# Chapter 2

# Hyperbolic Systems–general facts

This chapter is designed to give a general idea of hyperbolic theory. Since such a theory covers a rather vast landscape, and it contains very technical results our exposition is bound to be quite sketchy.

## 2.1 Hyperbolicity

Our goal in this section is to introduce and discuss a class of systems for which we can hope to investigate the properties introduced in the previous section. As we have seen, the chief property that we used in the study of the Arnold cat were the expanding and contracting properties of the map. These are generalized in the following definition.

**Definition 7** *By Hyperbolic System (with discrete time) we mean a Dynamical System* $(X,\ f,\ \mu)$ *such that* $X$ *is a smooth compact Riemannian manifold (possibly with boundary),* $f$ *is* $\mu$*-almost everywhere differentiable and there exists two measurable families of invariant*[1] *subspaces* $E^u(x)$, $E^s(x) \in \mathcal{T}_x X$ *almost surely transversal,*[2] *and measurable functions* $\nu(x) > 1$, $c(x) > 0$ *such that for almost all* $x \in X$

$$\begin{aligned} \|D_x f^n v\| &\geq c(x)^{-1}\nu(x)^n \|v\| \quad \forall v \in E^u(x) \\ \|D_x f^n v\| &\leq c(x)\nu(x)^{-n} \|v\| \quad \forall v \in E^s(x). \end{aligned}$$

[1] That is $D_x f E^{s(u)}(x) = E^{s(u)}(fx)$.

[2] That is, $E^u(x) \cap E^s(x) = \{0\}$ and $E^u(x) \oplus E^s(x) = \mathcal{T}_x X$ a.e.

*If the functions $c$, $\nu$ can be chosen constant and the distributions are transversal at each point, then the system is called* Uniformly Hyperbolic. *In addition, if $f$ is a diffeomorphism and $E^u$, $E^s$ vary with continuity, then the system is called* Anosov *(or sometimes $C$ or $U$ systems).*

The condition in Definition 7 is essentially equivalent to saying that two very close initial conditions almost certainly will grow apart at an exponential rate. This corresponds to a strong instability with respect to the initial conditions and characterizes the sense in which the dynamics of hyperbolic systems is a very complex one. Such complex behaviour has captured the popular fantasy under the ambiguous name of chaos.

### 2.1.1 Examples

#### *Rotations*

Clearly the rotations are not hyperbolic since $Df = 1$.

#### *Dilation*

One can easily see that such a system is expanding, hence $E^u = \mathbb{R}$ and $E^s = \emptyset$.

#### *Arnold cat*

We have seen it in detail in the previous chapter.

#### *Baker*

In this case one direction is expanding and one is contracting, $\dim E^u = \dim E^s = 1$

A more general notion of hyperbolicity is the one of *hyperbolic set*.

**Definition 8** *Given a diffeomorphism $f$ of a manifold $X$, we say that $\Lambda \subset X$ is hyperbolic if $\Lambda$ is compact, $f(\Lambda) = \Lambda$ and there exists two measurable families of invariant subspaces $E^u(x)$, $E^s(x) \in \mathcal{T}_x X$ transversal at each point and measurable functions $\nu(x) > 1$, $c(x) > 0$ such that for all $x \in \Lambda$*

$$\begin{aligned} \|D_x f^n v\| &\geq c(x)^{-1}\nu(x)^n \|v\| \quad \forall v \in E^u(x) \\ \|D_x f^n v\| &\leq c(x)\nu(x)^{-n} \|v\| \quad \forall v \in E^s(x). \end{aligned}$$

*If the constants $c$, $\nu$ can be chosen independently of $x \in \Lambda$ then $\Lambda$ is called* Uniformly Hyperbolic.

### 2.1.2 Examples

#### *Smale Horseshoe*

In this case the set $\Lambda$ is the one constructed in Examples 1.4.1 and $\dim E^s = \dim E^u = 1$.

#### *Forced pendulum*

Same situations as for the horseshoe, see Examples 1.8.1.

Definition 7 it is not particularly helpful in concrete cases since, in general, it is not clear how to verify if a systems is hyperbolic or not.

## 2.2 Lyapunov exponents and invariant distributions

We start by a different and very helpful characterization of hyperbolicity obtained by introducing the so called Lyapunov Exponents (LE).

**Definition 9** *For each $x \in X$, $v \in \mathcal{T}_x X$ we define*

$$\lambda(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\|.$$

*If $\lambda(x, v)$ exists it is called "Lyapunov exponent" (LE).*

It is interesting to notice that $\lambda(f(x), D_x f v) = \lambda(x, v)$ (see Problem 2.1). Moreover, it should be clear that, if the system is ergodic and the map invertible, then $\lambda(x, v)$, if it exists, can assume only finitely many values (see Problem 2.3).

The existence and properties of the LE have been intensively studied and have given rise to a multitude of results. Here, we content ourselves with the following theorem, which is by far not the most general version, but it suffices for our needs. See [62] for a more extensive presentation of the theorem and its proof. I will provide some ideas related to the proof at the end of the section.

**Theorem 2.2.1 (Oseledets [45])** *Let $(X, \mu)$ be a probability space and $f : X \to X$ a measure-preserving transformation. Let $L : X \to GL(d, \mathbb{R})$ be a measurable mapping from $X$ to the invertible $n \times n$ matrices such that $\ln \|L(\cdot)^{\pm 1}\| \in L^1(X, \mu)$. Then for $\mu$-almost all $x \in X$ there are subspaces*

*$\{0\} = V_x^0 \subset V_x^1 \subset \cdots \subset V_x^d = \mathbb{R}^d$ and numbers $\lambda_1(x) \leq \cdots \leq \lambda^d(x)$ such that, for all $i \in \{1, \ldots, d\}$,*[3]

$$\lim_{n\to\infty} \frac{1}{n} \ln \|L(f^{n-1}(x)) \cdots L(f(x))L(x)v\| = \lambda_i(x)$$

*if $v \in V_x^i \setminus V_x^{i-1}$.*

The above theorem is taylored to study *cocylces*, that is the dynamical systems $(X \times \mathbb{R}^d, F)$ where $F(x, v) = (f(x), L(x)v)$. Indeed one can easily check that $F^n(x, v) = (f^n(x), L(f^{n-1}(x)) \cdots L(x)v))$.
Given a measurable dynamical system $(X, f, \mu)$, where $X$ is a $d$ dimensional Riemannian manifold (possibly with boundary) and $T$ is almost surely differentiable we have the natural cocycle $(X \times \mathbb{R}^d, F)$ where $F(x, v) = (f(x), D_x f(v))$. Note that, by the chain rule, $F^n(x, v) = (f^n(x), D_x f^n v)$. So, the Lyapunov exponents of $f$ are exactly the numbers given by Oseledets' Theorem for the associated cocycle. The connection between Lyapunov exponents and hyperbolicity is illustrated by the following.

**Theorem 2.2.2** *A system $(X,\ f,\ \mu,)$, where $X$ is a Riemannian manifold and $f$ is a diffeomorphism. Then, $f$ is hyperbolic iff for almost all $x \in X$*

$$\lambda(x,\ v) \neq 0 \quad \forall v \in \mathcal{T}_x X, \quad v \neq 0.$$

Proof. Clearly, if the system is hyperbolic, then all the LE are different from zero. The other implication is almost as trivial. Define $E^s(x) = \{v \in \mathcal{T}_x \mid \lambda(x, v) < 0\}$; then consider the Dynamical system $(X, f^{-1}, \mu)$ and its LE $\lambda^-(x, v)$ and define $E^u(x) = \{v \in \mathcal{T}_x \mid \lambda^- < 0\}$. Next, let

$$\rho(x) = \sup\{\lambda(x, v),\ \lambda_-(x, w) \mid v \in E^s(x),\ w \in E^u(x)\}$$

clearly $\rho(x) < 0$ a.e.. Then setting $\nu(x) = e^{-\rho(x)/2}$ and

$$c(x) = \sup_n \{\nu(x)^n \|D_x f^n v\|,\ \nu(x)^n \|D_x f^{-n} w\| \mid v \in E^s(x);\ w \in E^u(x)\}_{n\in\mathbb{N}},$$

which is almost surely finite by construction, hence proving the theorem. □

To conclude the section, let me provide a few ideas related to the proof of Theorem 2.2.1 to give a feeling of what is involved. Note that, thanks to the ergodic decomposition, we can assume w.l.o.g. that $\mu$ is ergodic. Let us define

$$\bar{\lambda}(x, v) = \limsup_{n\to\infty} \frac{1}{n} \ln \|L(f^{n-1}(x)) \cdots L(f(x))L(x)v\|. \tag{2.2.1}$$

[3]Note that the $\mathbb{V}_i$ and the $\lambda_i$ are not necessarily distinct.

Note that, for each $\alpha \in \mathbb{R}$,

$$\bar\lambda(x, \alpha v) = \limsup_{n\to\infty} \frac{1}{n} \ln \left\{ \|L(f^{n-1}(x)) \cdots L(f(x))L(x)v\| |\alpha| \right\} = \lambda(x, v).$$

In addition,

$$\begin{aligned}\bar\lambda(x, v) &\leq \limsup_{n\to\infty} \frac{1}{n} \ln \|L(f^{n-1}(x)) \cdots L(f(x))L(x)\| \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \|L(f^{k-1}(x))\| = \int_X \ln \|L(x)\| \mu(dx) = \lambda_+(x).\end{aligned}$$

where the last inequality follows by Birkhoff's theorem. In addition, for $\|v\| = 1$,

$$\begin{aligned}0 &= \ln \|L(x)^{-1} L(f(x))^{-1} \cdots L(f^{n-1}(x))^{-1} L(f^{n-1}(x)) \cdots L(x)v\| \\ &\leq \ln \|L(x)^{-1} L(f(x))^{-1} \cdots L(f^{n-1}(x))^{-1}\| + \ln \|L(f^{n-1}(x)) \cdots L(x)v\|.\end{aligned}$$

Which, arguing as before, yields

$$\bar\lambda(x, v) \geq \int_X \ln \|L(x)^{-1}\|^{-1} \mu(dx).$$

Hence, the numbers $\bar\lambda(x, v)$ are almost surely bounded. Next, for $v, w \in \mathbb{R}^d$ let

$$\begin{aligned}v_n &= \sup_{m\geq n} \|L(f^{n-1}(x)) \cdots L(f(x))L(x)(v + w)\| \\ w_n &= \sup_{m\geq n} \|L(f^{n-1}(x)) \cdots L(f(x))L(x)(v + w)\|\end{aligned}$$

then

$$\begin{aligned}\bar\lambda(x, v + w) = &\limsup_{n\to\infty} \frac{1}{n} \ln \frac{\|L(f^{n-1}(x)) \cdots L(f(x))L(x)(v + w)\|}{\max\{v_n, w_n\}} \\ &+ \max\{\bar\lambda(x, v), \bar\lambda(x, w)\} \leq \max\{\bar\lambda(x, v), \bar\lambda(x, w)\}\end{aligned}$$

since $\|L(f^{n-1}(x)) \cdots L(f(x))L(x)(v + w)\| \leq 2 \max\{v_n, w_n\}$. Finally, by definition,

$$\bar\lambda(x, L(x)v) = \bar\lambda(f(x), v).$$

It follows that if we define $\mathbb{V}(x, \alpha) = \{v \in \mathbb{R}^d : \bar\lambda(x, v) \leq \alpha\}$, then the $\mathbb{V}(x, \alpha)$ are vector spaces and $L(x)\mathbb{V}(x, \alpha) = \mathbb{V}(f(x), \alpha)$. Two technical issues remain: to show that the functions $\mathbb{V}(\cdot, \alpha)$ are measurable;[4] to show that the limsup in (2.2.1) is indeed a limit. I refer to [62] for a proof.

[4] Here we see $\mathbb{V}(\cdot, \alpha)$ as elements of the union of Grassmanian, which is a topological space. If you do not want to be very fancy given $\mathbb{V}_1, \mathbb{V}_2 \subset \mathbb{R}^d$ let their distance be the Hausdroff distance between their intersection with the unit sphere. This gives a topology and we consider the associated Borel $\sigma$-algebra.

## 2.3 Comments on the non-smooth case

The results of the last section can be applied to non-smooth systems, however to develop a useful theory the singularities of the system cannot be arbitrary. As we will see in the following, systems that are quite natural both from the mathematical point of view and from the physical one are not smooth–typically they have discontinuities. In this section we will discuss a class of systems called *smooth systems with singularities*. Although the theory of such systems has been done in great generality, here we will give a restrictive definition, just sufficient for our later purposes. See the notes at the end of the chapter for information on more general settings.

**Definition 10** *By Smooth Dynamical System with singularities we mean a Dynamical Systems $(X, T, \mu)$, where*

- *$X$ is the union of finitely many compact pieces $X_i$ of $\mathbb{R}^n$, $\partial X_i$ is the union of finitely many $n-1$ dimensional smooth manifolds.*
- *$T$ is smooth outside a compact set $\mathcal{S}$. The singularity set $\mathcal{S}$ is the finite union of smooth $n-1$ dimensional manifolds with boundary $\mathcal{S}_i$, $\mathcal{S}_i \cap \mathcal{S}_j \not\subset \partial\mathcal{S}_i \cap \partial\mathcal{S}_j$ implies $i = j$. In addition, the boundary $\partial\mathcal{S}_i$ is the finite union of smooth $n-2$ dimensional manifolds.*
- *There exists $c_1, c_2 > 0$ such that*
$$\|D_x T\| + \|D_x^2 T\| \leq c_1 \, dist\,(x, \mathcal{S})^{-c_2}.$$
- *The measure $\mu$ is absolutely continuous with respect to Lebesgue.*

**Remark 2.3.1** *Note that the fact that $(X, T, \mu)$ is a Smooth Dynamical System with singularities does not implies immediately that the same holds for $(X, T^k, \mu)$. The problem is that the map $T$ can be very wild near the set $\mathcal{S}$, so it is not clear that the singularity set of $T^k$ will satisfy our requirements. Nevertheless, in the examples we will consider, all the Dynamical System $(X, T^k, \mu)$ will always be Smooth Dynamical System with singularities.*

**Remark 2.3.2** *We will call a smooth Dynamical System with singularities invertible if $T^{-1}$ is densely defined and $(X, T^{-1}, \mu)$ is itself a smooth Dynamical System with singularities.*

Note that the above conditions imply the applicability of Oseledets Theorem.

### 2.3.1 Examples

#### *Backer map*

It is easy to check that the Backer map is a Smooth Dynamical System with singularities.

#### *Discontinuous Arnold cat*

If we consider $(\mathbb{R}^2, L, m)$ where

$$L\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & a\\ a & 1+a^2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \tag{2.3.2}$$

with $a \notin \mathbb{Z}$, then it is not possible to project the system down to a torus preserving the continuity of the map. Yet, we can construct a discontinuous version of the Arnold cat.

Consider $\mathcal{M}_+ = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x + ay < 1;\ 0 \leq y < 1\}$ and $\mathcal{M}_- = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < 1;\ 0 \leq ax - y < 1\}$. It is easy to see that, if $\Pi$ is the projection from the universal cover $\mathbb{R}^2$ to the torus $\mathbb{T}^2$ ($\Pi\xi = \xi \mod 1$), then $\Pi$, restricted to $\mathcal{M}_\pm$, is one-one and onto. Moreover, $L\mathcal{M}_+ = \mathcal{M}_-$. This means that we can define $T : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T = \Pi L(\Pi|_{\mathcal{M}_+})^{-1}.$$

Of course $T$ is discontinuous on $\mathcal{S}_+ := \partial\mathcal{M}_+$ and $T^{-1}$ is discontinuous on $\mathcal{S}_- := \partial\mathcal{M}_-$. In addition, the Lebesgue measure is invariant and the map is hyperbolic since $DT = L$.

The question arises if there exists stable and unstable manifolds. A moment of thought shows that this is equivalent to the following question: there exist segments in the stable (unstable) direction such that their images in the future (past) never meet the discontinuity set $\mathcal{S}_+$ ($\mathcal{S}_-$)?

Let us analyze the unstable manifolds. Call $\mathcal{S}_\delta$ the $\delta$ neighborhood of $\mathcal{S}_+$. Consider a segment $J$ centered at $x$, in the unstable direction, and suppose that $T^{-n}J \cap \mathcal{S}_+ \neq \emptyset$, then $J$ cannot be the unstable manifold since its points do not have the same asymptotic trajectory in the past. Let $\lambda > 1$ the eigenvalue of $L$, then $T^{-n}J$ has total length $\lambda^{-n}|J|$, so the trajectory of $x$ can be fairly close to $\mathcal{S}_+$ without having a problem. This discussion leads naturally to considering the set

$$G_\delta = \{x \in \mathbb{T}^2 \mid \text{dist}\ (T^{-n}x, \mathcal{S}_+) \geq \lambda^{-n}\delta\}.$$

On the one hand, it is clear that if $x \in G_\delta$, a segment in the unstable direction of size $\delta$ is indeed an unstable manifold. On the other hand,

$m(G_\delta) \leq c\delta$. Thus almost all the points do have an unstable manifold of some positive size. This it is encouraging, yet it is clearly not sufficient to perform the Hopf argument. For the time being it suffices to notice that what we have seen so far implies that the discontinuous Arnold cat has, at most, countably many ergodic components.

## 2.4 Flows

All what we have described so far has a rather straightforward generalization in the case of flows, yet some natural changes are called for.

To appreciate the problem let us consider a flow, on a compact Riemannian manifold, generated by a smooth non-zero vector field $V$. By definition $\frac{d}{dt}\phi^t|_{t=0} = V(x)$ and $d\phi^t V(x) = V(\phi^t x)$, thus $\lambda(x, V(x)) = 0$. This is a rather general fact: the Lyapunov exponent in a flow, with a nonvanishing vector field, is zero in the flow direction. The only relevant exception is constituted by hyperbolic fixed points (think of the unstable equilibrium point of the pendulum) that, in the previous example, was ruled out by the assumption that the vector field be non zero. We will consider only such case.

Consequently a flow is hyperbolic if the tangent space is split in three transversal subspaces $E^s, E^u, E^0$, where $E^0$ is the flow direction and corresponds to a zero Lyapunov exponents.

Oseledec Theorem (Theorem 2.2.1) holds unchanged with the $L^1$ condition on the cocycle obviously replaced by

$$\int_X \|\log d\phi^t\| d\mu < \infty.$$

For a smooth flow coming from a non vanishing vector field Theorem 2.2.2 holds unchanged as well.

### 2.4.1 Examples

#### *Smooth flows with collisions*

Let $M$ be a smooth manifold with piecewise smooth boundary $\partial M$. We assume that the manifold $M$ is equipped with a symplectic structure $\omega$.[5] Given a smooth function $H$ on $M$ with *non vanishing* differential we obtain the non vanishing Hamiltonian vector field $F = \nabla_\omega H$ on $M$ by $\omega(\nabla_\omega H, v) =$

[5] That is a non–degenerate closed antisymmetric two form.

$dH(v)$. The vector field $F$ is tangent to the level sets of the Hamiltonian $M^c = \{z \in M | H(z) = c\}$.

We distinguish in the boundary $\partial M$ the regular part, $\partial M_r$, consisting of the points which do not belong to more than one smooth piece of the boundary and where the vector field $F$ is transversal to the boundary. The regular part of the boundary is further split into "outgoing" part, $\partial M_-$, where the vector field $F$ points outside the manifold $M$ and the "incoming" part, $\partial M_+$, where the vector field is directed inside the manifold. Suppose that additionally we have a piecewise smooth mapping $\Gamma : \partial M_- \to \partial M_+$, called the collision map. We assume that the mapping $\Gamma$ preserves the Hamiltonian, $H \circ \Gamma = H$, and so it can be restricted to each level set of the Hamiltonian.

We assume that all the integral curves of the vector field $F$ that end (or begin) in the singular part of the boundary lie in a codimension 1 submanifold of $M$.

We can now define a flow $\Psi^t : M \to M$, called a flow with collisions, which is a concatenation of the continuous time dynamics $\Phi^t$ given by the vector field $F$, and the collision map $\Gamma$. More precisely a trajectory of the flow with collisions, $\Psi^t(x),\ x \in M$, coincides with the trajectory of the flow $\Phi^t$ until it gets to the boundary of $M$ at time $t_c(x)$, the collision time. If the point on the boundary lies in the singular part then the flow is not defined for times $t > t_c(x)$ (the trajectory "dies" there). Otherwise the trajectory is continued at the point $\Gamma(\Psi^{t_c} x)$ until the next collision time, i.e., for $0 \le t \le t_c\left(\Gamma(\Psi^{t_c(x)} x)\right)$

$$\Psi^{t_c + t} x = \Phi^t \Gamma \Psi^{t_c} x.$$

We define a flow with collisions to be symplectic, if for the collision map $\Gamma$ restricted to any level set $M^c$ of the Hamiltonian we have

$$\Gamma^* \omega = \omega.$$

More explicitly we assume that for every vectors $\xi$ and $\eta$ from the tangent space $T_z \partial M^c$ to the boundary of the level set $M^c$ we have

$$\omega(D_z \Gamma \xi, D_z \Gamma \eta) = \omega(\xi, \eta).$$

We restrict the flow with collisions to one level set $M^c$ of the Hamiltonian and we denote the resulting flow by $\Psi_c^t$. This flow is very likely to be badly discontinuous but we can expect that for a fixed time $t$ the mapping $\Psi_c^t$ is piecewise smooth, so that the derivative $D\Psi_c^t$ is well defined except for a finite union of codimension one submanifolds of $M^c$. We will consider only such cases.

The symplectic volume $\wedge^d\omega$ is clearly invariant for the flow, so will be the measure $\mu_c$ obtained by restricting the symplectic volume to the manifold $M^c$. Clearly for such an invariant measure all the trajectories that begin (or end) in the singular part of the boundary have measure zero. With respect to the measure $\mu_c$ the flow $\Psi_c^t$ is a measurable flow in the sense of Definition 2 and we obtain a measurable derivative cocycle $D\Psi_c^t : T_xM^c \to T_{\Psi_c^t x}M^c$. We can define Lyapunov exponents of the flow $\Psi_c^t$ with respect to the measure $\mu_c$, if we assume that[6]

$$
\begin{aligned}
&\int_{M^c} \log_+ \|D_x\Psi_c^t\| d\mu_c(x) < +\infty \\
&\int_{\partial M^c_-} \log_+ \|D_y\Gamma\| d\mu_{cb}(y) < +\infty
\end{aligned}
\tag{2.4.3}
$$

(cf.[45]).

## Problems

**2.1** Prove that $\lambda(Tx, D_xTv) = \lambda(x, v)$.

**2.2** Prove that $\lambda(x, v+w) \leq \max\{\lambda(x, v), \lambda(x, w)\}$ and $\lambda(x, \alpha v) = \lambda(x, v)$ for each $\alpha \in \mathbb{R}$, if they all exist. (Hint: Just apply the definition of LE and note that

$$\lambda(x, v+w) \leq \lim_{n\to\infty} \max\{\frac{1}{n}\log\|D_xT^nv\|, \frac{1}{n}\log\|D_xT^nw\|\}. \quad )$$

**2.3** Assuming only that the LE are well defined a.e., prove that, if $(X, T, \mu)$ is ergodic, $X$ is a $d$ dimensional manifold and $T$ a diffeomorphism, then there exists $d$ numbers $\{\lambda_i\}$ such that the Lyapunov exponents $\lambda(x, v) \in \{\lambda_i\}$ a.e.. (Hint: For each $\alpha \in \mathbb{R}$ define $V_\alpha(x) := \{v \in \mathcal{T}_xX \mid \lambda(x, v) \leq \alpha\}$. By Problem 2.2 $V_\alpha(x)$ is a linear vector space and, by Problem 2.1 the distribution $V_\alpha$ is invariant. Then $d_\alpha(x) := \dim V_\alpha(x)$ is an invariant function, thus a.e. constant for each $\alpha$. In addition, $d_\alpha$ is an increasing function of $\alpha$ and can assume only the values $\{0, \ldots, d\}$. Thus there are at most $s \leq d$ $\{\alpha_j\}$ where $d_\alpha$ jumps. But this means that the LE are discrete. In fact, let $v \in V_\alpha(x)\backslash V_\beta(x)$,

[6] Here $\mu_{cb}$ is the restriction of the volume to $\partial M^c_-$.

$\alpha > \beta$, then for each $w \in \operatorname{span}\{v, V_\beta(x)\}$ it is easy to compute that $\lambda(x,w) = \lambda(x,v) > \beta$, which means: the LE is constant over $V_\alpha(x)$ apart for lower dimensional subspaces. In addition, we have a flag of subspaces $\{V_i\}_{i=0}^s$, $s \leq d$, such that $V_\alpha \in \{V_i\}_{i=0}^s$ for each $\alpha \in \mathbb{R}$. Hence, if $V_\alpha \supset V_i$ but $V_\alpha \not\supset V_{i+1}$ it must be $V_\alpha = V_i$, thus if $v \in V_\alpha$ but $v \notin V_{i-1}$ $\lambda(x,v) = \alpha_i$ where $\alpha_i = \inf\{\alpha \in \mathbb{R} \mid V_\alpha \supset V_i\}$.)

**2.4** Show that, if $T$ is invertible, $\{\lambda_i(x)\}$ is equal a.e. to $\{-\lambda_i^-(x)\}$ where $\{\lambda_i^-(x)\}$ are the LE of $(X, T^{-1}, \mu)$.

**2.5** Show that

$$\lim_{n\to\infty} \frac{1}{n} \log|\det(D_x T^n)|$$

exists almost everywhere. (Hint: Apply BET.)

**2.6** Let $(X, T, \mu)$ be a Dynamical Systems, $X$ a compact Riemannian manifold and $T$ a.e. differentiable. Suppose that there exists a one-dimensional distribution $E(x)$ such that $D_x T E(x) = E(Tx)$. Prove, without using Oseledets theorem, that for each $v \in E(x)$ the LE $\lambda(x,v)$ is well defined. (Hint: Let $v(x) \in E(x)$, $\|v(x)\| = 1$, then $D_x T v(x) = \alpha(x) v(Tx)$ and thus $D_x T^n v(x) = \prod_{i=1}^n \alpha(T^i x) v(T^i x)$. Then the result follows by the BET.)

**2.7** Define a cocycle associated with a flow with collision, which yields all the Lyapunov exponents, but the one in the flow direction. (Hint: The derivative of the flow with collisions can also be naturally factored onto the quotient of the tangent bundle $TM^c$ of $M^c$ by the vector field $F$, which we denote by $\widehat{T}M^c$. Note that for a point $z \in \partial M^c$ the tangent to the boundary at $z$ can be naturally identified with the quotient space. We will again denote the factor of the derivative cocycle by

$$A^t(x) : \widehat{T}_x M^c \to \widehat{T}_{\Psi_c^t x} M^c.$$

We will call it the transversal derivative cocycle. If the derivative cocycle has well defined Lyapunov exponents then the transversal derivative cocycle has also well defined Lyapunov exponents which coincide with the former ones except that one zero Lyapunov exponent is skipped.)

The theory of foliations for piecewise continue maps is developed in great generality in [33].