and show that it is not measurable. (Hint: For each trajectory choose a point x. At such a point choose the standard cone $\mathcal{C}_+$, let $\mathcal{C}_n^- = \{(v_1,v_2)\in\mathbb{R}^2 \mid 1+\frac{1}{n}\leq\frac{v_2}{v_1}\leq 2+\frac{1}{n}\}$ and $\mathcal{C}_n^+ = \{(v_1,v_2)\in\mathbb{R}^2 \mid -2-\frac{1}{n}\leq\frac{v_2}{v_1}\leq -1-\frac{1}{n}\}^c$. Then set $\mathcal{C}(T^nx)=\mathcal{C}_n^+$ and $\mathcal{C}(T^{-n}x)=\mathcal{C}_n^-$. Such a cone family is strictly monotone by construction (since $D_xT=1$), yet the system has obviously zero Lyapunov exponents. Since all the other hypothesis of Theorem 4.2.1 are satisfied, it follows that the above cone family cannot be measurable.)

5.2 Show that for two dimensional symplectic maps the sum of the Lyapunov exponent is zero (*pairing of the Lyapunov exponents*). (Hint: If $\omega(v,w)=1$ then $1=\omega(DT^nv,DT^nw)\sim\|DT^nv\|\|DT^nw\|$.)

5.3 Check that $\inf_{v\in\mathcal{C}_+}\sqrt{\frac{Q(Lv)}{Q(v)}} = \inf_{v\in\mathcal{C}_-}\sqrt{\frac{Q(L^{-1}v)}{Q(v)}}$, remember that $\mathcal{C}_- = \overline{(\mathcal{C}_+)^c}$. (Hint: see [52, Proposition 6.2])

5.4 Consider $\mathbb{R}^2$ endowed with the scalar product $\langle v, w\rangle_G := \langle v, Gw\rangle$, where $\langle\cdot,\cdot\rangle$ is the standard scalar product and $G > 0$. Show that there exists a change of coordinates $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that, in the new coordinates $\langle\cdot,\cdot\rangle_G$ becomes the standard scalar product.

5.5 Consider the cone $\mathcal{C}$ defined by the two transversal vectors $v_1, v_2 \in \mathbb{R}^2$. This means that $v \in \mathbb{R}^2$ belongs to the cone iff $v = \alpha v_1 + \beta v_2$ with $\alpha\beta \geq 0$. Show that there is a linear change of coordinates $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that $M\mathcal{C} = \mathcal{C}_+$ and $\det M = 1$.

5.6 Show that, in a two dimensional area preserving systems, if the LE are different from zero then there exists and eventually strictly invariant cone family. (Hint: By Oseledets there exists the unstable distributions, then construct the cones around it.)

5.7 Prove that if M is the two by two matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

with $a, b, c \in \mathbb{Z}$, then $M > 0$ iff $a, c > 0$ and $c > \frac{b^2}{a}$.

5.8 Prove that is L is symplect then $\det L = 1$. (Hint: The determinant of a matrix is nothing else than the volume of the parallelepiped of sides $(Le_1, \ldots, Le_{2d})$ (where $e_1, \ldots, e_{2d}$ is the standard orthonormal basis of $\mathbb{R}^{2d}$). On the other hand the volume form can be written has $\wedge^d\omega$ (since that is a $2d$ form with the right normalization and the space of $2d$ forms is one dimensional). Thus $\det L = \wedge^d\omega(Le_1, \ldots, Le_{2d}) = \wedge^d\omega(e_1, \ldots, e_{2d}) = 1$ where we have used the fact that $\omega(Lv, Lu) = \omega(v, u)$. The reader that wants to appreciate the power of the above geometrical interpretation of the determinant and of the external forms can try to prove the statement by purely algebraic means.)

5.9 Show that all symplectic Q-isometrys L (that is $Q(Lv) = Q(v)$) have the form

$$L = \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}.$$

(Hint: Start by considering the vector $(0, u)$, $U \in \mathbb{R}^d$, clearly $Q((0, u)) = 0$ thus $Q(L(0, u)) = 0$ if L is a Q-isometry. But if

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

it follows $\langle Bu, Du\rangle = 0$ for each $u \in \mathbb{R}^d$, that is $B^*D = 0$. The same argument applied to the vector $(u, 0)$ yields $A^*C = 0$. Accordingly, by symplecticity

$$\begin{aligned} Q(L(v,u)) =& \langle Au + Bv,\, Cu + Dv\rangle = \langle u, (A^*D + C^*B)v\rangle \\ =& \langle u, (\mathbb{1} + 2C^*B)v\rangle \end{aligned}$$

thus $Q(L(v,u)) = Q(v,u)$ iff $C^*B = 0$ which implies $A^*D = \mathbb{1}$.)

5.10 Show that if the matrix

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is symplectic then

$$L^{-1} = \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix}$$

5.11 Show that the symplectic matrices form a multiplicative group. (Hint: Use the definition and the above problems.)

5.12 A symplectic map L is a Q-isometry iff $L\mathcal{C} = \mathcal{C}$. (Hint: One direction is trivial. On the other hand, if $L\mathcal{C} = \mathcal{C}$ it follows that L maps the boundary, of $\mathcal{C}$, to the boundary. Accordingly, if $\langle v, u\rangle = 0$ it must be

$$0 = \langle Av + bu,\, Cv + Du\rangle. \tag{5.5.9}$$

Choosing in 5.5.9 $u = 0$ yields $A^*C = 0$, choosing $v = 0$ shows that it must be $B^*D = 0$. Thus 5.5.9 yields

$$0 = \langle u,\, (A^*D + C^*B)v\rangle = 2\langle u,\, C^*Bv\rangle.$$

The above equality shows that C^*Bv is parallel to v for each $v \in R^d$, that is $C^*B = \alpha\mathbb{1}$ for some $\alpha \in \mathbb{R}$. If $\alpha = 0$, then $A^*D = \mathbb{1}$ and thus $C = 0$ which is the wanted result. If $\alpha \neq 0$, then B is invertible and $C = \alpha B^{*1}$. But this implies $A = 0$ and hence $-\mathbb{1} = C^*B = \alpha\mathbb{1}$, that is $\alpha = -1$. Accordingly the matrix would have the form

$$L = \begin{pmatrix} 0 & B \\ -B^{*-1} & 0 \end{pmatrix}$$

which sends $\mathcal{C}$ in its complement, contrary to our requirement.)

5.13 Show that a strictly monotone symplectic matrix can be put into the form

$$\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ M & \mathbb{1}+M \end{pmatrix}$$

by multiplying it by Q-isometries on the left and on the right.

5.14 Show that all the Lagrangian subspaces transversal to $V = \{(0,\eta) \in \mathbb{R}^{2d} \mid \eta \in \mathbb{R}^d\}$ can be represented as $\{(\xi, U\xi) \in \mathbb{R}^{2d} \mid \xi \in \mathbb{R}^d\}$ for some symmetric matrix U. (Hint: Let $V_U := \{(\xi, U\xi) \in \mathbb{R}^{2d} \mid \xi \in \mathbb{R}^d\}$, then $\omega((\xi, U\xi), (\zeta, U\zeta)) = 0$, thus V_U is Lagrangian. On the other hand, if $\tilde{V}$ is Lagrangian, then it is a d dimensional space. Let $\{(\xi_i, \eta_i)\}_{i=1}^d$ be a base for $\tilde{V}$, then $\xi_i \neq 0$ by the transversality assumption and we can define the matrix U via $U\xi := \eta_i$. It is immediate that $\tilde{V}$ Lagrangian implies $U = U^*$.)

5.15 Show that $V_U := \{(\xi, U\xi) \in \mathbb{R}^{2d} \mid \xi \in \mathbb{R}^d\}$, $U = U^*$, belongs to the standard cone iff $U \geq 0$.

5.16 Show that given any two transversal lagrangian subspaces V_1, V_2,[5] there exists a symplectic map L such that $LV_1 = \{(\xi, 0)\}$ and $LV_2 = \{(0,\eta)\}$. (Hint: choose coordinates in which V_i are transversal to $V = \{(0,\eta) \in \mathbb{R}^{2d} \mid \eta \in \mathbb{R}^d\}$, then we can write $V_i = \{(\xi, U_i\xi)\}$. Note that, since V_1 and V_2 are transversal, $U_1 - U_2$ must be invertible. The, e.g., set $D = \mathbb{1}$ and $B = (U_1 - U_2)^{-1}$ and check the algebra.)

5.17 Find a symplectic change of coordinates that transforms the standard form Q into the form Q_h defined by:

$$Q_h((x,y)) = \frac{1}{2}(\langle x, x\rangle - \langle y, y\rangle),$$

and draw the associate cone. (Hint: Consider

$$\begin{aligned} x &= \tfrac{x'-y'}{\sqrt{2}} \\ y &= \tfrac{x'+y'}{\sqrt{2}}.) \end{aligned}$$

5.18 Hilbert metric for a disc and the half plane–hyperbolic geometry.

5.19 Show that the Perron-Frobenius operator associated to a smooth expanding map of the circle has a spectral gap as an operator on $Lip(\mathbb{T}^2)$. (Hint: Check that there exists $b \in \mathbb{R}^+$ such that the norm

$$\|h\| := \|h\|_\infty + b\|h\|_{Lip}$$

[5] Recall that two space are transversal iff $V_1 \cap V_2 = \emptyset$.

is adapted to the cone. Define $\mathbb{V} = \{h \in Lip(\mathbb{T}^2) \mid \int h = 0\}$, notice that $\mathcal{L}\mathbb{V} = \mathbb{V}$. Then, for each $h \in \mathbb{V}$ there exists$\rho \in \mathbb{R}^+$ such that $h + \rho h_* \in \mathcal{C}_\alpha$, so

$$\|\mathcal{L}^n h\| = \|\mathcal{L}^n(h + \rho h_*) - \rho h_*\| \leq K\Lambda^n \rho.$$

Thus the spectral radius of $\mathcal{L}|_{\mathbb{V}}$ is less than Λ.)

5.20 Estimate the rate of mixing for Lipschitz functions for a smooth expanding map of the circle (Hint: use the spectral gap of the previous Problem.)

5.21 Prove that any continuous fraction of the form

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

$a_i > 0$ is convergent provided the series $\sum_{n=1}^{\infty} a_n$ is divergent. (Hint: Let

$$\prod_{i=1}^{n} \begin{pmatrix} 1 & a_{2(n-i)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{2(n-1)+1} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} = \begin{pmatrix} \beta_n \\ \alpha_n \end{pmatrix}$$

and verify, by induction, that $\frac{\alpha_n}{\beta_n}$ is exactly the $2n$ truncation of the continuous fraction. Thus the continuous fraction is a projective coordinate for the vector (α_n, β_n). Consider the cone $\mathcal{C}_+ = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0; y \geq 0\}$. Then, for each $a, b \in \mathbb{R}^+$, holds

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{C}_+ \subset \mathcal{C}_+.$$

The result follows by computing the Hilbert metric contraction, see [21, Appendix D] for details on the Hilber metric and its properties.

For a different approach, see [73, Th14.1].)

Chapter 6

Billiards (two dimensional)

Billiards are very widely studied model systems. The study of billiards has a double parallel history. On the one hand, starting at least with G. Birkhoff, they are seen as simple examples of dynamical systems and a tool to understand issues of integrability (billiard in an ellipse, polygonal billiards) and tool to understand strongly irregular motion (Sinai and Bunimovich Billiards). We will concentrate on the second class of models here.

In general, billiards consist of a material point confined to some region of $\mathbb{R}^n$ or $\mathbb{T}^n$ with piecewise smooth boundaries;[1] in the simplest situation such a point moves with constant velocity until it reaches the boundary, and at the boundary it undergoes an elastic reflection. Such models include, e.g., a system of n hard spheres that interacts via elastic collisions (see section 6.4); the importance of such a system as a basic model in statistical mechanics can be hardly overestimated.

These systems are conceptually extremely simple, yet they have an unpleasant feature: they lack smoothness. As we will see in the following there are three main type of non-smoothness: a) tangent collisions; b) collision with a corner; c) accumulation of infinitely many collisions in a finite time. Due to such pathologies these models, in spite of their simplicity, may present some incredibly annoying complications in their treatment.

Let $\mathbb{B}$ be the region in which the point is allowed to move and suppose that $\partial\mathbb{B}$ is a finite union of smooth manifolds with boundary. Clearly, the motion can be seen as a flow ϕ_t on the unitary tangent bundle of $\mathbb{B}$ (in fact, given the initial position and the initial velocity the following motion is

[1] Although one can easily consider billiards in a region of a Riemannian manifold with piecewise smooth boundaries, in this case the motion in the interior is just the geodesic flow; see [12] for such a general setting.

uniquely determined, moreover the modulus of the velocity will be constant through the motion, so it can be assumed equal to one without loss of generality).[2]

It can be checked directly that the flow is symplectic (Hamiltonian) in $\widetilde{X} := \mathcal{B} \times \mathbb{R}^n$ (see problem Problem 6.1). So, calling m the measure induced by Lebesgue on $X \equiv \mathcal{B} \times \{v \in \mathbb{R}^n \mid \|v\| = 1\}$, (X, ϕ_t, m) is a smooth flow with collisions (crf. Examples 4.4.1).

6.0.1 Examples

Polygonal Billiards

The name is self-explanatory: the domain $\mathcal{B}$ is a polygon. The simplest case is probably a rectangle: $\mathcal{B} = [0, a] \times [0, b] \subset \mathbb{R}^2$. Although the notion is fairly trivial, to study it we will employ a neat trick that has many other applications. Consider a trajectory $x+vt$ that reaches the wall $\ell_1 := \{(a, y)\}$. The law of reflection states that, if $v = (v_1, v_2)$, the reflected velocity is $(-v_1, v_2)$. Now define the map $R_a(x, y) = (2a - x, y)$. This is a reflection (R_a^2 = identity) with respect to the wall $\{(a, y)\}$. Remark that $R_a\mathcal{B} = [a, 2a] \times [0, b]$, moreover $DR_a(-v_1, v_2) = v$. This means that, in the reflected box $R_a\mathcal{B}$, the reflected velocity is equal to the velocity before reflection.

The above algebraic discussion corresponds to a very intuitive geometrical fact: if the wall is a mirror, then the trajectory in the mirror is the continuation of the trajectory before collision.

After noticing this it is quite clear that one can understand better the trajectory in the "universal covering' of the box obtained be reflecting the box repeatedly with respect to its walls. In this covering the trajectory is simply a straight line and the trajectory in the original box is obtained by undoing the reflections (for the more mathematical inclines let us say that the plane is covered by equal boxes that are identified via reflections, see Problem 6.2). It is then obvious that, given the original velocity v only four velocities are possible: $(\pm v_1, \pm v_2)$. In fact, if we identify the opposite sides we obtain exactly a flat torus with sides twice as long as the ones of the original rectangle. In addition, the motion on such a torus corresponds precisely to the flow at unit speed in direction v. In other words, the motion is equivalent to rigid translations (geodesic flow) of the associated torus.

Accordingly, the motion is ergodic only if $\frac{v_1 b}{v_2 a}$ is irrational.

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[2] A little thought will convince the reader that two motions with initial velocities that differ only in modulus will be exactly the same apart from the fact that they are run at different speeds.

Circular Billiards

In this case $\mathcal{B}$ is a disk of radius r. For convenience, let us center it at the origin of a Cartesian coordinate frame. Let us consider a point that has just collided with the boundary at the position $rn(\theta) := r(\cos\theta, \sin\theta)$, where θ is the angle with the x axis counted counterclockwise, and has velocity $v(\theta-\varphi) := (-\sin(\theta-\varphi), \cos(\theta-\varphi))$, which means that the velocity forms an angle φ with the tangent at the collision point. Accordingly, the trajectory will move along the cord of length $2r\sin\varphi$ and collide with the angle $\pi-\varphi$ which, after reflection, will be φ again.

This phenomena is nothing else than the conservation of the angular momentum (for the mechanical inclined) or of the Claroit integral (for the differential geometers).

All the above implies that, if $\frac{\varphi}{\pi} \in \mathbb{Q}$, then the motion will be periodic, otherwise the collision point will perform an irrational rotation on the boundary. In fact, let us choose as coordinates the distance τ form the last collision point computed along the trajectory; the distance s, computed along the circumference, of the last collision point from a fixed point on the circumference; and the angle φ. Then the phase space is

$$X = \{(\tau, s, \varphi) \in [0, r] \times S^1 \times [0, \pi] \mid 0 \leq \tau \leq 2r\sin\varphi\}$$

and the flow is noting else than a suspension flow with ceiling function $2r\sin\varphi$ constructed on the map T defined by

$$T(s, \varphi) = (s + r(\pi - 2\varphi), \varphi).$$

At the same time the middle point of the cords between two consecutive collisions will describe an irrational rotation on the circle of radius $r\cos\varphi$. This last circle is called *caustic*; the name derives from optic because if the trajectory is run by a beam of light that is the place with the highest luminosity.[3] Note that this means that the trajectory under consideration (if $\varphi/\pi \notin \mathcal{Q}$) covers densely a two dimensional torus in the three dimensional space and it is ergodic restricted to it.

[3] In ancient Greek caustic ($\kappa\alpha\upsilon\sigma\tau\iota\kappa\acute{o}\zeta$) means “that burns”. Of course, that would be an important concept if you want, e.g., burn a Roman ship (to be honest, we do not know if Archimedes really knew and used burning mirrors against the Romans. Nor if he had the knowledge to do so, since his work on optic, if ever existed, has been lost. Yet, his work on conics shows that he was not so far off [2, On the sphere and the cylinder and Quadrature of the parabola]).

The above examples correspond to very regular motions ("integrable motion") that is exactly the opposite of what we mean to investigate. Unfortunately, to progress in the direction we are interested in many more technical tools are needed. Yet, before going on with general facts and definitions, let us anticipate two concrete examples that will be particularly relevant.

6.1 Sinai Billiard

The simplest example of Sinai billiards (introduced in [69] and studied in [70]) are given when $\mathbb{B} \subset \mathbb{T}^2$. More precisely, given a disk D, centered at the origin and with diameter $r < \frac{1}{2}$, let $\mathbb{B} = \mathbb{T}^2 \backslash D$. Calling $(x, v) \in \mathbb{B} \times \mathbb{R}^2$ the position and the velocity, respectively, the motion is described by a free flow

$$\phi_t(x, y) = (x + vt, v), \tag{6.1.1}$$

provided $\|x + vt\| \geq r$, that is provided the motion does not exist $\mathcal{B}$. When $x \in \partial\mathcal{B} = \partial D$ a collision takes place. Of course, at the collision, it must be $\langle x, v\rangle \leq 0$, the velocity points toward D, otherwise the point would not have reached the obstacle D but rather would be flowing away from it. The collision law is, as already said, an elastic collision–namely, the total energy and the momentum tangential to the collision plane must be preserved. Thus, calling v_- the velocity before collision and v_+ the velocity after collision, we require

$$\|v_+\| = \|v_-\| \; ; \quad \langle Jx, v_-\rangle = \langle Jx, v_+\rangle,$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that $\langle Jx, x\rangle = 0$, that is $r^{-1}Jx$ is a unit vector tangent to the disk and oriented counterclockwise. This implies:

$$v_+ = v_- - \frac{2}{r^2}\langle x, v_-\rangle x. \tag{6.1.2}$$

6.1.1 Flow

From the above discussion, it is clear that (X, ϕ^t, m) is a smooth flow with collisions, the only property that needs to be checked is (4.4.3).

Let us call $V(x, v) = (v, 0)$ the vector field generating ϕ^t. A useful fact is the following.

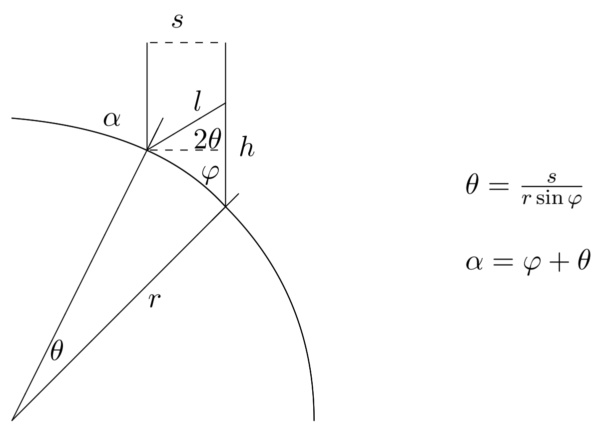


Figure 6.1: Collision

Lemma 6.1.1 *If $w \in \mathcal{T}_\xi X$ and $\langle w, V(\xi)\rangle = 0$, then $\langle d\phi^t w, V(\phi^t(\xi))\rangle = 0$.*

PROOF. If no collision takes place, then the statement it is obvious by equation (6.1.1) and since for each $w = (w_1, w_2) \in \mathcal{T}X$ it must be $\langle w_1, v\rangle = 0$ (just differentiate $\|v\|^2 = 1$). Let us see what happens at collision.

Given the tangent vector $w = (w_1, w_2)$ at the point $\xi \in X$, we can consider the curve $\gamma(s) = \xi + ws$ that generates it ($\gamma'(0) = w$). Suppose that the next collision takes place with an angle φ. If we refer to the Figure 6.1 all we need to compute is the relation between h and l. A bit of geometry shows that

$$h = s \arctan\varphi + \mathcal{O}(s^2); \quad l = \frac{s}{\cos\theta}\arctan\varphi + \mathcal{O}(s^2) = s\arctan\varphi + \mathcal{O}(s^2).$$

Thus, if τ is the collision time of the trajectory starting at ξ and $\tilde{\gamma}(s) = \phi^{\tau_+}(\gamma(s))$, we have $\tilde{\gamma}'(0) = d_\xi\phi^{\tau_+}w := \tilde{w}$, and, calling v_+ the velocity after reflection, $\langle v_+, \tilde{w}\rangle = 0$, which proves the lemma. □

This means that in this case there is a particularly simple way to quotient out the flow direction: consider only vectors perpendicular to the flow.

6.1.2 Reeb flows

A more general way to understand and contextualize Lemma 6.1.1 is to realize that billiards are an example of Reeb flow.

Definition 16 *Given a $2d+1$ dimensional manifold M equipped with a one form ω such that $\omega \wedge (d\omega)^d \neq 0$ (that is, a* contact manifold*), we call a Reeb* *flow a flow generated by a vector field V such that*

$$\omega(V) = 1$$
$$\omega(V, v) = 0 \text{ for all tangent vectros } v$$

Let ϕ_t be the Reb flow, generated by the vector field V.

Lemma 6.1.2 *For each $t \in \mathbb{R}$ we have $\phi_t^*\omega = \omega$.*

PROOF. By Cartan formula, we have

$$\begin{aligned}\frac{d}{dt}(\phi_t^*\omega) &= L_V(\phi_t^*\omega) = d(i_V[\phi_t^*\omega]) + i_V(d[\phi_t^*\omega]) \\ &= d(i_V[\phi_t^*\omega]) + i_V([\phi_t^* d\omega]) = d([\phi_t^* i_V\omega]) + ([\phi_t^* i_V d\omega]) = 0\end{aligned}$$

Thus ω is invariant for $D\phi_t$. □

As we want to extend the idea of Reeb flows to piecewise smooth flows, it is natural to say that a piecewise smooth flow is Reeb is $\omega(V) = 1$, where it makes sense and $\omega(D\phi_t w) = \omega(w)$, again where it makes sense.

We can then prove that Billiards are Reeb flows on the constant energy surface. First of all, note that the energy is just the Kinetic energy, hence $M_E = \{(q,p) \in \mathbb{R}^2 \ : \ \|p\|^2 = 2E\}$ is an invariant surface for the flow. Note that $(\delta q, \delta p) \in TM_E$ iff $\langle \delta p, p\rangle = 0$. We then consider the one form $\omega(\delta q, \delta p) = \frac{1}{2E}\langle p, \delta q\rangle$. Note that the vector field $V = (p, 0)$ generates the flow away from collisions, and $\omega(V) = 1$. Note that

$$\langle V, (\delta q, \delta p)\rangle = \langle p, \delta q\rangle = \omega((\delta q, \delta p)),$$

thus being Reeb automatically implies the result in the previous section as a special case. It remains to check the invariance. If the flow does not experience collisions, then

$$D\phi_t(\delta q, \delta p) = (\delta q + t\delta p, \delta p).$$

Hence,

$$\omega(D\phi_t(\delta q, \delta p)) = \langle p, \delta q + t\delta p\rangle = \langle p, \delta q\rangle = \omega((\delta q, \delta p)).$$

It remains to see what happens at a collision. First of all, note that a curve with tangent vector $(\delta q, \delta p)$ in general consists of trajectories that collide at different times. We want then to flow each point along the flow direction the exact amount that makes the curve collide simultaneously. This means that the new curve will have the tangent vector $(\widetilde{\delta q}, \widetilde{\delta p}) = (\delta q, \delta p) + \tau V$, for some τ determined by the condtion $\langle \widetilde{\delta q}, q\rangle = 0$. Clearly,

$$\omega((\widetilde{\delta q}, \widetilde{\delta p})) = \omega((\delta q, \delta p)) + \tau.$$

By (6.1.2) it follows that

$$\begin{aligned}
\widetilde{\delta q}_+ &= \widetilde{\delta q} \\
\widetilde{\delta p}_+ &= \widetilde{\delta p} - \frac{2}{r^2}\left[\langle \widetilde{\delta q}, p_-\rangle q + \langle q, \widetilde{\delta p}\rangle q + \langle q, p_-\rangle \widetilde{\delta q}\right].
\end{aligned}$$

Thus, using (6.1.2) again,

$$\omega((\widetilde{\delta q}_+, \widetilde{\delta p}_+)) = \langle p_+, \widetilde{\delta q}_+\rangle = \langle p_-, \delta q_+\rangle = \omega((\widetilde{\delta q}, \widetilde{\delta p})).$$

After the collision, we have to subtract the time shear that we introduced, and this yields

$$\omega((\delta q_+, \delta p_+)) = \omega((\delta q, \delta p))$$

which is the wanted time invariance.

6.1.3 Poincaré map

For many purposes it is useful to view the Sinai billiards as a symplectic map from a two dimensional domain to itself. Such a reduction is obtained via a general technique widely used in dynamical system: a Poincaré section (see 2.2). A Poincaré section consists in introducing some codimension one manifolds in the phase space X and then defining a map from such manifolds to themselves in such a way that to each point is associated its first return to the manifolds (if it exists). Let us be more concrete.

Historically the choice of the section to realize a Poincaré map as been based on $\partial\mathcal{B}$. In our case this consists of the boundary of the disk, that is a circle. Of course, it is also necessary to specify the velocity. Clearly there are two possibilities: one can consider velocities just before collision, which means $\langle x, v\rangle \leq 0$, (this is the Poincaré map from before collision to before collision) or one can consider the velocity just after collision, meaning $\langle x, v\rangle \geq 0$, (that is the Poincaré map from just after collision to just after the next collision). The two choices are equivalent, let us make the second.

If we define the velocity by the angle φ between v and the tangent (directed clockwise) to the disk at the collision point, then the phase space is $\mathcal{M} = S^1 \times [0, \pi]$.

We can then define a map $T : \mathcal{M} \to \mathcal{M}$ in the following way: for each $\xi \in \mathcal{M}$, let $T\xi$ be the point just after the next reflection (if such a reflection exists). Note that, if no reflections would occur, almost all the trajectories would fill $\mathbb{T}^2$ densely,[4] hence T is defined almost everywhere.

It is natural to use as coordinate on the boundary the distance s, computed counterclockwise along the circle, from a given point. If we want to compute the induced invariant measure on the Poincaré section $\mathcal{M}$, we can to introduce the change of coordinates $\Xi : \mathcal{M} \times [0,\delta] \to X$ defined by

$$\Xi(s,\varphi,t) = (rn(sr^{-1}) + v(sr^{-1}+\varphi)t, sr^{-1}+\varphi-\frac{\pi}{2}),$$

where $n(\theta) = (\cos\theta, \sin\theta)$, $v(\theta) = (\sin\theta, -\cos\theta)$. In this coordinates a point is determined by its collision data (s,φ) and the time t past from the last collision.

A direct computation shows that

$$\begin{aligned}
\det \Xi &= \begin{vmatrix} -v(sr^{-1}) + r^{-1}n(sr^{-1}+\varphi)t & n(sr^{-1}+\varphi) & v(sr^{-1}+\varphi) \\ r^{-1} & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} -v(sr^{-1}) & n(sr^{-1}+\varphi) & v(sr^{-1}+\varphi) \\ 0 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} -v(sr^{-1}) & v(sr^{-1}+\varphi) \end{vmatrix} = \sin\varphi.
\end{aligned}$$

So, given a set $A \subset \mathcal{M}$, calling $A_\varepsilon = \cup_{t=0}^{\varepsilon} \phi^t A$,

$$\mu(A) := \frac{1}{\varepsilon} m(A_\varepsilon) = \frac{1}{\varepsilon}\int_{A_\varepsilon} |det(\Xi)| ds d\varphi dt = \int_A \sin\varphi ds d\varphi.$$

Thus $d\mu = \sin\varphi ds d\varphi$ and $(\mathcal{M}, T, \mu)$ is a Dynamical Systems.

It is interesting to notice that μ becomes degenerate for $\varphi \in \{0,\pi\}$, which correspond to tangent collisions. Another annoying feature of the above choice is that some trajectories never meet the boundary of the disk (for example consider the initial condition $x = (1,0)$, $v = (0,1)$) and other will travel an arbitrarily long time before the next collision.[5] These facts, although not catastrophic, may look unpleasant to someone. It is therefore relevant to notice that there are several other possible choices for the Poincaré section, each one with is own advantages and disadvantages. Let us see a couple of them.

Consider the fundamental domain $Q = [-\frac{1}{2}, \frac{1}{2}]^2$ of $\mathbb{T}^2$, choose ∂Q as the basis of the Poincaré section. Of course, ∂Q is not a smooth manifold (it consists of four lines). This problem is easily solved by extending the

[4] Since for almost all velocities v we would have an irrational translation on $\mathbb{T}^2$.

[5] This property is called *infinite horizon*. We will discuss it further in the sequel.

concept of Poincaré section to the case in which the section Σ is a finite (or even countable) union of smooth manifolds; the reader can see that this generalization is indeed immediate. This section has the advantage of a simple structure, that there is a maximal time from Σ to itself, yet it does not solve the problem of the degeneration of the measure. Here, we also have problems with the trajectories that meet the section at very small angles.

To overcome such a problem one can choose the section $\Sigma \times [\delta, \pi - \delta]$. It is easy to see that if δ is chosen small enough then the only effect is to skip at most one crossing of the boundary Σ.

We will keep using the relation between the two dynamical systems (X, m, ϕ_t) and $(\mathcal{M}, \mu, T)$. In particular it is convenient to define the cone family on all $\mathcal{T}X$ instead that only on $\mathcal{T}\mathcal{M}$. We will see that an invariant cone family on $\mathcal{T}X$ induces an invariant cone family on $\mathcal{T}\mathcal{M}$.

6.1.4 Singularity manifolds

In this subsection we will study more precisely the singularities of the system and we will verify that they belong to the general setting developed in 4.3. We will consider two different Poincaré section to give the reader a more complete idea of the situation.

Let us start with the classical section *just after collision*. As already mentioned, the phase space is $\mathcal{M} = S^1 \times [0, \pi]$. Clearly, the only singularities of the map correspond to coordinates where the next collision is a tangent one. To analyze such a pathology, it is more convenient to look at the billiard on the universal covering of the torus. In such a covering, the obstacles will form a lattice and the particles moves along a straight line between collisions.[6]

The particle with coordinates (s, φ), just after collision, will move in the direction $v(r^{-1}s + \varphi)$ with unit speed.[7] Hence, if $C \in \mathbb{R}^2$ is the coordinate of the center of the obstacle with which the next collision will take place, the condition for a tangent collision reads

$$rn(r^{-1}s) + tv(r^{-1}s + \varphi) = C \pm rn(r^{-1}s + \varphi). \tag{6.1.3}$$

Where $t = t(s, \varphi)$ is the collision time. From (6.1.3) we can immediately extract two interesting informations multipling it by $n(r^{-1}s + \varphi)$ and

[6] This trick is very similar to the one used at the beginning of the chapter to discuss rectangular billiards, only now we take advantage of the periodicity of the torus rather than the invariant properties of the domain with respect to the reflections.

[7] Remember the convention $n(\theta) := (\cos\theta, \sin\theta)$ and $v(\theta) := (\sin\theta, -\cos\theta)$.

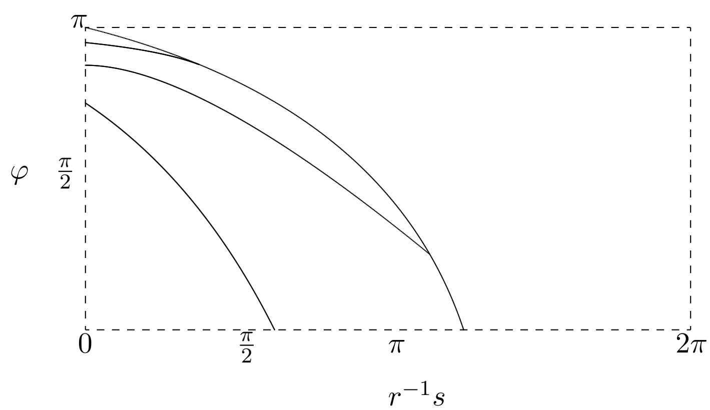


Figure 6.2: Few discontinuity lines in the Poicaré map

$v(r^{-1}s + \varphi)$ respectively

$$\langle C, v(r^{-1}s + \varphi)\rangle = t + r\sin\varphi > 0$$
$$F(s,\varphi) := \langle C, n(r^{-1}s + \varphi)\rangle - r\cos\varphi \pm r = 0$$

Taking the derivative of F with respect to φ we get

$$-r\sin\varphi + \langle C, v(r^{-1}s + \varphi)\rangle = t > 0,$$

thus we can apply the implicit function theorem and conclude that the manifold corresponding to this discontinuity can be represented as the graph of a function $\varphi(s)$. In addition,

$$\frac{d\varphi}{ds} = -(\frac{1}{r} + \frac{\sin\varphi}{t}) < 0. \tag{6.1.4}$$

Since there are infinitely many obstacles with which the next collision can take place, there must be countably many discontinuity line (some of them are schematically represented in figure 6.2)

To analize the preimages of the boundary of the section one can proced in analogy with what we have done before, equation (6.1.3) in this case beomes

$$rn(r^{-1}s) + tv(r^{-1}s + \varphi) = C \pm rn(r^{-1}s + \varphi \pm \delta). \tag{6.1.5}$$

From (6.1.5) we obtain

$$\frac{d\varphi}{ds} = -(\frac{1}{r} + \frac{\sin\varphi}{t + r\sin\delta}) < 0. \tag{6.1.6}$$

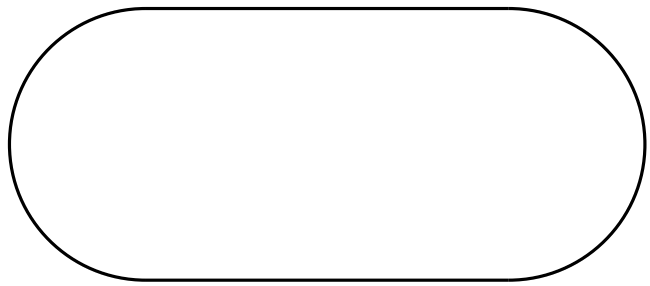


Figure 6.3: Bunimovich stadium

Clearly, the map is smooth up to, and including, this type of discontinuity, not so for the tangencies. In fact, it is easy to verify that the map is continuously crossing a tangency line (see Problem 6.7) but we will see immediately that it is not differentiable. We will see in section 7.3 (see in particular fromulae (7.3.1) and (7.3.2)) that if the next collision takes place with an angle $\varphi \notin [\delta, \pi - \delta]$, then calling τ_1 the time up to the tangent collision and τ_2 the time from the tangent collision to the next, we have the formula

$$DT = \begin{pmatrix} 1 + \frac{2\tau_1}{r \sin\varphi} & \frac{2}{r\sin\varphi} \\ \tau_2(1 + \frac{2\tau_1}{r\sin\varphi}) & 1 + \frac{2\tau_2}{r\sin\varphi} \end{pmatrix}.$$

Clearly, the norm of DT is bounded by a constant time $\frac{1}{\sin\varphi}$ (if in doubt do Problem 6.8). Now, if a point has distance ε from the singularity line, it will land at a distance $\sqrt{\varepsilon}$ from the tangency, which means that there exists a constant $c_t > 0$ such that, calling $\mathcal{S}$ the singularity line and ξ the point

$$|\sin\varphi| \geq c_t \sqrt{d(\xi, \mathcal{S})}.$$

This means that the Derivative blows up only as a square root getting close to the singularity. By similar considerations, it is possible to verify also that the second derivative blows up polynomially (see Problem 6.9).

6.2 Bunimovich Stadium

These billiards have been introduced [10] and first studied [11] by Bunimovich. In this case the table of the billiard is a convex subset of $\mathbb{R}^2$. The simplest, and original, one consists in two half circles joined by two straight lines (see Figure 6.3).

The name "stadium" is due to the shape of the domain $\mathbb{B}$ in which the motion takes place. The only difference is that now the curvature of the boundary is either zero (collisions against the straight segments) or negative (collision against the half circles).

6.2.1 Flow

We have seen that the flow in a square or in a circle is well defined and rather regular. Clearly the only relevant discontinuity in the Bunimovich Billiard arise when a trajectory hit the joining between the circumference and the straight lines.

6.3 Different Tables, different games

Let us start with a bit of classification.

Definition 17 *Here are some standard classes of billiards:*

- dispersing *billiards are billiards with the boundary $\partial\mathbb{B}$ of the table is a finite union of stricly convex manifolds with boundary (this are often called Sinai billiards as well)*
- semi-dispersing *billiards are billiards with the boundary $\partial\mathbb{B}$ of the table is a finite union of stricly convex manifolds with boundary*
- convex *billiards are billiards where the tale $\mathbb{B}$ is a convex set.*

The remainder of the section is devoted to a more explicit description of several concrete examples of the above cases.

6.3.1 Dispersing

We have already seen the standard Sinai billiard in section 6.1. In general several convex obstacles may be present and they are not necessarily disjoint. One main issue in this class of billiards is the distinction between finite and infinite horizon. Finite Horizon means that there is a maximal time after which a collision must take place, infinite horizon means that there exists trajectories that never experience a collision.[8] The relevance of such a concept stems from the fact that orbit with no collision have zero Lyapunov exponents, hence the corresponding billiards cannot be uniformly hyperbolic.

[8]Note that the other possibility (all the trajectories experience a collision in finite time, but there does not exist an upper bound for such a time) cannot take place (see Problem 6.11).

Infinite Horizon

As already mentioned we have already seen the prototypical example in this class, yet it may be instructive to analize its properties a bit further. Consider the Sinai billiard described in section 6.1 and let r_1 the radius of the obstacle. Clearly it is necessary $r_1 < 1/2$ to have no self intersections of the obstacle. It is also obvious that if $r_1 < 1/2$ then there are trajecotirs that never collide. Let us study such trajectories a bit more in detail. First of all, since the sistem has a square simmetry, it is enough to consider trajectories with velocity in the first half of the first quadrant, i.e. velocities parallel to the directions $(1, \omega)$ with $\omega \in [0, 1]$.[9]

Let us consider the motion with no obstacle (a traslation on the torus) and see if there are trajectories that never enter in the region $\|(x, y)\| \leq r_1$. Clearly such trajetories are trajectories for the billiard systems as well and precisely the trajecotries that never experince a collision. For such trajectories it is particularly convenient to consider the poincarè section determined by the line $S : \{x = -1/2\}$. If we look at the motion only when the particle intersects such a line we have that the corresponding map is given by $Ty = y + \omega \mod 1$, that is a rotation by ω of the circle $(-1/2, 1/2]$. If $\omega \notin \mathcal{Q}$ then the map T is ergodic and this means that the trajectory will eventually collide. On the contrary, if $\omega = p/q$, $p, q \in \mathbb{N}$ and with no common divisors, then all the orbits will be periodic of period q and it may be possible that some of them never collide.

Notice that a point in S with velocity parallel to $(1, \omega)$ will experience before a collision before meeting S again only if $y \in [-\omega/2 - r_1\sqrt{1+\omega^2}, -\omega/2 + r_1\sqrt{1+\omega^2}]$. On the other hand, since the orbit of the point y has lenght q and because it is restricted to points of the type $y + n/q \mod 1$, which are exactly q, it follows that the orbit consists exactly of all such points. Accordingly, the orbit can avoid only intervals of size less than $1/q$. We can then conclude that there are orbits of the type p/q that never collide if and only if

$$2r_1\sqrt{1 + \frac{p^2}{q^2}} < \frac{1}{q}. \tag{6.3.7}$$

If $q = 1$ then for $p = 0$, we have the already know result $r_1 < 1/2$; for $p = 1$ there can be no collisions only if $r_1 < \frac{1}{2\sqrt{2}}$. For $q \leq 2$ there are always collisions if $r_1 > [2\sqrt{5}]^{-1}$.

[9]If this it is not clear, read again the discussion of polygonal billiards in section 6.0.1.

Finite Horizon

The simplest case of Sinai billiard with a finite horizon is obtained by employing two circular obstacles. We have thus a torus of size one together with a circular obstacle at the point $(0, 0)$ with radius r_1 and a circular obstacle at the point $(1/2, 1/2)$ with radius r_2.[10] Clearly

$$r_1 + r_2 < \frac{1}{\sqrt{2}}$$

in order for the obstacels not to intersect each other. By the discussion of the infinite horizon case it follows that we can choose $r_1 > 1/(2\sqrt{2})$ and $0 < r_2 < 1/\sqrt{2} - r_1$ to have a Sinai billiard with disjoint obstacles and finite horizon. For example one could choose $r_1 = 3/7$ and $r_2 = 1/4$.

In the following we will need a more in depth undertanding of the above model. Let us consider a regularized Poincarè section of the type introduced in section 6.1.4 and discuss the structure of the singularity lines for such a section.

The first step is to understand multiple consecutive tangencies. Let us start with a double tangency, the first of which is with the central obstacle. By simmetry, one can limit the analysis to the case in which the second takes place with the upper right copy of the obstacle. The position of the particle at time t is given by $r_1 n(\theta) + v(\theta)t$, where $n(\theta) = (\cos\theta, \sin\theta)$ and $v(\theta) = (\sin\theta, -\cos\theta)$, so we have the next two equations

$$\begin{aligned} &\|r_1 n(\theta) + v(\theta)t - p\| = r_2 \\ &\langle r_1 n(\theta) + v(\theta)t - p, v(\theta)\rangle = 0, \end{aligned}$$

where $p = (1/2, 1/2)$ are the coordiantes of the center of teh second obstacle (of course we are working in the universal covering). The first equation determine the value of t for which the second collision takes place while the second impose that the collision is tangent. Solving the above equations yields

$$\frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{4} - \theta\right) = \langle n(\theta), p\rangle = r_1 \pm r_2.$$

Accordingly we have four solutions: $\frac{\pi}{4} - \theta = \pm(\cos^{-1}\sqrt{2}(r_1 \pm r_2))$. In fact, only two are really relevant since the other two are obtained by simmetry around the line joining the two centers. It remains to check that the above tajectories do not intersect any other obstacle between the two

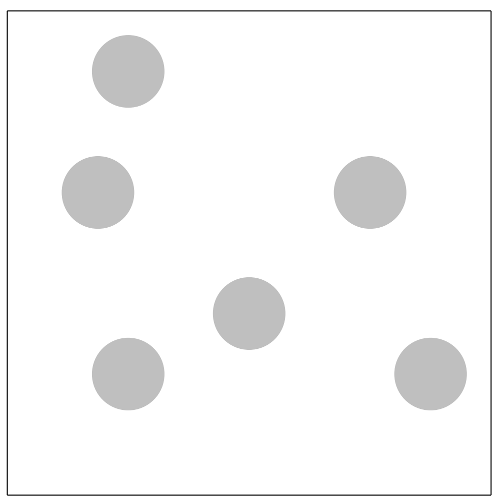
[10] Remember that the coordinates are in the universal covering of the torus and that the points $(1/2, -1/2)$, $(1/2, 1/2)$, $(-1/2, 1/2)$, $(-1/2, -1/2)$ are identified.

tangencies. In fact, it turns out that the trajectories of the type $\frac{\pi}{4} - \theta = \pm(\cos^{-1}\sqrt{2}(r_1 - r_2))$ have a tangent collision with the central scatterer before colliding with the corner one. It is then easy to see that there can be at most four consecutive tangencies after that the next collision will take place with an angle of more than 70 degree.

6.4 A gas with two particles

We have already mentioned that the motions of several discs or balls that collide elastically among themselves are an example of billiards. The study of such models goes back at least to Boltzmann, who proposed studying the properties of a gas, imagining that it consists of balls colliding elastically.

We start by looking at the simplest possible case.



A two dimensional gas of particles in a box

The (seemingly ridiculous) simplest case is a gas of two particles in two dimensions. For simplicity, let us consider two particles of radius $r < \frac{1}{2}$ in a torus of size one. Let $x_1, x_2 \in \mathbb{T}^2$ be the coordinate of the center of the disks, the velocity changes at collision according to the law

$$\begin{cases} v_1^+ = v_1^- - \langle n, v_2^- - v_1^- \rangle n \\ v_2^+ = v_2^- + \langle n, v_2^- - v_1^- \rangle n \end{cases} \tag{6.4.8}$$

where n is a unit vector in the direction $x_2 - x_1$.[11]

Here, there are three integrals of motion: the energy $E = \frac{1}{2}(\|v_1\|^2 + \|v_2\|^2)$ and the total momentum $P = v_1 + v_2$. Thus, if we want to obtain an ergodic system, we have to reduce the system. We will then consider that phase spaces

$$X_{E,P} = \left\{ (x_1, x_2, v_1, v_2) \in \mathbb{T}^4 \times \mathbb{R}^4 \;\middle|\; \frac{1}{2}(\|v_1\|^2 + \|v_2\|^2) = E;\; v_1 + v_2 = P \right\}.$$

Since, in the velocity space, the previous conditions correspond to the intersection between the surface of a four-dimensional sphere (S^3) and a two-dimensional linear space, the velocity vectors $(v_1 + v_2)$ are contained in a one-dimensional circle. Thus, topologically, $X_{E,P} = \mathbb{T}^4 \times S^1$.[12] It is then natural to choose an angle θ as coordinate on S^1, moreover, since

$$2E = \|v_1\|^2 + \|v_2\|^2 = \frac{1}{2}\|v_1 - v_2\|^2 + \frac{1}{2}\|P\|^2,$$

it is hard to resist setting $v_2 - v_1 = v(\theta)$.[13] Hence,

$$\begin{cases} v_1 = \frac{1}{2}(P - v(\theta)) \\ v_2 = \frac{1}{2}(P + v(\theta)). \end{cases}$$

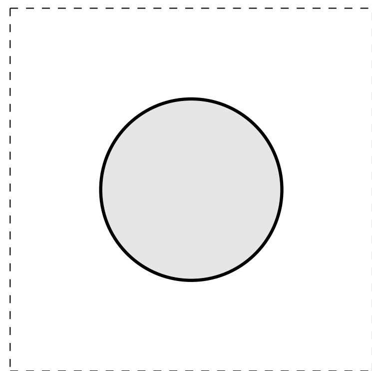
The free motion is then given by

$$\begin{cases} x_1(t) = x_1(0) + \frac{1}{2}(P - v(\theta))t \\ x_2(t) = x_2(0) + \frac{1}{2}(P + v(\theta))t. \end{cases}$$

Accordingly,

$$\begin{cases} x_1(t) + x_2(t) = x_1(0) + x_2(0) + Pt \mod 1 \\ x_2(t) - x_1(t) = x_2(0) - x_1(0) + v(\theta)t \mod 1. \end{cases}$$

It is then clear the need to introduce the two new variables $Q = x_1 + x_2$ and $\xi = x_2 - x_1$. The variable Q performs a translation on the torus, such a motions are completely understood, and we can then disregard it. The only relevant motion is the one in the variables (ξ, θ). The reduced phase space is then $\mathcal{B} \times S^1$ where $\mathcal{B} = \mathbb{T}^2 \setminus \{\|\xi\| \leq 2r\}$, that is, the torus minus a disk of radius $2r$. The domain $\mathcal{B}$ is represented in the next Figure and, apart from the different choices of the fundamental domain, it corresponds exactly to



The Siani billiard $\mathcal{B}$

Figure 6.4: Sinai Billiard with infinite horizon

the simplest Sinai billiard. The free motion corresponds to the free motion of a point as well, while at collision, from (6.4.8), we have

$$v(\theta^+) = v(\theta^-) - 2\left\langle \frac{\xi}{2r}, v(\theta^-)\right\rangle v(\theta^-)$$

that is exactly the elastic reflection from the disk!

It is then natural to consider the general problem of a particle moving in a region with reflecting boundary conditions. Let $\mathcal{B} \subset \mathbb{R}^d$ (or $\mathcal{B} \subset \mathbb{T}^d$) be the region and suppose that the boundary $\partial\mathcal{B}$ is made of finitely many smooth manifolds. Calling $(x, v) \in \mathcal{B} \times \mathbb{R}^d$ the position and the velocity, respectively, the motion inside $\mathcal{B}$ is described by a free flow

$$\phi_t(x, v) = (x + vt, v), \tag{6.4.9}$$

When $x \in \partial\mathcal{B}$, a collision takes place. If $n \in \mathbb{R}^d$, $\|n\| = 1$, is the normal to $\partial\mathcal{B}$ at x, then, calling v_- and v_+ the velocities before and after collision, respectively, the elastic collision is described by

$$v_+ = v_- - 2\langle v_-, n\rangle n.$$

[11] To be precise $x_2 - x_1$ has no meaning since $\mathbb{T}^2$ it is not a linear space. Yet, at collision, the distance between the two disks is $2r$, so the global structure of $\mathbb{T}^2$ is irrelevant, and we can safely confuse it with a piece of $\mathbb{R}^2$.

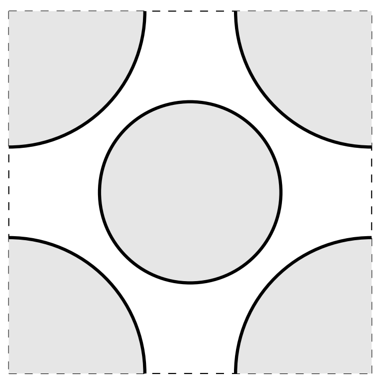
[12] Of course, we are considering only the cases $E \neq 0$.

[13] As usual $v(\theta) = (\sin\theta, \cos\theta)$.

Remark 6.4.1 *Here, I will provide a few ideas on billiards and hyperbolicity. This should allow the reader to be able to easily takle a more complete account of the theory (in particular [17]).*

6.5 Some Billiard tables

In the two-dimensional case, there are many possible billiards tables that have been studied. The two most famous are the Sinai Billiard and the Bunimovich Stadium.



Sinai Billiard with finite horizon

Bunimovich stadium

Further interesting billiard tables can be found in [75, 14, 17] and references therein.

Problems

6.1 Check that the maps ϕ^t generated by a billiard flow are symplectic. (Hint: It is obvious for the free flow, but it remains to be checked for the reflections. This can be done by using formulae like (7.3.3)).

6.2 Given a rectangular box $\mathbb{B}$, with its sides labeled by $\{1, 2, 3, 4\}$ and let $R_i(\mathbb{B})$ be the reflection with respect to the side i of the box $\mathbb{B}$.[14] Let R_0 be the identity. Consider $G = \cup_{n=0}^{\infty}\{0, 1, 2, 3, 4\}^n$, if $g \in G$ then we define $R_g(\mathbb{B}) = R_{g_1}(\cdots R_{g-n}(\mathbb{B})\cdots)$ and, for each $g^i \in \{0, 1, 2, 3, 4\}$, $i \in \{1, 2\}$, $g = g^2 \circ g^1 \in \{0, 1, 2, 3, 4\}^{n_1+n_2}$, is defined by $g_k = g^1_k$ for $k \leq n_1$ and $g_k = g^2_{n-k}$, for $k > n_1$. Verify that $R_{g^2}(R_{g^1}(\mathbb{B})) = R_g(\mathbb{B})$. Introduce the equivalence relation $g_1 \sim g_2$ iff $R_{g^1}(\mathbb{B}) = R_{g^2}(\mathbb{B})$. Let $\tilde{G}$ be the collection of the equivalence classes. Verify the $\tilde{G}$ is a commutative group with respect to the operation $\circ$. (hint: Note that the geometrical meaning is simply that the final position of the box after a certain number of reflections does not depend on the order of the reflections. Clearly, it suffices to check such a property for two reflections.)

6.3 Study the motion in a triangular billiard when the angles defining the triangle are all rational multiples of π. (Hint: use reflections again)

6.4 Study the motion in an elliptical billiard. (Hint: Verify that there exists an integral of motion.)

6.5 Verify that the caustics correspond to a two dimensional torus.

6.6 Find a change of variable that transforms the symplectic form in a regularized boundary section in the standard symplectic form.

6.7 Verify that, in a regularized boundary section, the map is continuous across a singularity line corresponding to a tangency.

6.8 Prove that, given an $n \times n$ matrix A the norm $\|A\| := \sup_{v\in\mathbb{R}^n} \frac{\|Av\|}{\|v\|}$ where $\|v\| := \sqrt{\sum_n v_n^2}$ satisfies

$$\|A\| \leq \text{constant} \max_{ij} |A_{i,j}|$$

and compute explicitely the optimal constant.

[14] The labels attached to the sides of the reflected boxes are the ones obtained naturally from the old ones.

6.9 Determine the rate at which the second derivative explode as one gets near a tangency singularity in the Sinai billiard with one circular obstacle in the torus.

6.10 Compute the number of collisions in a convex angle.

6.11 Show that in a Sinai billiard on $\mathbb{T}^2$ for which there exist trajectories that spend an arbitrarily long time without colliding, there must exist trajectories that never collide. (Hint: some continuity...)

6.12 Study two disks with different masses.

6.13 Prove that the Poincarè map for the Sinai billiard is piecewise Hölder of Hölder exponent $\frac{1}{2}$.

Chapter 7

Billiards (higher dimensions)

Let $\mathbb{B} \subset \mathbb{R}^{d_1} \times \mathbb{T}^{d_2}$, $d_1 + d_2 = d \geq 2$,[1] be the region in which the point is allowed to move and suppose that $\partial\mathbb{B}$ is a finite union of smooth manifolds with boundary $\mathcal{M} := \{M_i\}_{i=1}^{K_*}$. We assume that the elements of $\mathcal{M}$ are pairwise transversal and their intersections consists of finitely many smooth submanifolds. Since the energy is conserved, the motion can be seen, by possibly rescaling time, as a flow ϕ_t on the unitary tangent bundle of $T_1\mathbb{B}$, that is $\mathbb{B} \times S^{d-1}$, where $S^{d-1} = \{v \in \mathbb{R}^d \ : \ \|v\| = 1\}$. For $q \in M_i$ let $n_i(q)$ be the unitary normal to M_i pointing toward the interior of $\mathbb{B}$. Then, calling $(q,p) \in T_1\mathbb{B}$ the coordinates of the point, a collision happens at time t if $q(t) \in M_i$ and $\langle p(t), n(q(t))\rangle \leq 0$. Using $q(t_+), p(t_+))$ for the coordinates just after the collision we can write

$$\begin{aligned}
&q(t_+) = q(t) \\
&p(t_+) = p(t) - 2\langle p(t), n_i(q(t))\rangle n_i(q(t)).
\end{aligned}$$

While, in between collisions, the flow is given by

$$(q(t+s), p(t+s)) = (q(t) + sp(t), p(t)).$$

We call ϕ_t the above flow. Note ϕ_t is not defined at $x = (q,p)$ if $q \in M_i \cap M_j$, for some $i \neq j$. Let

$$\mathcal{S}_n = \{(q,p) \in T_1\mathbb{B} \ : \ \exists \{k_i\}_{i=1}^n, i \neq j \to k_i \neq k_j \text{ such that } q \in \cap_{i=1}^n M_{k_i}\}.$$

It is then natural to define

$$\mathcal{S}_* = \{x \in \mathbb{T}_1\mathbb{B} \ : \ \exists \tau \in \mathbb{R} \text{ such that } \phi_\tau(x) \in \mathcal{S}_0\}.$$

[1] We use the convention $\mathbb{R}^{d_1} \times \mathbb{T}^{d_2} = \mathbb{R}^d$ if $d_2 = 0$ and $\mathbb{R}^{d_1} \times \mathbb{T}^{d_2} = \mathbb{T}^d$ if $d_1 = 0$.

Remark 7.0.1 *In the following, we always assume that* $\mathbb{B}$ *has a finite Lebesgue measure.*

Lemma 7.0.2 *Let* m *be the product of the (normalized) Lebesgue measure on* $\mathbb{B}$ *times the Haar measure on* S^{d-1}*, then* $m(S_*) = 0$ *and* m *is invariant by the billiard flow.*

PROOF. Note that $m(S_0) = 0$, by the transversality of the manifolds in $\mathcal{M}$. Then .... □

Let $\mathcal{K}$ be the curvature of the manifolds in $\mathcal{M}$. We assume that the billiard is *dispersing*, that is, $\mathcal{K} \geq 0$ and the complement is locally convex.

Lemma 7.0.3 *Assume* $\mathbb{B}$ *to be compact. There exists* $C(\mathbb{B}) \in \mathbb{N}$ *such that the billiard flow can have at most* $C(\mathbb{B}) \in \mathbb{N}$ *collisions per unit time.*

PROOF. By the transversality asumption there exists $\eta \in \mathcal{C}^0(\partial\mathbb{B}, S^{d-1})$ such that $\langle \eta(q), n_i(q) \rangle > 0$ for each i such that $q \in M_i$. By compactness there exists $\beta > 0$ such that $\langle \eta(q), n_i \rangle \geq \beta$ for all $q \in \partial\mathbb{B}$.
Let $S_n(\varepsilon) = \{q \in \mathbb{B} \ : \ B_\varepsilon(q) \cap S_n \neq \emptyset\}$.[2] Fix $\varepsilon > 0$ small enough and $a > 1$ large enough so that for each $q \in S_m$, the set $B_{a^m\varepsilon}(q) \cap \{S_m \setminus \cup_{n>m} S_n(a^n\varepsilon)\}$ is connected. This is possible since, by transversality, the distance between the manifolds that intersect at a point of S_l grows at least linearly with the distance from S_l. Next, we consider a covering $\mathcal{B}$ of $\mathbb{B}$ by balls of radius ε. If $q \notin \mathcal{S}_1(\varepsilon)$, then it will flow for a time ε before it can experience a collision. If $q \in S_m(a^m\varepsilon) \setminus \cup_{n>m} S_n(a^n\varepsilon)$, then there exists $\bar{q} \in \mathcal{S}_m$ such that $q \in B_{a^m\varepsilon}(\bar{q})$ and $B_{a^m\varepsilon}(\bar{q}) \cap \mathcal{S}_m$ is connected. Let $n_i = n(\bar{q})$ be the unit normal to M_i at $\bar{q}$, without loss of generality, we can assume that the manifolds meeting at $\bar{q}$ are $\{M_1, \dots, M_m\}$. If $q \notin B_{a^{m-1}\varepsilon}(\bar{q})$ then the particle must experience a free flight of at least $c\varepsilon$, for some $c > 0$, either before or after the first collision. Otherwise,the particle cannot have more than ε^{-2} collisions before existing $B_{a^m\varepsilon}(\bar{q})$. Suppose otherwise and let $\{M_{i_1}, \dots, M_{i_k}\}$ be the walls with which the first ε^{-1} collisions take place. If $\langle p, n_{i_j} \rangle \leq A\varepsilon$ for all $j \in \{1, \dots, k\}$, then

let $M_* = \mathcal{S}_m \cap B_{a^{m-1}\varepsilon}(\bar{q})$. By hypothesis, M_* is a manifold, let $\kappa \in \mathbb{N}$ be its dimension. Then there exists a $\kappa + 1$-dimensional linear subspace P such that $P \cap B_{a^{m-1}\varepsilon}(\bar{q}) \subset \mathbb{B}$.

.... Boh, must think ....

□

[2] We use the notation $B_r(q) = \{z \in \mathbb{B} \ : \ d(z, q) < r\}$.

7.1 Poincarè map

Let us consider a collision between two boundary manifolds M_1, M_2. That is, a trajectory between $(q, p) \in T_1M_1$ and $(\bar{q}, \bar{p}) \in T_1M_2$. Let (U_i, ϕ_i) be a chart in M_i such that $\phi_1(q) = 0 = \phi_2(\bar{q})$. Let $\gamma_i = \phi_i^{-1}$, then

$$\gamma_2(0) = \gamma_1(0) + p\tau_0$$

for some function $\tau_0 \in \mathbb{R}_+$. Let $n(s)$ the unit normal to M_1 at $\gamma_1(s)$ and $\bar{n}(\bar{s})$ the unit normal to M_2 at $\gamma_2(\bar{s})$. For nearby points the map T, from just after collision to just after the next collision $(\bar{s}, \bar{p}) = T(s, p)$ is defiend by

$$\begin{aligned}
&\gamma_2(\bar{s}) = \gamma_1(s) + p\tau(s, p)\\
&\bar{p} = p - 2\langle p, \bar{n}(\bar{s})\rangle \bar{n}(\bar{s})\\
&\cos\varphi = \langle p, n(s)\rangle\\
&\cos\varphi_1 = \langle \bar{p}, \bar{n}(\bar{s})\rangle = -\langle p, \bar{n}(\bar{s})\rangle.
\end{aligned}$$

Differentiating and setting $p = \cos\varphi n(s) + \sin\varphi\eta$, where $\langle \eta, n(s)\rangle = 0$, $\|\eta\| = 1$, we obtain

$$\begin{aligned}
&\sin\varphi_1\partial_\varphi\varphi_1 = \partial_\varphi\langle p, \bar{n}(s_1)\rangle = -\sin\varphi\langle n(s), \bar{n}(\bar{s})\rangle\\
&\sin\varphi_1\partial_s\varphi_1 = \cos\varphi\langle K\partial_s\gamma, \bar{n}(\bar{s})\rangle + \langle p, \bar{K}(\bar{s})\partial_{\bar{s}}\gamma_2(\bar{s})\rangle\partial_s\bar{s}\\
&\partial_{\bar{s}}\gamma_2(\bar{s})\partial_s\bar{s} = \partial_s\gamma + \cos\varphi K(s)\partial_s\gamma\tau + p\partial_s\tau
\end{aligned}$$

. next we multiply the last by a vector orthogonal to p as in CM page 34 .

7.2 The n particle gas in $\mathbb{T}^d$

Let us briefly discuss the case of n particles in $\mathbb{T}^d$.

..................show that it fits in the above class ....

Two new possible issues arise

a) the collision of more than two balls

b) the possibility of infinitely many collisions in finite time

If (a) happens, then it is not clear how to continue the dynamics. Hence, we must consider them as a singularity of the map in the same way we have done for tangent collisions. If m balls collide at the same time, then we have a codimension $m - 1$ singularity, thus for $m = 3$, we have the same codimension as tangent collisions, while for $m > 3$, the codimension is higher. The situation (b) instead cannot happen, although the proof is not straightforward, see [15].

7.3 Collision map and Jacobi fields

To compute, in general, the collision map it is helpful to introduce appropriate coordinates in $\mathcal{T}X$. A very interesting choice is constitute by the *Jacobi fields*.[3] Let X_- be the set of configurations just before collision. For each $(x,\, v) \in X \backslash X_-$ there exists $\delta > 0$ such that

$$\phi_t(x,\, v) = (x + vt,\, v) \quad 0 \leq t \leq \delta.$$

Let us consider the curve in $\mathcal{X}$

$$\xi(\varepsilon) = (x(\varepsilon),\, v(\varepsilon)),$$

with $\xi(0) = (x,\, v)$ and $\|v(\varepsilon)\| = 1$.

For each t such that $\phi_t(\xi(0)) \notin X_-$, let

$$\xi(\varepsilon,\, t) = (x(\varepsilon,\, t),\, v(\varepsilon,\, t)) = \phi_t(\xi(\varepsilon)).$$

The Jacobi field $J(t)$ is defined by

$$J(t) \equiv \left.\frac{\partial x}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

Note that, since $x(0,\, t) \notin X_-$, for $s < \delta$

$$\xi(\varepsilon,\, t+s) = \xi(\varepsilon,\, t) + (v(\varepsilon,\, t)s,\, 0),$$

so

$$J'(t) = \frac{dJ(t)}{dt} = \left.\frac{\partial v(\varepsilon,\, t)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

That is, $(J(t),\, J'(t)) = d\phi_t \xi'(0)$.

At each point $\xi = (x,\, v) \in X$ we choose the following base for $\mathcal{T}_\xi X$:[4]

$$\eta_0 = (v,\, 0); \;\; \eta_1 = (v^\perp,\, 0); \;\; \eta_2 = (0,\, v^\perp);$$

where $\|v^\perp\| = 1$, $\langle v,\, v^\perp \rangle = 0$.

---

[3] The Jacobi Fields are a widely used instrument in Riemannian geometry (see [26]) and have an important rôle in the study of Geodetic flows, although we will not insists on this aspect at present. Here they appear in a very simple form.

[4] Here $v^\perp = Jv$ with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

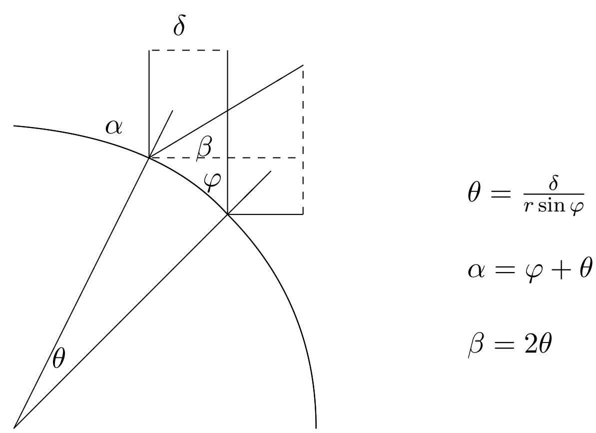


Figure 7.1: Collision

The vector η_0 corresponds to a family of trajectories along the flow direction, and it is clearly invariant; η_1 to a family of parallel trajectories and η_2 to a family of trajectories just after focusing. It is very useful the following graphic representation. We represent a tangent vector by drawing a curve that is tangent to it. A curve in $\mathcal{T}X$ is given by a base curve that describes the variation of the x coordinate equipped with a direction at each point (specified by an arrow), which shows how the velocity varies (see Figure 7.1).

A direct check shows that each vector η perpendicular to the flow direction will stay so (see Lemma 6.1.1), i.e.

$$\langle d\phi_t\eta,\, (v_t,\, 0)\rangle = \langle d\phi_t\eta,\, d\phi_t(v,\, 0)\rangle = \langle \eta,\, (v,\, 0)\rangle = 0.$$

So the free flow is described by

$$d\phi^t\eta_0 = \eta_0; \quad d\phi^t\eta_1 = \eta_1; \quad d\phi^t\eta_2 = \eta_2 + t\eta_1,$$

that is, in the above coordinates

$$d\phi^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}. \tag{7.3.1}$$

Let us see now what happens at a collision.

Let $x_0 \in \partial\mathbb{B}$ be the collision point and let $\xi_c = (x_0, v)$ be the configuration at the collision. We want to compute $R_\varepsilon := d_{\phi^{-\varepsilon}\xi_c}\phi^{2\varepsilon}$, that is the tangent map from just before to just after the collision. Clearly $R_\varepsilon\eta_0 = \eta_0$.

From the Figure 7.1 follows that, if $\gamma(s)$ is the curve associated to η_1 at the point $\phi^{-\varepsilon}\xi_c$,

$$d\phi^{2\varepsilon}\gamma(s) = \left(v_+^\perp\left[s+\varepsilon\frac{2s}{r\sin\varphi}\right], \frac{2s}{r\sin\varphi}\right) + \mathcal{O}(s^2)$$

where r is the radius of the osculating circle (that is the circle tangent to the boundary up to second order) which is the inverse of the curvature $K(x_0)$ of the boundary at the collision point.

The above equation means that

$$J(\varepsilon) = (1 + \frac{2\varepsilon K(x_0)}{\sin\varphi})v_+^\perp.$$

Accordingly, calling $R = \lim_{\varepsilon\to 0} R_\varepsilon$ the collision map, we have

$$R\eta_1 = \eta_1 + \frac{2K}{\sin\varphi}\eta_2; \quad R\eta_2 = \eta_2.$$

Hence,

$$DR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{2K}{\sin\varphi} \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.3.2}$$

The above computations provide the following formula for the derivative of the Poincaré section from the boundary of the obstacle, just after the collision, to the boundary of the obstacle just after the next collision

$$DT = \begin{pmatrix} 1 & \frac{2K}{\sin\varphi} \\ \tau & 1 + \frac{2\tau K}{\sin\varphi} \end{pmatrix}, \tag{7.3.3}$$

where τ is the flying time between the two collisions and φ the collision angle. Formula (7.3.3) is sometimes called *Benettin formula* (e.g., [41]).

7.3.1 An alternative derivation

Let us see how we can do the above computation in dimension $d \geq 2$ for a particle that collies against a sphere. The phase space is $M = \{(q,p) \in \mathbb{T}^d \times \mathbb{R}^d \;:\; \|q\| \geq r\,,\; \|p\| = 1\}$. Then a vector $(\delta q, \delta p) \in \mathbb{R}^{2d}$ is an element of TM only if $\langle p, \delta p\rangle = 0$. When there are no collisions, the tangent vector evolves a

$$\begin{aligned} \delta p(t) &= \delta p(0) \\ \delta q(t) &= \delta q(0) + t\delta p(0). \end{aligned}$$

To study what happens at collision, consider the case $\|q\| = r$ and $\langle q, p\rangle < 0$, that is, the particle just before the collision. To compute what happens at the collision, we proceed as in section 6.1.2. We shift the vector along the flow direction so as to have a vector tangent to the sphere, that is a vector that represents an infinitesimal family of trajectories that collide all at the same time. That is, we look for $\tau \in \mathbb{R}$ such that $(\widetilde{\delta q}, \widetilde{\delta p}) = (\delta q + \tau p, \delta p)$ is such that $\langle \delta q + \tau p, q\rangle = 0$. Hence

$$\tau = -\frac{\langle \delta q, q\rangle}{\langle p, q\rangle}.$$

To such a vector we can apply the derivative of the collision map (6.1.2)

$$\begin{aligned}
\widetilde{\delta q}^{\,+} &= \widetilde{\delta q} \\
\widetilde{\delta p}^{\,+} &= \widetilde{\delta p} - 2r^{-2}\langle \widetilde{\delta p}, q\rangle q - 2r^{-2}\langle p, \widetilde{\delta q}\rangle q - 2r^{-2}\langle p, q\rangle \widetilde{\delta q}
\end{aligned}$$

To conclude we have to shift back along the flow direction to determine the evolution of the original vector:

$$\begin{aligned}
\delta q^+ &= \widetilde{\delta q}^{\,+} - \tau p^+ \\
\delta p^+ &= \delta p - 2r^{-2}\langle \delta p, q\rangle q - 2r^{-2}\langle p, \widetilde{\delta q}\rangle q - 2r^{-2}\langle p, q\rangle \widetilde{\delta q}.
\end{aligned}$$

As we have already seen in section 6.1.2 we have $\langle \delta q^+, p^+\rangle = \langle \delta q, p\rangle$. We can then quotient along the flow direction, which is equivalent to considering only vectors such that $\langle \delta q, p\rangle = 0$.

To connect with the previous section, let us consider the case $d = 2$. Then we have the natural base $\eta_2 = (p^\perp, 0)$ and $\eta_3 = (0, p^\perp)$ where $\langle p^\perp, p\rangle = 0$ and $\|p^\perp\| = 1$. Then η_3 is mapped into η_3 and η_2 into $\eta_2 + \|\delta p^+\|\eta_3$. Since $\langle \widetilde{\delta q}, q\rangle = 0$, $\langle \delta q, p\rangle = 0$ we have $\|\delta q\|^2 = 1 + \frac{\cos^2\varphi}{\sin^2\varphi} = \frac{1}{\sin^2\varphi}$ and

$$\begin{aligned}
\|\delta p^+\|^2 &= 4r^{-2}\langle p, \widetilde{\delta q}\rangle^2 + 4r^{-4}\langle p, q\rangle^2 \|\widetilde{\delta q}\|^2 \\
&= 4r^{-2}\frac{\cos^2\varphi}{\sin^2\varphi} + 4r^{-2} = \frac{4}{r^2\sin^2\varphi}.
\end{aligned}$$

which agrees with (7.3.2).

Chapter 8

Hyperbolicity of Billiards

The issue of establishing hyperbolicity for billiards is a vast subject. Let us start with the basic examples.

8.1 Hyperbolicity of Sinai Billiard

As an example let us consider the Sinai Billiard depicted in Figure 6.4. Note that the system cannot be uniformly hyperbolic since there are trajectories that never hit the obstacle, and hence have clearly zero Lyapunov exponents. We define a cone family in the plane perpendicular to the flow direction $(v, 0)$, that is in the plane η_1, η_2, this plane is naturally isomorphic to the tangent space of $\mathcal{M}$ (just project along the flow direction) in each non-tangent point.

In the case in which no collision takes place, we have seen that the parallel family η_1 stays parallel, while the most divergent family (the vector η_2) becomes less divergent (a linear combination of η_1 and η_2 with positive coefficients). This means that the first quadrant (in the η_i coordinates goes into itself but the η_1 side stays put). Let us study what happens at a reflection. Any divergent family of trajectories will be divergent after the collision, and in particular, the parallel family will be strictly divergent. To be more precise the η_2 family will go into itself from just before to just after the collision, while the parallel one will be strictly divergent. Again the cone goes strictly inside itself but one side (the η_2 one this time) stays put. Nevertheless, the combination of free motion and reflection clearly sends the cone strictly inside itself.

Note that if a trajectory has a velocity with components with irrational ratios, then the flow without the obstacle is ergodic. This means that it is impossible that the trajectory does not hit the obstacle. Since the set of trajectories with velocities having components with rational ratios are of zero measure, it follows that almost all trajectories experience a collision. Hence, the billiard cocycle is eventually strictly monotone, and Wojtkowski's theorem applies. Accordingly, all the Lyapunov exponents are different from zero almost everywhere for the dynamical system $(\mathcal{M}, T, m)$.

8.2 Hyperbolicity of Bunimovich stadium

The naïve understanding of the previous example is that the obstacle acts as a defocusing mirror and thus makes the trajectories diverge, whereby creating instability. This idea was already present in Krylov work [44] and was considered the natural mechanism producing hyperbolicity. With this point of view in mind it seems that a table with convex boundaries (in which parallel trajectories are focused after reflections) is unlikely to yield hyperbolic behavior. This impression can be only confirmed by the presence of caustics in smooth convex billiards [48]. It came then as a surprise the discovery by Bunimovich that perturbations of the circle[1] could be hyperbolic.

The main intuition behind it is that, although the trajectories after reflection maybe focusing, after some time, they focus and then become divergent, so if there is enough time between two consecutive collisions, we can have divergent families going into divergent families, again (provided we look at the right place). Another equivalent point of view is that the instability is measured not just by the change in position but also, by the change in velocity, from this point of view, a very strong convergence is not so different from a strong divergence.

To find a new invariant family of cones, let us consider first a circular billiard. The collision angle is a conserved quantity of the motion. It is then natural to consider, at each point in phase space, the tangent vector η_3 associated to a family of trajectories that, at the next collision with one of the two half circles, will have the same collision angle.

We have defined η_3 in geometrical terms, clearly its expression in terms of η_1 and η_2 changes from point to point. Yet, there are special points (the middle of the cord between two consecutive collision with the same half-circle) in which η_3 coincides with η_2 (this is seen immediately by geometric considerations).

Clearly, in a sequence of collisions with the same circumference the vector

[1]Clearly non smooth perturbations such as the stadium, otherwise the KAM theorem would apply, see [31].

η_3 is invariant. Also, from the above considerations follows that before collisions η_3 corresponds to a diverging family, while immediately after a collision it corresponds to a convergent family.

What happens to the parallel family η_1? Since divergent families becomes convergent it is obvious that the parallel family, after reflection, becomes even more convergent. Hence, it will focus before the middle point to the next collision (the point where the family η_3 focuses).

The previous considerations suggest to consider the cone $\mathcal{C}(x, v) = \{\xi \in \mathcal{TM} \mid \alpha\eta_1 + \beta\eta_3 \text{ with } \alpha\beta \geq 0\}$.

Hence, for a trajectory that collides only with a half circle the cone just defined is invariant but not strictly invariant. Since this would be true also for a billiard inside a circle it is clearly not sufficient (the billiard inside a circle has zero Lyapunov exponents, since, as we have already remarked, the motion is integrable).

Let us go back to the Bunimovich stadium. Clearly, it will behave as a circular billiard for trajectories colliding only with a half circle. So we need to see what happens if a trajectory goes from one circumference to the other (which will happen with probability one). In this case, the infinitesimal motion is the same that would happen if the straight line would be not present. In fact, if we reflect the billiard table along the straight lines we can imagine that the motion proceeds in a straight line.

Hence the family η_3 will first focus and than diverge for a longer time (and so get closer to the parallel family) than would happen if the collision would be in the same circle. This is exactly what we need to get strict invariance of the cone family.

In conclusion the cone family is strictly invariant each time that the trajectory goes from one half circle to the other. Since this happens almost surely, again we have proven hyperbolicity of the system.

It is interesting to notice that the cone family coincide with the one used in the Sinai Billiard (divergent trajectories) if one looks at it at the right point: the point laying in the intersection between the trajectory before collision and the circle of radius $r/2$ (if r is the radius of the half-circles forming the table) tangent to the the boundary at the next collision with a half-circle (but nowhere else).[2]

[2] If the last collision was with a flat wall, then the point is obtained by reflecting the billiard so the trajectory backward looks straight, determining the point and then reflecting back to find the real point on the trajectory.

Problems

8.1 Let

$$L_i = \begin{pmatrix} 1 & 0 \\ t_i & 1 \end{pmatrix} \quad \text{and} \quad R_i = \begin{pmatrix} 1 & k_i \\ 0 & 1 \end{pmatrix},$$

for each $u \in \mathbb{R}^+$ write

$$\prod_{i=0}^{n} R_i L_i \begin{pmatrix} 1 \\ u \end{pmatrix} = \lambda_n \begin{pmatrix} 1 \\ u_n \end{pmatrix}.$$

Show that

$$u_n = k_n + \cfrac{1}{t_n + \cfrac{1}{k_{n-1} + \cfrac{1}{t_{n-1} + \ddots + \cfrac{1}{t_1 + \frac{1}{k_1 + \frac{1}{u}}}}}}$$

And find conditions for the convergence of the continuous fraction. (Hint: see Problem 5.21)

Chapter 9

Hyperbolicity of hard spheres

For hard balls or radius $\frac{1}{2}$, and mass one, in dimension d, the flow is given by $\phi_t(q,p) = q + tp$ if no collision occurs. If the ball i collides with the ball j, then let p_i^-, p_j^- and p_i^+, p_j^+ be the velocities just before and after the collision, respectively. Note that for the balls to collide is must be that before the collision

$$0 > \frac{d}{dt}\|q_i - q_j\|^2 = \langle q_i - q_j, p_i - p_j\rangle.$$

Thus, at collision, $\langle q_i - q_j, p_i - p_j\rangle \leq 0$. Let $n = q_i - q_j$, then

$$\begin{aligned} p_i^+ &= p_i^- - \langle n, p_i^- - p_j^-\rangle n \\ p_j^+ &= p_j^- + \langle n, p_i^- - p_j^-\rangle n. \end{aligned} \tag{9.0.1}$$

Let us compute $d_{(q,p)}\phi_t(\delta q, \delta p)$ across a collision. If τ is the collision time of the trajectory then $\|q_i(\tau) - q_j(\tau)\| = 1$. If we consider the trajectories $\phi_t((q,p) + s(\delta q, \delta p))$, then the collision time $\tau(s)$ satisfies

$$\langle q_i(\tau) - q_j(\tau), \delta q_i(\tau) - \delta q_j(\tau)\rangle + \langle q_i(\tau) - q_j(\tau), p_i(\tau) - p_j(\tau)\rangle \tau'(0) = 0.$$

If the collision is non tangent (i.e. $\langle n, p_i(\tau) - p_j(\tau)\rangle \neq 0$), then,

$$\tau'(0) = -\frac{\langle n, \delta q_i(\tau) - \delta q_j(\tau)\rangle}{\langle n, p_i(\tau) - p_j(\tau)\rangle}.$$

To compute $d\phi_t$ is then convenient to shift along the flow direction by τ so all the trajectories $(q,p) + s(\delta q(\tau(s)), \delta p) + \tau(s)(p, 0)$ collide simultaneously.

Let us call $(\tilde{\delta}q^-, \tilde{\delta}p^-) = (\delta q(\tau(0)) + \tau'(0)p_i, \delta p_i)$, the shifted tangent vector. For such a tangent vector, we have that (9.0.1) yields

$$\begin{aligned}
\tilde{\delta}q_i^+ =&\tilde{\delta}q_i^- \\
\tilde{\delta}q_j^+ =&\tilde{\delta}q_j^- \\
\delta p_i^+ =&\delta p_i^- - \langle \tilde{\delta}q_i^- - \tilde{\delta}q_j^-, p_i^- - p_j^- \rangle n - \langle n, p_i^- - p_j^- \rangle (\tilde{\delta}q_i^- - \tilde{\delta}q_j^-) \\
&- \langle n, \delta p_i^- - \delta p_j^- \rangle n \\
\delta p_j^+ =&\delta p_j^- + \langle \tilde{\delta}q_i^- - \tilde{\delta}q_j^-, p_i^- - p_j^- \rangle n + \langle n, p_i^- - p_j^- \rangle (\tilde{\delta}q_i^- - \tilde{\delta}q_j^-) \\
&+ \langle n, \delta p_i^- - \delta p_j^- \rangle n.
\end{aligned} \tag{9.0.2}$$

The derivative is then obtained by shifting back along the flow direction, by $-\tau'(0)$. By taking the case $\tau(0) = 0$ we can thus obtain the behavior of a tangent from just before to just after a collision:

$$\begin{aligned}
\delta q_i^+ =&\delta q_i + \langle p_i - p_j, \delta q_i - \delta q_j \rangle n \\
\delta q_j^+ =&\delta q_j - \langle p_i - p_j, \delta q_i - \delta q_j \rangle n \\
\delta p_i^+ =&\delta p_i - \langle \delta q_i - \delta q_j, p_i^- - p_j^- \rangle n - \tau'(0) \|p_i^- - p_j^-\|^2 n \\
&- \langle n, p_i^- - p_j^- \rangle (\delta q_i - \delta q_j) - \tau'(0) \langle n, p_i^- - p_j^- \rangle (p_i^- - p_j^-) \\
&- \langle n, \delta p_i - \delta p_j \rangle n \\
\delta p_j^+ =&\delta p_j + \langle \delta q_i - \delta q_j, p_i^- - p_j^- \rangle n + \tau'(0) \|p_i^- - p_j^-\|^2 n \\
&+ \langle n, p_i^- - p_j^- \rangle (\delta q_i - \delta q_j) + \tau'(0) \langle n, p_i^- - p_j^- \rangle (p_i^- - p_j^-) \\
&+ \langle n, \delta p_i - \delta p_j \rangle n.
\end{aligned} \tag{9.0.3}$$

To apply Theorem 5.5.1, we have to construct the quadratic form Q. We choose the lagrangian spaces $\mathbb{V}_1 = \{\delta q = 0\}$ and $\mathbb{V}_2 = \{\delta p = 0\}$. The energy is only kinetic energy, then the vectors tangent to the constant energy are $\langle p, \delta p \rangle = 0$. This yields the form $Q(\delta q, \delta p) = \langle \delta q, \delta p \rangle$. The vector field is $(p, 0)$, and $Q(\delta q + \alpha p, \delta p) = Q(\delta q, \delta p)$, so Q is well defined on the quotient and we can restrict ourselves to the vectors $\{(\delta q, \delta p) \;:\; \langle p, \delta p \rangle = \langle p, \delta q \rangle = 0\}$. Note that

$$Q((\delta q + t\delta p, \delta p)) = Q(\delta q, \delta p) + t\|\delta p\|^2 \geq 0.$$

It remains to compute the change in the quadratic form from just before to just after a collision. Since Q is invariant along the flow direction, we can use formula (9.0.2) and, since by construction $\langle \tilde{\delta}q_i^- - \tilde{\delta}q_j^-, n \rangle = 0$, obtain

$$Q((\delta q^+, \delta p^+)) = Q(\tilde{\delta}q^+, \tilde{\delta}p^+) = Q(\delta q, \delta p) - \langle n, p_i^- - p_j^- \rangle \|\tilde{\delta}q_i^- - \tilde{\delta}q_j^-\|^2 \geq 0.$$

The invariance of the cone follows.

Note that we have strict invariance if $\delta p \neq 0$. If $\delta p = 0$, then we have the strict invariance if $\tilde{\delta}q_i^- \neq \tilde{\delta}q_j^-$. This fails only if

$$\delta q_i^- - \frac{\langle n, \delta q_i - \delta q_j\rangle}{\langle n, p_i(\tau) - p_j(\tau)\rangle} p_i = \delta q_j^- - \frac{\langle n, \delta q_i - \delta q_j\rangle}{\langle n, p_i(\tau) - p_j(\tau)\rangle} p_j,$$

i.e. there exists $z \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} \delta q_i &= z + \lambda p_i \\ \delta q_j &= z + \lambda p_j. \end{aligned} \tag{9.0.4}$$

To see how to use the above facts, it is convenient to introduce a bit of notation.

9.1 Collision graphs and their decorations

First of all, I will introduce a *collision graph* to describe pictorially the relevant features of a trajectory, it will be a directed graph (the direction being given by time). The graph starts with n roots (each one representing one ball), from each root starts an edge (representing the path of a ball). A collision is represented by a vertex in the graph (I will indicate it pictorially by a star, so as not to confuse it with edges that cross due to the two-dimensional representation). If the collision involves k balls, then the vertex will have degree $2k$ with k entering edges–representing the incoming particles–and k exiting edges–representing the outcoming particles. We arrange the height of the vertices vertically proportionally to the time.

Lemma 9.1.1 *The graphs with a degree higher than two or with two vertices at the same height happen only on a set of codimension* 2.

carla▶ I should say something more, so it can imply the same for the poincarè map

PROOF. By the implicit function theorem, the collision of two balls is a codimension one condition, while of three balls is codimension two. The same holds for the case of two collisions that happen at the same time. □

See figure 9.1 for the case of four balls in which number one collides with two, then two with four, and finally two with three.[1]

[1] The rule for tracing the graph is that the order of the balls is not changed at collision, so the line on the left represents the particles entering the collision vertex from the left. Remark that the collision graph is only a symbolic device and does not respect the geometry of the actual collisions, so the ordering of the balls is only a device to tell them apart and has no relation with the actual geometry of the associated configuration. Keeping this in mind, in figure 9.1 the final disposition of the balls is: one, four, two, three.

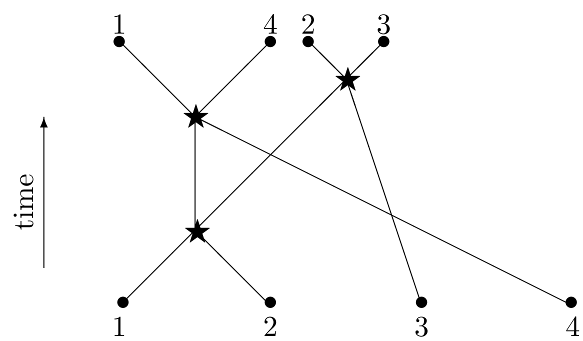


Figure 9.1: A simple collision graph (the stars are the collisions)

Next, let us call $\mathcal{G}$ a collision graph and let $V(\mathcal{G})$ be the collection of its vertexes, $\tilde{E}(\mathcal{G})$ the collection of its edges and $E(\mathcal{G})$ the collection of edges that connect starred vertices. In addition for each edge $b \in E(\mathcal{G})$ let $\nu(b), \nu_+(b)$ be the two vertices joined by the edge.[2]

Definition 18 *Define $\pi : \widetilde{E}(\mathcal{G}) \to \{0, \dots, n\}$ so that $\pi(b)$ is the particle associated to the vertex $b \in \widetilde{E}(\mathcal{G})$. Also, $q^{\pm}_{\pi(b)}$, $p^{\pm}_{\pi(b)}$, $\delta q^{\pm}_{\pi(b)}$, $\delta p^{\pm}_{\pi(b)}$ are the position, velocity, and components of the tangent vector just after the collision represented by $\nu_-(b)$ and just before the collision represented by $\nu_+(b)$, respectively.*

To follow the history of a vector of type $(\delta q, 0)$ that stubbornly refuses to enter strictly in the cone it is convenient to specify at each vertex the values (λ_ν, z_ν) appearing in the associated equation (9.0.4). Of course, to recover the tangent vectors from the $\{(\lambda_\nu, z_\nu)\}_{\nu \in V(\mathcal{G})}$, it is necessary to specify the velocities. To this end we specify for each edge the velocity $v(b)$ of the particle associated to such a line. We can then decorate a graph with the above informations and we obtain a full description of the history of a tangent vector that keeps being not increased by the dynamics in the trajectory piece described by the graph (of course provided such a vector exists at all).

Now consider a edge $b \in E(\mathcal{G})$, if it represents the trajectory of the particle j between the collision corresponding to the the vertex $\nu(b)$ and the one corresponding to the vertex $\nu_+(b)$, then the corresponding component of the tangent vector at such times can be written both as $\delta q_j = z(\nu(b)) + \lambda(\nu(b))v(b)$ and $\delta q_j = z(\nu_+(b)) + \lambda(\nu_+(b))v(b)$. Accordingly, the following compatibility condition must be satisfied:

$$z(\nu(b)) - z(\nu_+(b)) = [\lambda(\nu_+(b)) - \lambda(\nu(b))]v(b). \tag{9.1.5}$$

[2]By convention $\nu(b)$ corresponds to the lower collision and $\nu_+(b)$ to the upper.

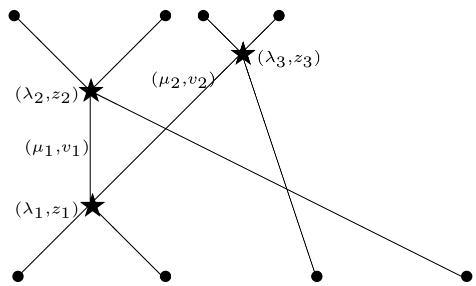


Figure 9.2: A decorated collision graph

It is then natural to define another decoration, this time associated to edges that connect two collision vertexes,

$$\mu(b) := \lambda(\nu_+(b)) - \lambda(\nu(b)). \tag{9.1.6}$$

By decorated collision graph, we will mean a graph with $(\lambda(\nu), z(\nu))$ attached to each vertex and $\mu(b), v(b)$ to each edge connecting two collisions, with a mild abuse of notations we will call such decorated graph $\mathcal{G}$ as well, see figure 9.2.[3]

9.2 Close path formula and cycles

As time progresses, the graph will grow more complex, in particular it may develop *cycles.*

Definition 19 *Let $m > 1$. A* path *of lenght m is a sequence of edges $\{b_1, \cdots, b_m\}$ and vertices $\{\nu_0, \cdots, \nu_m\}$ such that $\{\nu_{i-1}, \nu_i\} = \{\nu_\pm(b_i)\}$. A path is* simple *if $i, j \in \{1, \ldots, m\}$, with $i \neq j$, implies $b_i \neq b_j$. A* closed path *of lenght m is a path such that $\nu_0 = \nu_m$. A* cycle *of length m, is a closed path of length $m > 2$ for which if $i, j \in \{0, \ldots, m-1\}$ with $i \neq j$, then $\nu_i \neq \nu_j$, or of lenght $m = 2$ for which $b_1 \neq b_2$. (e.g. the thick edges in the graph of figure 9.3). We call close paths of type $\{b, b\}$* null-path.

Note that if we erase on edge from a cycle, then we do not have a cycle anymore. Next, we show that cycles capture all the closed path.

Lemma 9.2.1 *The edges of any closed path can be seen as the edges of the union of cycles and null path.*

[3] Note that the above description is quite redundant due to (9.1.5), yet we will see in the following that such a description is quite convenient.

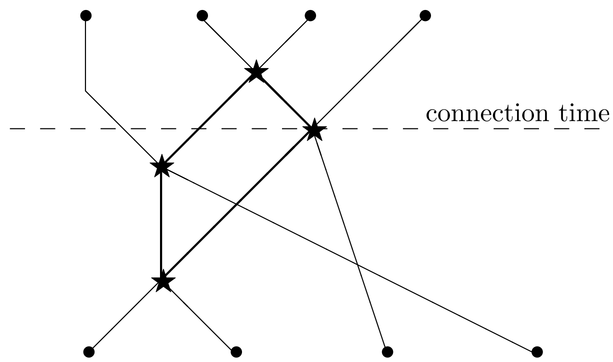


Figure 9.3: A cycle

Proof. Let $\{b_1, \cdots, b_m\}$ be a closed path which is not a cycle. This implies that, for some $m \geq i > j > 0$, $\nu_j = \nu_i$. This implies that the edges can be seen as the edges of the two closed paths $\{b_{i+1}, \ldots, b_m, b_1, \ldots b_j\}$ and $\{b_{j+1}, \cdots, b_{i-1}\}$ We can then continue the decomposition until we are left only with cycles or null-paths. Note that the same cycle or null-path can appear several time, so it is enough to consider each only once. □

Given a closed path a remarkable compatibility condition can be derived. In fact, let $C \subset \mathcal{G}$ be a cycle, let us run it counterclockwise and define, for each edge $b \in C$, $\varepsilon_C(b) = 1$ if the edge is run from bottom to top and $\varepsilon_C(b) = -1$ if it is run from top to bottom. We have, by definition (9.1.6), $\sum_{b\in E(C)} \varepsilon_C(b)\mu(b) = 0$. In addition, we can sum equation (9.1.5) for each edge in the cycle and obtain

$$\begin{aligned} &\sum_{b\in E(C)} \mu(b)\varepsilon_C(b)v(b) = 0 \\ &\sum_{b\in E(C)} \varepsilon_C(b)\mu(b) = 0. \end{aligned} \tag{9.2.7}$$

The above formula is essentially the *closed path formula* introduced by Simanyi in [65]. Notice that, given a graph, the set of closed paths contains many more objects than we are interested in. In fact, a closed path can contain an edge that is run consecutively in opposite directions, in particular, a null-path. This adds zero to the constraint (9.2.7). Hence, by Lemma 9.2.1, we can consider only cycles. Also, a cycle can be run in opposite directions, which gives rise to the same closed path formula; hence, we will always use the counterclockwise orientation.[4]

[4] Since, by definition, a cycle has no self-intersections, it is orientable, and, by Jordan's

The formula 9.2.7 expresses a compatibility condition that puts a clear restriction on the possible existence of the decorated collision graph, and hence of the corresponding nonincreasing vector. Studying the combinatorics of such collisions, it is possible to establish the hyperbolicity and ergodicity of a gas of n particles. This has been done first in a series of papers by the *Hungarian team* [43, 65, 66, 67, 68]. In the following we will enter in some of the related details.

9.2.1 A cohomological point of view

Althogh not strictly necessary in the following, it is amusing to note that the above structure is very reminiscent of cohomology over a decorated graph as I am going to briefly explain.
Let $\mathcal{C}(\mathcal{G})$ be the set of cycles of the graph $\mathcal{G}$. Consider, the following set of fucntions on a decorated graph $\mathcal{G}$: $V_v(\mathcal{G}) = \{f : V(\mathcal{G}) \to \mathbb{R}^{d+1}\}$; $V_e = \{f : E(\mathcal{G}) \to \mathbb{R}^{d+1}\}$; $V_c = \{f : \mathcal{G}(\mathcal{G}) \to \mathbb{R}^{d+1}\}$. We can then define the operator $d : V_v(\mathcal{G}) \to V_e(\mathcal{G})$ and $d : V_e(\mathcal{G}) \to V_c(\mathcal{G})$ as

$$(df)(b) = f(\nu_+(b)) - f(\nu_-(b))$$
$$(df)(C) = \sum_{b\in C} \varepsilon_C(b) f(b).$$

Note that $d^2 = 0$, so we can define a cohomology on decorated graphs. Note that if $f \in V_e(\mathcal{G})$ is closed, that is $df = 0$, then it is exact, that is there exists $g \in V_v(\mathcal{G})$ such that $dg = f$.

With a small twist with respect to the usual convention we define the restricted space

$$V_e^*(\mathcal{G}) = \{f \in V_e(\mathcal{G}) \ : \ f(b) = \mu(b)\bar{v}(b), \mu(b) \in \mathbb{R}\},$$

and say that a function $f \in V_v(\mathcal{G})$ is closed iff $df(b) \in V_e^*(\mathcal{G})$. While, as usual and already stated earlier, $f \in V_e(\mathcal{G})$ is closed iff $df = 0$. Also, note that for a function in $V_e^*(\mathcal{G})$ being closed means exaclty that it satisfies the cycle contraints (9.2.7).

With such a notation, for a trajectory to be hyperbolic (i.e. with all the Lyapunov exponent non zero) is equivalent to saying that there exists a time such that the associated graph $\mathcal{G}$ has the following property: the only closed functions in $V_v(\mathcal{G})$ are the constant ones.

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theorem, it has an interior. By counterclockwise orientation we mean that it is run so that the interior lies on the left.

9.3 Hyperbolicity: some examples

Having developed a general theory to describe the evolution of tangent vectors that fail to become hyperbolic, we are going to check the effectiveness of the theory by analyzing a series of examples of increasing complexity.

9.3.1 2 balls in dimension $d \geq 2$

First of all, recall that there are always zero Lyapunov exponents connected with the flow direction and momenta conservation. To avoid that, we consider only the situation where the center of mass is at rest. This implies that $\sum_i \delta q_i = 0$.

Next, notice that the situation in which the two balls never collide is of zero measure: if there is no collision, the two balls just perform a translation on the torus. One can then see it as a translation on $\mathbb{T}^{2d}$, which is ergodic if the velocities have entries that are not rational among them, which is a zero-measure condition. Hence, the ball will collide with probability one.

Once the two balls collide for the second time, we have a close cycle made of just two bonds $\{b_1, b_2\}$. In this case (9.2.7) implies

$$\mu(b_1)(v(b_1) - v(b_2)) = 0,$$

which has solutions either $\mu(b_1) = 0$ or $v(b_1) = v(b_2)$. The latter condition is a codimension $d-1$ condition, hence with probability one $\mu(b_1) = 0$. But then the second of (9.2.7) implies $\mu(b_2) = 0$. It follows that $\delta q = (z, z)$, which implies $z = 0$, hence eventual strict monotonicity and hence hyperbolicity.

9.3.2 3 balls in dimension $d \geq 2$

Again we can assume $\sum_i \delta q_i = 0$. Also, we would like to know that the situation in which a ball does not collide with the other two has zero measure. This is more complex and needs the mixing of the two ball systems. Let us assume it and proceed.

After the first collision (say of particles $1, 2$), we wait until a collision involving particle 3, say a collision with particle 1. The next collision will close a cycle. If the cycle involves particles $1, 3$, then the previous discussion implies $\mu(b_1) = \mu(b_3) = 0$. This situation will persist until there is a collision with particle 2, say with particle 3. At that point (9.2.7) implies $\mu(b_2) = 0$. It follows that, almost surely $\delta q = (z, z, z)$ and hence $z = 0$.

The case in which the cycle involves all the particles remains to be analyzed, say particle 3 collides with particle 2. In this case (9.2.7) implies

$$\mu(b_1)\varepsilon_C(b_1)(v(b_1) - v(b_2)) + \mu(b_3)\varepsilon_C(b_3)(v(b_3) - v(b_2)) = 0$$

where b_1 is the edge associated to the particle 1 before the collisione with 3, b_3 is the edge associated to the partile 3 after the collision with 1 and b_2 the edge associated to the particle 2 after the collision with 1. On the other had, calling b_3^- the edge associated to 3 before the collision with 1, by (9.0.1) we have

$$v(b_3) = v(b_3^-) - \langle n, v(b_3^-) - v(b_1)\rangle n.$$

Note that $R := \langle n, v(b_3^-) - v(b_1)\rangle = 0$ is a codimension one condition hence it happens on a zero measure set. Accordingly, just before the collision of 1 and 3 we have

$$\mu(b_1)\varepsilon_C(b_1)(v(b_1) - v(b_2)) - \mu(b_3)\varepsilon_C(b_3)(v(b_1) - v(b_3^-) - Rn) = 0.$$

Again, for $v(b_1) - v(b_2)$ and $v(b_1) - v(b_3^-) - Rn$ to be linearly dependent is a codimension one condition. It follows $\mu(b_1) = \mu(b_3) = 0$ and then (9.2.7) implies $\mu(b_2) = 0$. So $\delta q_i = 0$, thus the form is eventually strictly increasing.

9.3.3 $d + 1$ balls in dimension $d \geq 2$

Given a decorated graph $\mathcal{G}$, we identify two vertices joined by an edge $b \in E(\mathcal{G})$ iff all the vectors for which the form does not increase have $\mu(b) = 0$. Let us call $\mathcal{G}_*$ the resulting graph. Notice that now the vertices can have several edges entering and leaving. The assertion that the form does not increase only for the vectors of the form $\delta q_i = z + \lambda p_i$, $\delta p_i = 0$ is equivalent to the fact that $\mathcal{G}_*$ has only one vertex.

Lemma 9.3.1 *If $\mathcal{G}_*$ has only one vertex at a certain time, the same will remain true forever.*

Proof. The Lemma follows immediately from the discussion before equation (9.0.4). □

Hence, our goal will be to show that there exists a time at which $\mathcal{G}_*$ has, almost surely, one vertex. Suppose $\mathcal{G}_*$ has more than one vertices, then it will eventually develop a cycle C_1. This, as seen in the previous example, assumes that the possibility that the particles can be separated into two groups that never collide among them, is a zero-measure event. In turn, this is a consequence of the mixing of all the systems of d balls, which, for the

moment, we assume.
Let $C_1 := \{b_1, \cdots, b_m\}$ be the edges of the cycle and $\{\nu_1, \cdots, \nu_m\}$ and the vertices such that b_i is the edge connecting the vertices ν_i and ν_{i+1}, with $\nu_{m+1} = \nu_1$. The form does not increase only if equation (**??**) is satisfied. if the $v(b_i)$ are linearly independent, then equation (**??**) has $\mu(b_i) = 0$ as the only solution contrary to the assumption that $\mathcal{G}_*$ had more than one vertices. Hence, the hyperbolicity follows from the next Lemma.

Lemma 9.3.2 *The vectors $\{v(b) \ : \ b \in E(C_1)\}$ are linearly independent outside a codimension one set.*

PROOF. Note that $m = \sharp E(C_1) \leq n+1$. Let $E_k \subset E(C_1)$ be a set of k edges. We proceed by induction on k. Suppose all the $\{v(b) \ ; \ b \in E_i\}$, $i \leq k$ are linearly independent a part for a zero-measure set. Note that the statement is trivially true for $k = 1$. We prove that the same holds for $k+1$. Let $E_{k+1} \subset E(C)$ be a set of $k+1$ edges. Choose $b \in E_{k+1}$ and let ν it lower vertice. By construction, there exists $b_1 \notin C$ such that b_1 enters in ν. Let $E_-(b_1)$ be the collection of edges in $E(\mathcal{G}_*)$ such that they belong to a path that connects to the lower vertex of b_1. Clearly $E_-(b_1) \cap C = \emptyset$ since otherwise C would not be the first cycle in $\mathcal{G}_*$. Let k be the particle associated to the bond b_1. At the time just before ν, we consider the tangent vector $\delta q_i = z$, $\delta p_i = 0$ for $i \neq k$ and $\delta q_k = -dz$, $\langle \delta p_k, p_k \rangle = 0$. It follows that, if l is the particle associated with edge b, using (9.0.2), we have, if $l \neq k$,

$$\delta p_l = an + bz$$

and, if $l = k$,

$$\delta p_l = \delta p_l - an - bz$$

for some $a, b \neq 0$ almost surely. In both cases, δp_l can span all the space. Next, there are two possibilities: either the edges exiting $\nu^+(b)$ do not belong to C or one of them, call it b_2, belongs to C. In the former case, all the tangent vectors associated to the edges in $E(C) \setminus \{b\}$ do not change their δp component. In the latter case, the other entering bond in $\nu^+(b)$, call it b_3, cannot belong to C. Let $\pi(b)$ be the particle associated to the bond b, then

$$\begin{aligned}\delta p_{\pi(b_2)} =& \delta p_{\pi(b_3)} + \langle \tilde{\delta} q^-_{\pi(b)} - \tilde{\delta} q^-_{\pi(b_3)}, p_{\pi(b)} - p_{\pi(b_3)} \rangle n \\ &+ \langle n, p_{\pi(b)} - p_{\pi(b_3)} \rangle (\tilde{\delta} q^-_{\pi(b)} - \tilde{\delta} q^-_{\pi(b_3)}) + \langle n, \delta p_{\pi(b)} - \delta p_{\pi(b_3)} \rangle n.\end{aligned}$$

if we can force $\delta p_{\pi(b_2)} = 0$, we are done

We can then use the implicit function theorem to prove that the vectors $\{v(b) \ : \ b \in E_{k+1}\}$ are linearly independent outside a codimension one manifold, hence proving the Lemma. □

Chapter 10

Geometry of foliations and ergodicity (very few words)

We have seen conditions that imply hyperbolicity. Once the map is hyperbolic Pesin Theory (e.g. see [40]) implies that there exists stable and unstable manifolds. However, the objects constructed in Pesin's theory have very poor regularity properties. For applications, a more explicit construction can be essential, especially if it provides extra information on the properties of the manifolds. For this, the mere existence of an eventually strictly invariant cone field is not enough; from now on, we will assume that the cone field is continuous.

10.1 Cones and invariant distributions

We have seen that the growth of an appropriate quadratic form implies the contraction of a cone. A natural question is if such a contraction can be described in a more quantitative way. This is possible, a general theory can be found in [51], here we give only a quick overview.

Definition 20 *The symplectic angle between two vectors* $u, w \in \operatorname{int}(\mathcal{C})$ *is the real number* $\Theta(u, w)$ *defined by*

$$\omega(u, v) = \sqrt{Q(u)}\sqrt{Q(v)} \sinh \Theta(u, v)$$

Definition 21 *The distance* $s(U, W)$ *of two Lagrangian subspaces* $U, V \subset \operatorname{int}(\mathcal{C})$ *is equal to the supremum of absolute values of symplectic angles be-*

tween nonzero vectors from the two Lagrangian subspaces i.e.

$$s(U,V) = \sup_{\substack{0\neq u\in U\\ 0\neq v\in V}} |\Theta(u,v)|.$$

It turns out that s is a metric on the set of Lagrangian subspaces contained in the cone (e.g. see [51]). Here, we just note that if $s(U,V)=0$ it follows, by definition, that $\omega(v,u)=0$ for all $u\in U$ and $v\in V$, but this implies that $V=U$.
Note that if $s(U,V)<\infty$ and L is a monotone symplectic matrix, then

$$\begin{aligned}\sqrt{Q(u)}\sqrt{Q(v)}\sinh\Theta(u,v) &= \omega(u,v)=\omega(Lu,Lv)\\ &= \sqrt{Q(Lu)}\sqrt{Q(Lv)}\sinh\Theta(Lu,Lv)\end{aligned}$$

Thus,

$$\begin{aligned}\sinh s(LU,LV) &= \sup_{\substack{0\neq u\in U\\ 0\neq v\in V}} \sinh\Theta(u,v)\sqrt{\frac{Q(u)}{Q(Lu)}}\sqrt{\frac{Q(v)}{Q(Lv)}}\\ &\leq \sigma(L)^{-2}\sinh s(U,V).\end{aligned} \tag{10.1.1}$$

The following Theorem gives a criterion for $s(U,V)$ to be finite.

Theorem 10.1.1 (Theorem 2 [51]) *For a strictly monotone map L the diameter of $L\mathcal{C}$, where $\mathcal{C}$ is determined by the transversal lagrangian spaces V_1, V_2, is equal to the s distance of LV_1 and LV_2, Moreover, for each Lagrangian spaces $V, U\subset\mathcal{C}$*

$$\tanh\left(\frac{s(LV,LU)}{2}\right) = \sigma(L)^{-2}.$$

Lemma 10.1.2 *Given a smooth Symplectic Dynamical Systems with syngularities (X,T,μ), X a symplectic two dimensional manifold, μ the symplectic volume, if the systems is eventually strictly monotone, then $\{E^u(x)\}$ is almost everywhere well defined. Moreover, if $\mathcal{C}(x)$ is continuous, then $\{E^u(x)\}$ is continuous (where it is defined). In addition, if the cone family is strictly monotone, then $\{E^u(x)\}$ is everywhere defined.*

Proof. Let $\mathcal{C}_n(x) := D_{T^{-n}x}T^n\mathcal{C}(T^{-n}x)$ and $\Delta_n(x) := \mathrm{diam}(\mathcal{C}_n(x))$, then Δ_n is decreasing, thus we can define

$$\Delta_\infty(x) := \lim_{n\to\infty}\Delta_n(x).$$

By Theorem 10.1.1

$$\begin{aligned}\Delta_\infty(T^m x) &= \lim_{n\to\infty} \operatorname{diam}(D_{T^{m-n}x}T^n\mathcal{C}(T^{-n+m}x)) \\ &= \lim_{n\to\infty} \operatorname{diam}(D_x T^m D_{T^{-n}x}T^n\mathcal{C}(T^{-n}x)) \\ &\leq \frac{1}{\sigma(D_xT^m)^2}\Delta_\infty(x).\end{aligned}$$

Next, let $\Omega = \{x \in X \mid \Delta_\infty(x) = \infty\}$, we claim that $\mu(\Omega) = 0$. In fact, let $B_m = \{x \in X \mid \sigma(D_xT^m) \geq 2\}$, by eventual strict monotonicity of the cone field it follows $\mu(\cup_{m\in\mathbb{N}}B_m) = \mu(X)$, recall (5.5.8). In addition, $B_m \supset B_{m_0}$ for all $m > m_0$. Moreover, if $x \in B_m$, then $\Delta_\infty(T^m x) < \infty$ (see Theorem 10.1.1). Thus $T^{-n}\Omega \cap B_m = \emptyset$ for all $n \geq m$, and

$$\mu(\Omega) = \lim_{n\to\infty}\mu(T^{-n}\Omega) \leq \lim_{n\to\infty}\mu(X\setminus\cup_{m\leq n}B_m) = 0.$$

Finally, let $\Omega_L = \{x \in X \mid \frac{L}{2} \leq \Delta_\infty(x) \leq L\}$ and suppose $\mu(\Omega_L) > 0$. Then, there exists $m \in \mathbb{N}$ such that $\mu(\Omega_L \cap B_m) > 0$. Consequently, for almost all $x \in \Omega_L \cap B_m$ there exists a return time $\bar{n}m \in \mathbb{N}$ in the past (that is $T^{-m\bar{n}}x \in \Omega_L \cap B_m$). Accordingly,

$$\frac{L}{2} \leq \Delta_\infty(x) \leq \frac{1}{\sigma(D_xT^m)^2}\Delta_\infty(T^{-m\bar{n}}x) \leq \frac{L}{4},$$

which is a contradiction unless $L = 0$. We have so proven that $\mu(\Omega_0) = \mu(X)$. In other words the cones $\mathcal{C}_\infty = \cap_{n\geq 0}\mathcal{C}_n(x)$ is almost everywhere degenerate since, having zero diameter, means that the cone is a Lagrangian suspace which is precisely the unstable direction.

To prove the continuity of the above distribution note that the cone family $\mathcal{C}_n(x)$ is continuous. Let x be such that $\Delta_\infty(x) = 0$, then, for each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\Delta_m(x) < \frac{\varepsilon}{2}$. Then one can chose δ such that the edges of $\mathcal{C}_m(y)$ vary by an amount less than $\frac{\varepsilon}{2}$ if $d(x, y) < \delta$. The result follows then taking into account that the Hilbert metric bounds the angle and that the unstable distribution is contained in $\mathcal{C}_n$ for each $n \in \mathbb{N}$.

The proof of the last fact is obvious: just a simplification of the above arguments. □

Let us conclude with an interesting simple fact.

Lemma 10.1.3 *A smooth two-dimensional Symplectic Dynamical System (X, T, μ) is Anosov iff it admits a strictly monotone continuous cone family.*

Proof. By Lemma 10.1.2 it follows that the stable and unstable distribution are continuous. But then, by continuity, there exists $\alpha > 0$ and $\sigma > 1$ such that

$$\begin{aligned}&\alpha\sqrt{Q(v)} \leq \|v\| \leq \alpha^{-1}\sqrt{Q(v)} \quad \forall x, \in X \text{ and } v \in E^u(x)\\&\sigma(D_xT) \geq \sigma \quad \forall x \in X.\end{aligned}$$

Thus,

$$\|D_xT^nv\| \geq \alpha\sqrt{Q(D_xT^nv)} \geq \alpha\sigma^n\sqrt{Q(v)} \geq \alpha^2\sigma^n\|v\|.$$

Analogously one can obtain the statement for the stable direction by using the cone family given by the complementary cones (see Problem 5.3).

The proof that an Anosov systems admit a continuous, strictly invariant cone family is obvious and it is left to the reader. □

10.2 Invariant foliations

Once we know that the system is hyperbolic, we can try to take advantage of hyperbolicity: the first step is to construct stable and unstable manifolds.

The strategy is the usual one: e.g., to construct the unstable manifold at x, consider the trajectory $f^{-n}(x)$ (for simplicity, we consider the Poincarè map). If the trajectory does not meet a discontinuity, then we can consider a manifold W, with tangent space in the unstable cone, centered at $f^{-n}(x)$ and push it forward with the dynamics. In this way, we obtain a sequence of manifolds $W_n = f^n(W)$ that we expect to converge to a limit object. Yet, one has to take into account that the manifold can be cut by singularities, and this could be a serious problem.

In the uniformly hyperbolic case, the analysis is especially simple: since the manifold W expands exponentially ($|W_n| \geq e^{\lambda n}|W|$), we have that the manifolds are cut at a distance shorter than δ only if the distance of $f^{-n}(x)$ from the singularities is less than $\delta e^{-\lambda n}$. This means that the manifold is cut short only if $f^{-n}(x)$ belongs to a neighborhood $\mathcal{S}_n$ of measure $\delta e^{-\lambda n}$. But since the measure is preserved, we have

$$\text{Leb}\left(\cup_{n=0}^{\infty} f^n(\mathcal{S}_n)\right) \leq \sum_{n=0}^{\infty} e^{-\lambda n}\delta \leq C\delta.$$

It follows that there exists a set of measure $1 - C\delta$ in which the unstable manifold has a length larger than δ.

Implementing the above basic idea can be technically challenging, especially since the formula (7.3.3) shows that the derivative blows up near tangencies. Yet, it can be done, for details, see [41, 17]. A technical tool used to deal with the blow-up of the differential at tangent collisions is the introduction, by Sinai, of homogeneity strips. See [17] for details.

The above construction provides a stable foliation, yet the foliation has very poor regularity properties, and this makes it very hard to use it; in general, it is only measurable. Luckily, the holonomy is absolutely continuous. Moreover, it turns out that it can be approximated by a foliation with much better properties that can be conveniently used, see [6, Section 6] for details.

The next step is to prove ergodicity. Once we have an absolutely continuous foliation, you can try to copy Hopf's argument. Such an argument is based on the observation that the ergodic averages of continuous functions are constant along stable and unstable manifolds. This was achieved by Sinai [70]. But see [51] for a more general version. In addition, [51] discusses a piecewise linear example in which the technical difficulties are reduced to a bare minimum, and hence Sinai's argument can be easily understood. The idea is to prove local ergodicity, and then a global argument can be employed to prove ergodicity. The same argument proves that all the powers of the Poincare maps are ergodic, which implies mixing.

It remains the problem of flows. Since the flow can be seen as a suspension over the Poicnarè map, the ergodicity of the flow follows from the ergodicity of the map. Not so for mixing: think of a suspension with a constant ceiling. Mixing for the flows follows from the contact structure. Forgetting for one second the discontinuities, the fact that the flow is contact implies that is we do a cycle stable, unstable, stable, unstable, we move in the flow direction, see Figure 10.1.

Indeed, let α be the contact form, then if v is a strong unstable or a strong stable vector, then $\alpha(v) = 0$, while $\alpha((p, 0)) = 1$, where $(p, 0)$ is the flow direction, it follows that if the cycle in bold in figure 10.1, call it γ, has sides of length δ, then

$$\delta^2 = \int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha = \int_\gamma \alpha$$

which equals exactly the displacement in the flow direction, which is then non-zero. It follows that the stable and unstable foliations are not *jointly integrable*, and this property shows that the flow cannot be reduced to a constant flow suspension by a change of coordinates (since, in such a case, the foliations would indeed be jointly integrable). This suffices to prove the mixing of the flow, even though the argument is a bit more technical than this.

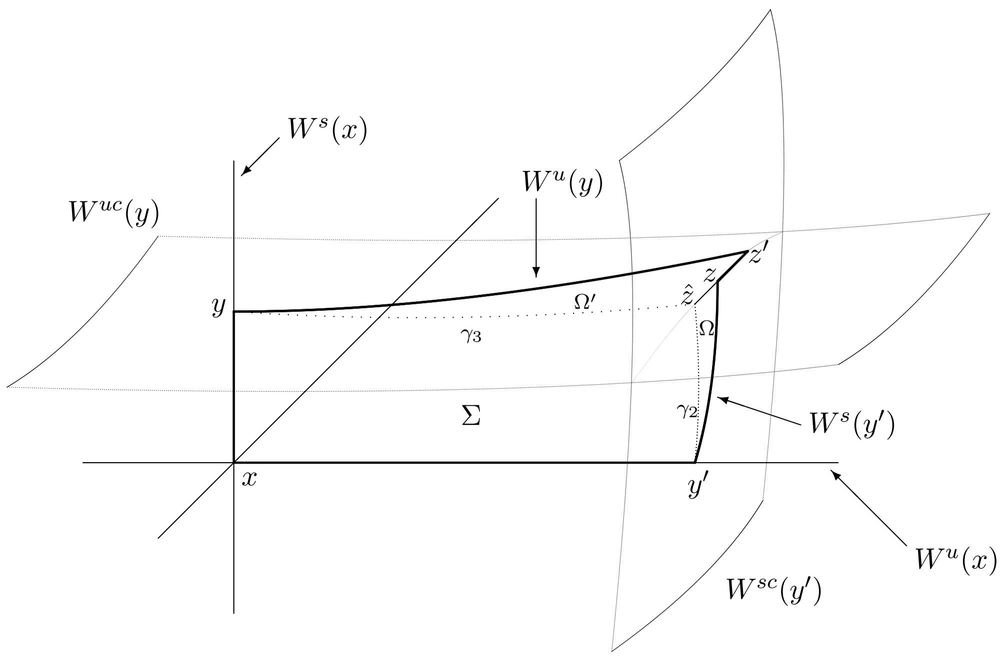

Figure 10.1: Definition of the *temporal function* $\Delta(y, y')$ and related quantities

Bibliography

[1] J.Aaronson *An introduction to infinite ergodic theory* American Mathematical Society, Mathematical Surveys and Monographs, **50** (1997)

[2] Archimedes, The works of Archimedes, by T.L. Heath, Cambridge University press (1897)

[3] V.I.Arnold, A.Avez *Problèmes ergodiques de la mécanique clasique* Gauthier-Villards Paris (1967)

[4] V.I.Arnold*Mathematical Methods in Celestial Mechanics* Springer Verlag, Berlin, Heidelgerg, New York, (1978, 1989)

[5] Baladi, Viviane; Tsujii, Masato *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms.* Ann. Inst. Fourier (Grenoble) 57 (2007), no. 1, 127–154.

[6] Baladi, Viviane; Demers, Mark F.; Liverani, Carlangelo *Exponential decay of correlations for finite horizon Sinai billiard flows.* Invent. Math. 211 (2018), no. 1, 39–177.

[7] Bálint, Péter; Gouëzel, Sébastien *Limit theorems in the stadium billiard.* Comm. Math. Phys. 263 (2006), no. 2, 461–512.

[8] Blank, Michael; Keller, Gerhard; Liverani, Carlangelo *Ruelle-Perron-Frobenius spectrum for Anosov maps.* Nonlinearity 15 (2002), no. 6, 1905–1973.

[9] Bunimovich, L. A.; Sinaĭ, Ya. G. *Statistical properties of Lorentz gas with periodic configuration of scatterers.* Comm. Math. Phys. 78 (1980/81), no. 4, 479–497.

[10] L.A. Bunimovich *On billiards close to dispersing* Math. USSR *sl* 95, 295–312 (1979)

[11] L. A. Bunimovich *On the ergodic properties of nowhere dispersing billiards* Comm.Math.Phys. **65** 295 – 312 (1979)

[12] L.A. Bunimovich *Dynamical Systems of Hyperbolic Type with Singularities*–in *Dynamical Systems II* Ya.G.Sinai ed., Encyclopaedia of Mathematical Sciences, Springer–Verlag, Berling, Heidelberg, New York, 151–178 (1989)

[13] Bunimovich, L. A.; Sinaĭ, Ya. G.; Chernov, N. I. *Statistical properties of two-dimensional hyperbolic billiards.* (Russian) Uspekhi Mat. Nauk 46 (1991), no. 4(280), 43–92, 192; translation in Russian Math. Surveys 46 (1991), no. 4, 47–106

[14] Bunimovich, Leonid A. *Mushrooms and other billiards with divided phase space.* Chaos 11 (2001), no. 4, 802–808.

[15] Burago, D.; Ferleger, S.; Kononenko, A. *Uniform estimates on the number of collisions in semi-dispersing billiards.* Ann. of Math. (2) 147 (1998), no. 3, 695–708.

[16] Chernov, N., *Decay of correlations and dispersing billiards.* J. Statist. Phys. 94 (1999), no. 3-4, 513–556.

[17] Chernov, Nikolai; Markarian, Roberto *Chaotic billiards.* Mathematical Surveys and Monographs, 127. American Mathematical Society, Providence, RI, 2006. xii+316 pp.

[18] Chernov, N.; Dolgopyat, D. *Brownian Brownian motion. I.* Mem. Amer. Math. Soc. 198 (2009), no. 927, viii+193 pp. ISBN: 978-0-8218-4282-9

[19] Baladi, Viviane; Demers, Mark F.; Liverani, Carlangelo *Exponential decay of correlations for finite horizon Sinai billiard flows.* Invent. Math. 211 (2018), no. 1, 39–177.

[20] Demers, Mark F.; Liverani, Carlangelo Stability of statistical properties in two-dimensional piecewise hyperbolic maps. Trans. Amer. Math. Soc. 360 (2008), no. 9, 4777–4814.

[21] Demers, Mark F.; Kiamari, Niloofar; Liverani, Carlangelo *Transfer operators in hyperbolic dynamics—an introduction.* 33 o Colóquio Brasileiro de Matemática. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2021. 238 pp. ISBN: 978-65-89124-26-9

[22] Demers, Mark F.; Zhang, Hong-Kun *Spectral analysis of the transfer operator for the Lorentz gas.* J. Mod. Dyn. 5 (2011), no. 4, 665–709.

[23] Demers, Mark F.; Zhang, Hong-Kun A functional analytic approach to perturbations of the Lorentz gas. Comm. Math. Phys. 324 (2013), no. 3, 767–830.

[24] Demers, Mark F.; Zhang, Hong-Kun *Spectral analysis of hyperbolic systems with singularities.* Nonlinearity 27 (2014), no. 3, 379–433.

[25] Demers, Mark F.; Liverani, Carlangelo Projective cones for sequential dispersing billiards. Comm. Math. Phys. 401 (2023), no. 1, 841–923. 37C83.

[26] do Carmo, Manfredo Perdigão *Riemannian geometry.* Translated from the second Portuguese edition by Francis Flaherty. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. xiv+300 pp. ISBN: 0-8176-3490-8

[27] Predrag Cvitanovic, *Chaos: Classical and Quantum*, https://chaosbook.org/chapters/ChaosBook.pdf.

[28] Dolgopyat, Dmitry *Limit theorems for partially hyperbolic systems.* Trans. Amer. Math. Soc. 356 (2004), no. 4, 1637–1689.

[29] Dolgopyat, Dmitry; Szász, Domokos; Varjú, Tamás *Recurrence properties of planar Lorentz process.* Duke Math. J. 142 (2008), no. 2, 241–281.

[30] Donnay, Victor; Liverani, Carlangelo, *Potentials on the two-torus for which the Hamiltonian flow is ergodic.* Comm. Math. Phys. 135 (1991), no. 2, 267–302.

[31] Donnay, Victor J. *Elliptic islands in generalized Sinai billiards.* Ergodic Theory Dynam. Systems 16 (1996), no. 5, 975–1010.

[32] Euclid, *The Thirteen Books of the Elements.* Translation by Thomas L. Heath, Dover, (1956). An online version is available at Elements.

[33] L.E.Evans, R.F.Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, New York, London, Tokyo (1992).

[34] Feller, William *An introduction to probability theory and its applications. Vol. II.* Second edition John Wiley & Sons, Inc., New York-London-Sydney (1971).

[35] Ferrero, P.; Schmitt, B. *Produits aléatoires d'opérateurs matrices de transfert.* (French) [Random products of transfer matrix operators] Probab. Theory Related Fields 79 (1988), no. 2, 227–248.

[36] Gallavotti, Giovanni, *The elements of mechanics*, Texts and Monographs in Physics, Springer-Verlag, New York, (1983)

[37] González-Tokman, Cecilia; Quas, Anthony *A concise proof of the multiplicative ergodic theorem on Banach spaces.* J. Mod. Dyn. 9 (2015), 237–255.

[38] Gouëzel, Sébastien; Liverani, Carlangelo *Banach spaces adapted to Anosov systems.* Ergodic Theory Dynam. Systems 26 (2006), no. 1, 189–217.

[39] Kato, Tosio, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften, Band **132**, Springer-Verlag New York, Inc., New York (1966)

[40] A.Katok, B.Hasselblatt *Introduction to the Modern Theory of Dynamical Systems* Camdridge University Press, Cambridge, (1995)

[41] Katok, Anatole; Strelcyn, Jean-Marie; Ledrappier; Przytycki, F. *Invariant manifolds, entropy and billiards; smooth maps with singularities.* Lecture Notes in Mathematics, 1222. Springer-Verlag, Berlin, 1986.

[42] Knauf, Andreas, *Ergodic and topological properties of Coulombic periodic potentials.* Comm. Math. Phys. 110 (1987), no. 1, 89–112.

[43] Krámli, A.; Simányi, N.; Szász, D. *A "transversal" fundamental theorem for semi-dispersing billiards.* Comm. Math. Phys. 129 (1990), no. 3, 535–560.

[44] Krylov, Nikolai Sergeevich *Works on the foundations of statistical physics.* Translated from the Russian by A. B. Migdal, Ya. G. Sinai [Ja. G. Sinaĭ] and Yu. L. Zeeman [Ju. L. Zeeman]. With a preface by A. S. Wightman. With a biography of Krylov by V. A. Fock [V. A. Fok]. With an introductory article "The views of N. S. Krylov on the foundations of statistical physics" by Migdal and Fok. With a supplementary article "Development of Krylov's ideas" by Sinaĭ. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1979.

[45] Hopf, Eberhard, *Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung*, Ber. Verh. Sächs. Akad. Wiss. Leipzig, **91**,261–304 (1939)

[46] Hopf, Eberhard, *Statistik der Lösungen geodätischer Probleme vom unstabilen Typus. II*, Mathematische Annalen, **117**, 590–608 (1940).

[47] Landau, L. D. and Lifshitz, E. M., Course of theoretical physics. Vol. **1**, *Mechanics, Translated from the Russian by J. B. Skyes and J. S. Bell*, Pergamon Press, Oxford (1976)

[48] Lazutkin, V.F. *On the existence of caustics for the billiard ball problem in a convex domain.* Math. USSR Izv. 7, 185–215 (1973)

[49] F.Ledrappier,*Propriètè ergodiques des mesures de Sinai*, C.R.Acad.Sci., Paris, Sèr. I, **294**, 593–595 (1982).

[50] E.H.Lieb, M.Loss *Analysis* Graduate studies in Mathematics, vol. **14**, American Mathematical Society (1996)

[51] Liverani, Carlangelo; Wojtkowski, Maciej P. *Generalization of the Hilbert metric to the space of positive definite matrices.* Pacific J. Math. 166 (1994), no. 2, 339–355.

[52] Liverani, Carlangelo; Wojtkowski, Maciej P. *Ergodicity in Hamiltonian systems.* Dynamics reported, 130–202, Dynam. Report. Expositions Dynam. Systems (N.S.), 4, Springer, Berlin, 1995.

[53] Liverani, Carlangelo *Decay of correlations.* Ann. of Math. (2) 142 (1995), no. 2, 239–301.

[54] Liverani, Carlangelo; Saussol, Benoît; Vaienti, Sandro *A probabilistic approach to intermittency.* Ergodic Theory Dynam. Systems 19 (1999), no. 3, 671–685.

[55] R. Mañe *Ergodic Theory an differentiable dynamics* Spring-Verlag (1987)

[56] Moser, Jürgen, *Stable and random motions in dynamical systems*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ (2001)

[57] Oseledec, V. I. *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems.* (Russian) Trudy Moskov.

Mat. Obšč. 19 (1968), 179–210. [Translation: Trans. Moscow Math. Soc. **19**, 197–231, (1968)]

[58] Palis, Jacob and Takens, Floris, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge Studies in Advanced Mathematics, **35**, Cambridge University Press, Cambridge, (1993)

[59] Ya. B. Pesin *Lyapunov Characteristic Exponents and Smooth Ergodic Theory* Russ. Math. Surveys **32**, 4, 55–114 (1977)

[60] Poincaré, Henri, *Le Méthodes Nouvelles de la Mécanique Céleste, Tome I,II, III*, le grand clasiques Gauthier-Villards, Blanchard, Paris (1987)

[61] M.Reed, B.Simon *Funtional Analysis, I*, Revised and enlarged edition, Accademic Press, New York,London,Toronto,Sydney,San Francisco (1980)

[62] H.L.Royden *Real Analysis*, III edition, Macmillan Publishng Company, New York, Collier Macmillan Publishing, London (1988)

[63] Rudin, Walter, *Functional Analysis*. International Series in Pure and Applied Mathematics. Vol. 8 (Second ed.). New York, NY (1991)

[64] Sarig, Omri *Subexponential decay of correlations.* Invent. Math. 150 (2002), no. 3, 629–653.

[65] Simányi, Nándor, *The K-property of N billiard balls. II. Computation of neutral linear spaces.* Invent. Math. 110 (1992), no. 1, 151–172.

[66] Simányi, Nándor, *The K-property of N billiard balls. I* Invent. Math. 108 (1992), no. 3, 521–548.

[67] Simányi, Nándor; Szász, Domokos *Hard ball systems are completely hyperbolic.* Ann. of Math. (2) 149 (1999), no. 1, 35–96.

[68] Simányi, Nándor *Proof of the Boltzmann-Sinai ergodic hypothesis for typical hard disk systems.* Invent. Math. 154 (2003), no. 1, 123–178.

[69] Ya.G.Sinai *Markov partitions and C-diffeomorphisms* Funct. Anal. Appl. **2**, 61–82, (1968)

[70] Sinaĭ, Ja. G. *Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards.* (Russian) Uspehi Mat. Nauk 25 (1970), no. 2(152), 141–192.

[71] Elias M. Stein and Rami Shakarchi, *Fourier Analysis: An Introduction*, Princeton Unversity press, (2003).

[72] Stenlund, Mikko; Young, Lai-Sang; Zhang, Hongkun *Dispersing billiards with moving scatterers*. Comm. Math. Phys. 322 (2013), no. 3, 909–955.

[73] H.S. *Algebraic theory of continuous fractions*, D.Van Nostrand Company, Inc. Toronto, New York, London (1948).

[74] Wojtkowski, Maciej *Invariant families of cones and Lyapunov exponents*. Ergodic Theory Dynam. Systems 5 (1985), no. 1, 145–161.

[75] Wojtkowski, Maciej *Principles for the design of billiards with nonvanishing Lyapunov exponents*. Comm. Math. Phys. 105 (1986), no. 3, 391–414.

[76] Viana, Marcelo *Lectures on Lyapunov exponents*. Cambridge Studies in Advanced Mathematics, 145. Cambridge University Press, Cambridge, 2014. xiv+202 pp.

[77] Young, Lai-Sang *Statistical properties of dynamical systems with some hyperbolicity*. Ann. of Math. (2) 147 (1998), no. 3, 585–650.

[78] Young, Lai-Sang *Recurrence times and rates of mixing*. Israel J. Math. 110 (1999), 153–188.