Chapter 3

Hyperbolicity: how to establish it

Here, we discuss how to establish hyperbolicity for symplectic maps and flows. The ideas put forward can also be used for more general systems, but simplecitiy provides an extra structure that allows the development of a much richer theory. Since Billiards are Hamiltonian systems, and hence give rise to symplectic flows and maps, this theory is relevant for Billiards.

The material of this chapter is taken from [63, 44], and the reader is referred to such articles for the full details. Here I just try to present the ideas in the simplest possible form.

3.1 Hamiltonian flows and Symplectic structure

Given the matrix $2d \times 2d$ defined by

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

Hamilton's equations can be written as[1]

$$\dot{x} = J\nabla H(x) \tag{3.1.1}$$

where $x = (q, p)$. Note that $J^2 = -\mathbb{1}$ e $J^T = -J$.[2] The matrix J plays a fundamental role in the Hamiltonian structure. In particular, one can define

[1]The gradient of a function $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ is given by the vector $\nabla f := (\partial_{x_i} f)$.

[2]Note the similarity with the imaginary number i, where the transpose takes the place of the complex conjugation; this is no accident!

the bilinear form on $\mathbb{R}^{2d}$

$$\omega(v,w) := \langle v, Jw\rangle. \tag{3.1.2}$$

The form ω is called the *symplectic form.* A matrix A with the property $\omega(Av, Aw) = \omega(v,w)$, for every $v, w \in \mathbb{R}^{2d}$, is called *symplectic.* A transformation $F \in \mathcal{C}^1(\mathbb{R}^{2d}, \mathbb{R}^{2d})$ such that $DF(x)$ is symplectic for every $x \in \mathbb{R}^{2d}$ is said to be *symplectic transformation.*

Lemma 3.1.1 *For each Hamiltonian H the Hamiltonian flow ϕ_t is a symplectic transformation.*

Proof. Let $\Xi(x,t) = D\phi_t$, then

$$\dot{\Xi}(x,t) = JD^2H \circ \phi_t(x) \cdot \Xi(x,t)$$

hence, for each $v, w \in \mathbb{R}^{2d}$,

$$\frac{d}{dt}\omega(\Xi v, \Xi w) = \omega(\dot{\Xi} v, \Xi w) + \omega(\Xi v, \dot{\Xi} w) = \langle JD^2H\Xi v, J\Xi w\rangle - \langle \Xi v, D^2H\Xi w\rangle = 0,$$

where we used the fact that D^2H is a symmetric matrix.[3] □

Lemma 3.1.2 *The set of symplectic matrices form a group (called $Sp(2d,\mathbb{R})$). Furthermore, if $L \in Sp(2d,\mathbb{R})$, then $L^T \in Sp(2d,\mathbb{R})$.*

Proof. First note that a matrix is symplectic if and only if $L^TJL = J$. Then it is trivial to verify that $\mathbb{1} \in Sp(2d,\mathbb{R})$. Furthermore, if $L, B \in Sp(2d,\mathbb{R})$, then

$$(LB)^TJLB = B^TL^TJLB = J,$$

therefore $LB \in Sp(2d,\mathbb{R})$. Moreover, $L[-JL^TJ] = \mathbb{1}$ shows that L is invertible and $L^{-1} = -JL^TJ$, furthermore

$$(L^{-1})^TJL^{-1} = (-JL^TJ)^TJL^{-1} = JLL^{-1} = J.$$

Hence $L^{-1} \in Sp(2d,\mathbb{R})$. Finally, if $L \in Sp(2d,\mathbb{R})$, then $L^{-1}J(L^T)^{-1} = J$ which impleis $(L^T)^{-1} \in Sp(2d,\mathbb{R})$ and $L^T \in Sp(2d,\mathbb{R})$. □

Next, we provide a useful decomposition.

[3]Obviously we are assuming that $H \in \mathcal{C}^2$ and symmetry follows from Schwartz's Lemma.

Lemma 3.1.3 *If* $L := \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \in Sp(2d, \mathbb{R})$*, where* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ *are* $d \times d$ *matrices, and* $\det(\boldsymbol{a}) \neq 0$*, then there exist symmetric* $d \times d$ *matrices* R, P *such that*

$$L = \begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ P & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & R \\ 0 & \mathbb{1} \end{pmatrix}, \tag{3.1.3}$$

PROOF. A direct computation shows that $L \in Sp(2d, \mathbb{R})$ if and only if

$$\boldsymbol{c}^T \boldsymbol{a} = (\boldsymbol{a}^T \boldsymbol{c})^T = \boldsymbol{a}^T \boldsymbol{c}\,; \quad \boldsymbol{d}^T \boldsymbol{b} = (\boldsymbol{b}^T \boldsymbol{d})^T = \boldsymbol{b}^T \boldsymbol{d}\,; \quad \boldsymbol{a}^T \boldsymbol{d} - \boldsymbol{c}^T \boldsymbol{b} = \mathbb{1}. \tag{3.1.4}$$

Since $\boldsymbol{a}$ is invertible, we can write

$$L = \begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbb{1} & R \\ P & H \end{pmatrix}, \tag{3.1.5}$$

where $R = \boldsymbol{a}^{-1}\boldsymbol{b}$, $P = \boldsymbol{a}^T \boldsymbol{c}$ and $H = \boldsymbol{a}^T \boldsymbol{d}$. Condition (3.1.4) implies that $\begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix}$ is symplectic. Then, by Lemma 3.1.2, also the matrix $\begin{pmatrix} \mathbb{1} & R \\ P & H \end{pmatrix}$ must be symplectic. Accordingly, (3.1.4) implies

$$P^T = P\,; \quad H = \mathbb{1} + P^T R = \mathbb{1} + PR.$$

On the other hand, by Lemma 3.1.2, also the matrix $\begin{pmatrix} \mathbb{1} & P \\ R^T & H^T \end{pmatrix}$ is symplectic, hence (3.1.4) implies

$$R^T = R$$

from which the Lemma follows. □

Note that $L^T J L = J$ implies $\det(L)^2 = 1$. In fact, since the symplectic group is connected, the above decomposition implies that $\det(L) = 1$ by continuity (see Problem 3.8 for a more direct proof of this latter fact).

3.2 Symplectic Poincarè sections and time one maps

Let $\tau : \mathbb{R}^{2d} \to \mathbb{R}_+$ be a piecewise differentiable function and define the map $f(x) = \phi_{\tau(x)}(x)$. Where f is differentiable, we have

$$D_x f = D_x \phi_\tau + J \nabla H(\phi_\tau(x)) \otimes \nabla \tau.$$

We restrict the map f to a constant energy surface $M_E = \{x \in \mathbb{R}^{2d} : H(x) = E\}$. Then, for $v \in TM_E$ we have $\langle \nabla H, v\rangle = 0$. It follows that, for $v, w \in TM_E$,

$$\begin{aligned}\omega(Dfv, Dfw) =&\langle D\phi_\tau v + J\nabla H(\phi_\tau(x))\langle\nabla\tau, v\rangle, J(D\phi_\tau w + J\nabla H(\phi_\tau(x))\langle\nabla\tau, w\rangle)\rangle\\ =&\omega(v,w) + \langle\nabla H(\phi_\tau(x)), D\phi_\tau w\rangle\langle\nabla\tau, v\rangle\\ &- \langle D\phi_\tau v, \nabla H(\phi_\tau(x))\rangle\langle\nabla\tau, w\rangle\\ &+ \langle\nabla H(\phi_\tau(x)), J\nabla H(\phi_\tau(x))\rangle\langle\nabla\tau, v\rangle\langle\nabla\tau, w\rangle = \omega(v,w).\end{aligned}$$

It is then natural to introduce the equivalence relation $v \sim w$ is $v - w = \lambda J\nabla H$ for some $\lambda \in \mathbb{R}$. Let $\mathbb{V}_x = T_x M_E / \sim$ be the vector space formed by the equivalence classes. Note that

$$\begin{aligned}D_x f(v + \lambda J\nabla H(x)) &= D_x f v + \lambda D_x\phi_\tau J\nabla H(x) + \lambda J\nabla H(\phi_\tau(x))\langle\nabla\tau, J\nabla H(x)\rangle\\ &= D_x f v + \lambda J\nabla H(f(x))\left[1 + \langle\nabla\tau, J\nabla H(x)\rangle\right].\end{aligned}$$

Hence, the action of Df from $T_x M_E$ to $T_{f(x)}M_E$ quotients naturally in an action between $\mathbb{V}_x$ and $\mathbb{V}_{f(x)}$. On the other hand, for $v \in \mathbb{V}_x$ we have

$$\omega(J\nabla H, v) = \langle\nabla H, v\rangle = 0.$$

Thus $\omega(v+\lambda J\nabla H, w+\mu J\nabla H) = \omega(v,w)$, that is we can quotient ω as well on $\mathbb{V}_x$. It follows that ω induces canonically a symplectic form, which we still call ω, on each $\mathbb{V}_x$. By the above discussion the d dimensional spaces $W_1^+ = \{(v,0) : v \in \mathbb{R}^d\}$ and $W_2^+ = \{(0,v) : v \in \mathbb{R}^d\}$ quotient to $d-1$ dimensional spaces W_i in each $\mathbb{V}_x$, moreover $\omega(w,w') = 0$ for each $w, w' \in W_1$ or $w, w' \in W_2$ (such subspaces, as we will see briefly, are called Lagrangian). Next, one can check that it is possible to choose basis $\{e_i\}$ in W_1 and $\{f_i\}$ in W_2 such that $\omega(e_i, f_j) = \delta_{ij}$. Then we can write any vector $a \in \mathbb{V}_x$ as $a = \sum_{i=1}^{d-1}\xi_i e_i + \sum_{i=1}^{d-1}\eta_i f_i$ and

$$\omega(a,a') = \sum_{i,j}\xi_i\eta'_j\omega(e_i,f_j) + \eta_i\xi'_j\omega(f_i,e_j) = \sum_i \xi_i\eta'_i - \xi'\eta_i = \langle(\xi,\eta), J(\xi',\eta').\rangle$$

That is, in such coordinates, the symplectic form has the standard form (3.1.2). We can thus identify all the spaces $\mathbb{V}_x$ and, in such coordinates, $Df|_{\mathbb{V}}$ is symplectic.

By choosing $\tau \equiv 1$, the map ϕ_1 can be seen as a $2d-2$ symplectic map. Moreover, if Σ is a Poincarè section for the flow, then we can choose τ to be the first return time and since $\mathbb{V}$ is naturally isomorphic to $T\Sigma$, again we have that the Poincarè map $f(x) = \phi_{\tau(x)}(x)$ is symplectic.

3.3 Two dimensions

We are interested in the case $L(x) = D_x\phi_1$, where ϕ_t is the billiard flow. Of course, the flow will have a zero Lyapunov exponent (the flow direction).

Definition 11 *A symplectic flow is hyperbolic if the only zero Lyapunov exponent is the one associated with the flow direction. Equivalently, a symplectic flow is hyperbolic if the Poincarè map has no zero Lyapunov exponent.*

The problem is to have a tool to establish hyperbolicity. The following theorem provides a very efficient tool (we do not provide the proof as it is a special case of Theorem 3.5.1).

Theorem 3.3.1 (Wojtkowski [63]) *Let X be a Riemannian manifold, possibly with boundaries, $\{\mathcal{C}(x) \subset \mathcal{T}_xX \;:\; x \in X\}$ a family of closed cones in the tangent space. Let $f : X \to X$ and $L : X \to SL(n, \mathbb{R})$ as in Theorem 2.2.1. If for μ almost $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $L(f^{n(x)-1}) \cdots L(x)\mathcal{C}(x) \subset \operatorname{int}(\mathcal{C}(f^{n(x)}(x)))$, then the maximal Lyapunov exponent is strictly positive.*

The above theorem suffices for planar billiards, where there are two Lyapunov exponents λ_i and, by volume conservation $\lambda_1 = -\lambda_2$. For higher dimensional billiard, it does not control all the Lyapunov exponents. To achieve this, we have to use more heavily the fact that the Billiards flows are Hamiltonian, and hence symplectic. In addition, while a two-dimensional cone is simply a sector, a higher-dimensional cone can have many different shapes, and it is not obvious what is a natural cone shape.

3.4 Higher dimensions: the symplectic structure

Given a symplectic form ω, which is left invariant by map $f : M \to M$, we have a symplectic flow. If $\mathcal{T}M = \mathbb{R}^{2d}$, then a d-dimensional subspace $V \subset \mathbb{R}^{2d}$ is called *Lagrangian* if $\omega|_V \equiv 0$. Given two transversal Lagrangian subspaces V_1, V_2, we can write uniquely $v \in \mathbb{R}^{2d}$ ad $v = v_1 + v_2$, with $v_i \in V_i$. we can then define the quadratic function

$$Q(v) = \omega(v_1, v_2).$$

This allows us to define special cones with remarkable properties:

$$\mathcal{C} = \{v \in \mathbb{R}^{2n} \;:\; Q(v) > 0\}. \tag{3.4.6}$$

Accordingly, if we specify a field of transversal Lagrangian subspace, we have the quadratic functions Q_x and the cone field $\mathcal{C}_x$.

Obviously, if $Q_{f(x)}(d_x f v) \geq Q_x(v)$, then $d_x f \mathcal{C}_x \subset \mathcal{C}_{f(x)}$, hence we have cone invarince. Such maps are called *monotone*.

If $Q_{f(x)}(d_x f v) > Q_x(v)$ for all $v \neq 0$, then $d_x f(\overline{\mathcal{C}_x} \setminus \{0\}) \subset \mathcal{C}_{f(x)}$, such maps are called *strictly monotone*.

Lemma 3.4.1 ([44], Sections 6) *A map is monotone if and only if the cone field is invariant. The same is true for strict monotonicity.*

Theorem 3.4.2 ([44] Sections 5, 6, or [43]) *If a map is eventually strictly monotone, then all its Lyapunov exponents are non-zero.*

This is proven exactly as Theorem 3.5.1, so we refer to the proof of the latter.
The above also has a continuous version: a Hamiltonian flow in a $2d+2$ dimensional manifold, is determined by a Hamiltonian

3.4.1 Lagrangian subspaces

By a symplectic change of variables, we can assume that the space is $\mathbb{R}^{2d}$, the vectors are written as (ξ, η), $\xi, \eta \in \mathbb{R}^d$ and the symplecit form is given by

$$\omega((\xi, \eta), (\eta', \eta') = \langle \xi, \eta' \rangle - \langle \eta, \xi' \rangle.$$

Then, $A \in GL(2d, \mathbb{R})$ is sympletc if and only if $\omega(Av, Aw) = \omega(v, w)$ for all $v, w \in \mathbb{R}^{2d}$. That is if

$$A^T J A = J$$

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

To introduce an appropriate higher dimensional formalism, it is convenient to discuss briefly *Lagrangian subspaces*.

Definition 12 *A d-dimensional subspace $\mathbb{V}$ of $\mathbb{R}^{2d}$ is Lagrangian iff*

$$\omega(v, w) = 0$$

for all $v, w \in \mathbb{V}$.

Lemma 3.4.3 *For each $d \times d$ matrix U, the space $\mathbb{V} = \{(v, Uv) : v \in \mathbb{R}^d\}$ is Lagrangian iff U is symmetric.*

PROOF. Clearly $\mathbb{V}$ is d-dimensional. To conclude, it suffices to compute

$$\omega((v, Uv), (w, Uw)) = \langle v, Uw\rangle - \langle w, Uv\rangle$$

which is zero only if U is symmetric. □

Let $V_1, V_2 \in \mathbb{R}^{2d}$ two transversal Lagrangian subspaces, then, for each $v \in \mathbb{R}^{2d}$ we can write uniquely $v = v_1 + v_2$ with $v_i \in V_i$. We then write

$$Q(v) := \omega(v_1, v_2)$$

By a symplectic change of variable, we can always reduce the general case to the case $V_1 = \{(v_1, 0) \ : \ v_1 \in \mathbb{R}^d\}$ $V_1 = \{(0, v_2) \ : \ v_2 \in \mathbb{R}^d\}$. In this case

$$Q((v_1, v_2)) = \langle v_1, v_2\rangle.$$

We say that a symplectic matrix L is monotone if $Q(Lv) \geq Q(v)$ for each $v \in \mathbb{R}^{2d}$, and we say that a symplectic matrix L is strictly monotone if $Q(Lv) > Q(v)$ for each $v \in \mathbb{R}^{2d} \setminus \{0\}$.

To measure precisely how much the quadric form increases, it is convenient to introduce the cones

$$\mathcal{C} = \{v \in \mathbb{R}^{2d} \ : \ Q(v) > 0\} \ ; \quad \overline{\mathcal{C}} = \{v \in \mathbb{R}^{2d} \ : \ Q(v) \geq 0\}.$$

Lemma 3.4.4 *A Lagrangian space $\mathbb{V}$ belongs to $\mathcal{C} \cup \{0\}$ iff it is of the form (v, Uv), with U strictly positive.*

PROOF. If $\pi_i(v_1, v_2) = v_i$, then $\pi_1 : \mathbb{V} \to \mathbb{R}^n$ is injective. If not, there exists $(v_1, v_2) \in \mathbb{V} \setminus \{0\}$ such that $v_1 = 0$. But then $Q((v_1, v_2)) = 0$ contrary to the hypothesis. We can then define $U := \pi_2 \circ \pi_1^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbb{V} = \{(v, Uv) \ : lv \in \mathbb{R}^d\}$. Then, by Lemma 3.4.3 U must be symmetric. Finally, for $v \neq 0$,

$$0 < Q((v, Uv)) = \langle v, Uv\rangle$$

hence U is strictly positive. The opposit implication is trivial. □

Lemma 3.4.5 *A symplectic matrix $L = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$ is strictly monotone if and only if* $\det \boldsymbol{a} \neq 0$ *and the matrices R, P in the factorization (3.1.3) are strictly positive.*

PROOF. Indeed, if $\det \boldsymbol{a} = 0$, then there exists $\xi \in \mathbb{R}^d \setminus \{0\}$ such that $\boldsymbol{a}\xi = 0$, but then

$$Q(L(\xi, 0)) = \langle \boldsymbol{a}\xi, \boldsymbol{c}\xi \rangle = 0 = Q((\xi, 0))$$

contrary to the hypothesis. We can then apply Lemma 3.1.3 to write

$$\begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ P & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & R \\ 0 & \mathbb{1} \end{pmatrix} (v_1, v_2) = (\boldsymbol{a}(v_1 + Rv_2), (\boldsymbol{a}^{-1})^T (Pv_1 + (\mathbb{1} + PR)v_2)).$$

Thus,

$$Q(L(v_1, v_2)) = \langle v_1 + Rv_2, Pv_1 + (\mathbb{1} + PR)v_2 \rangle \tag{3.4.7}$$

If $v_2 = 0$, then we have

$$0 < Q(L(v_1, 0)) = \langle v_1, Pv_1 \rangle$$

hence P is a strictly positive matrix. On the other and, for each $\mu > 0$ and $\|v\| = 1$, we have that

$$\mu < Q(L(v, \mu v)) = \langle v + \mu Rv, Pv + \mu(\mathbb{1} + PR)v \rangle.$$

We can then chose v to be an eigenvector of R, so $Rv = \lambda v$. Then we obtain

$$\mu < \langle (1 + \mu\lambda)v, Pv + \mu(\mathbb{1} + \lambda P)v \rangle = (1 + \lambda)\mu + (1 + \lambda\mu)^2 \langle v, Pv \rangle$$

that is

$$\lambda\mu + (1 + \lambda\mu)^2 \langle v, Pv \rangle > 0.$$

It follows that it must be $\lambda \geq 0$ otherwise we can choose $\mu = -\lambda^{-1}$ and obtain the contradiction $-1 > 0$. On the other hand, if $\lambda = 0$, then

$$0 < Q(L(0, v)) = \langle 0, v \rangle = 0$$

which is also impossible. Finally, if $\det(\boldsymbol{a}) \neq 0$ and the matrices P, R are strictly positive, then

$$Q(L(v_1, v_2)) = \langle v_1, v_2 \rangle + \langle v_2, Rv_2 \rangle + \langle v_1 + Rv_2, P(v_1 + Rv_2) \rangle > Q((v_1, v_2)).$$

□

The above implies that if L is strictly monotone, then $LV_i \subset \mathcal{C} \cup \{0\}$. There is a useful partial converse of this fact.[4]

[4]Note that [44, Proposition 8.4] is false as the example $L = \begin{pmatrix} \mathbb{1} & 0 \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & -2\mathbb{1} \\ 0 & \mathbb{1} \end{pmatrix}$ shows.

Lemma 3.4.6 *If $LV_i \subset \mathcal{C} \cup \{0\}$ and, for all $v \in \mathbb{R}^d$, $\omega(L(0, \boldsymbol{a}^T v), (0, v)) \geq 0$, then L is strictly monotone.*

Proof. First of all, note that

$$0 < Q(L(v,0)) = \langle Q((\boldsymbol{a}v, \boldsymbol{c}v)) = \langle v, \boldsymbol{c}^T \boldsymbol{a} v\rangle.$$

Since (3.1.4) implies that $\boldsymbol{c}^T \boldsymbol{a}$ is a symmetric matrix, it follows that $\boldsymbol{c}^T \boldsymbol{a}$ is strictly positive, hence $\det(\boldsymbol{a}) \neq 0$. We can then use the decomposition (3.1.3) which yields the expression (3.4.7) which implies

$$0 < Q(L(v,0)) = \langle v, Pv\rangle$$

which implies that P is a strictly positive matrix. This implies that

$$Q\left(\begin{pmatrix} \mathbb{1} & 0 \\ P & \mathbb{1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = Q((v_1, Pv_1 + v_2)) = Q((v_1, v_2)) + \langle v_1, Pv_1\rangle \geq Q((v_1, v_2)).$$

On the other hand

$$0 < Q(L(0,v)) = \langle Rv, (\mathbb{1} + PR)v\rangle = \langle v, (R + RPR)v\rangle,$$

that is $R + RPR$ is strictly positive matrix. Since R is symmetric it has d eigenvectors, let w, $\|w\| = 1$, and eigenvector and λ the corresponding eigenvalue, then

$$0 < \langle w, (R + RPR)w\rangle = \lambda + \lambda^2 \langle w, Pw\rangle$$

which implies $\lambda \neq 0$. Finally, setting $w = \boldsymbol{a}^T v$,

$$0 \leq \omega(L(0,w), (0, (\boldsymbol{a}^T)^{-1} w)) = \langle Rw, w\rangle$$

implies that R is positive and hence strictly positive. The Lemma follows then from Lemma 3.4.5. □

Let us define

$$\sigma(L) = \inf_{v \in \mathcal{C}} \sqrt{\frac{Q(Lv)}{Q(v)}}.$$

Lemma 3.4.7 *If a symplectic matrix $L = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$ is strictly monotone, then the eaigenvalues of $\boldsymbol{c}^T \boldsymbol{b}$ are all strictly positive and, calling t the minimal such eigenvalue, we have*

$$\sigma(L) \geq \sqrt{t} + \sqrt{1+t} > 1.$$

PROOF. We use the decomposition (3.1.3) and note that the matrix

$$\mathcal{R} = \begin{pmatrix} R^{-\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{pmatrix}$$

is a Q-isometry, that is $Q(\mathcal{R}v) = Q(v)$ for all $v \in \mathbb{R}^{2d}$. In particular, this implies that $\mathcal{R}\mathcal{C} = \mathcal{C}$. Hence, setting $\mathcal{L} = \begin{pmatrix} \mathbb{1} & R \\ P & \mathbb{1} + PR \end{pmatrix}$,

$$\inf_{v\in\mathcal{C}} \sqrt{\frac{Q(Lv)}{Q(v)}} = \inf_{v\in\mathcal{C}} \sqrt{\frac{Q(\mathcal{L}v)}{Q(v)}} = \inf_{v\in\mathcal{C}} \sqrt{\frac{Q(\mathcal{R}\mathcal{L}\mathcal{R}^{-1}(\mathcal{R}v))}{Q(\mathcal{R}v)}} = \inf_{v\in\mathcal{C}} \sqrt{\frac{Q(\mathcal{R}\mathcal{L}\mathcal{R}^{-1}v)}{Q(v)}}.$$

Setting $T = R^{\frac{1}{2}} P R^{\frac{1}{2}}$, we have

$$\mathcal{R}\mathcal{L}\mathcal{R}^{-1} = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ T & \mathbb{1} + T \end{pmatrix} =: \mathcal{T}$$

Note that T is a strictly positive matrix; hence, calling t_i its eigenvalues and w_i the associated eigenvector, we have $t_i > 0$. In addition, we have

$$PR(R^{-\frac{1}{2}}w_i) = R^{-\frac{1}{2}}Tw_i = t_i R^{-\frac{1}{2}}w_i.$$

That is, the eigenvalues of T are also the eigenvalues of $PR = \boldsymbol{c}^T\boldsymbol{a}\boldsymbol{a}^{-1}\boldsymbol{b} = \boldsymbol{c}^T\boldsymbol{b}$, where we have used (3.1.5) and the fact that $P^T = P$. To conclude, we note that, setting $v = (v_1, v_2)$ and calling t the minimal eigenvalue ot T,

$$\begin{aligned}
\frac{Q(\mathcal{T}v)}{Q(v)} &= \frac{\langle v_1, v_2\rangle + \langle v_2, v_2\rangle + \langle (v_1+v_2), T(v_1+v_2)\rangle}{\langle v_1, v_2\rangle} \geq 1 + \frac{\|v_2\|^2 + t\|v_1+v_2\|^2}{\langle v_1, v_2\rangle} \\
&= 1 + \frac{(1+t)\|v_1\|^2 + 2t\langle v_1, v_2\rangle + t\|v_2\|^2}{\langle v_1, v_2\rangle} \\
&= 1 + \frac{2t\langle v_1, v_2\rangle + (1+t)^{\frac{1}{2}}t^{\frac{1}{2}}\left[(1+t)^{\frac{1}{2}}t^{-\frac{1}{2}}\|v_1\|^2 + (1+t)^{-\frac{1}{2}}t^{\frac{1}{2}}\|v_2\|^2\right]}{\langle v_1, v_2\rangle} \\
&\geq 1 + \frac{2\left[t + (1+t)^{\frac{1}{2}}t^{\frac{1}{2}}\right]\langle v_1, v_2\rangle}{\langle v_1, v_2\rangle} = \left[\sqrt{t} + \sqrt{1+t}\right]^2.
\end{aligned}$$

□

3.5 Higher dimensions: hyperbolicity

We say that a Hamiltonian flow (M, ϕ_t) is hyperbolic on a constant energy surface M_E if, when restricted to such a surface, all his Lyponov exponents, but one (the one in the flow direction), are non-zero. For simplicity, we restrict to the case $M \subset \mathbb{R}^{2d}$, but the result holds for general symplectic manifolds. Also, we require that M_E is compact. Let μ be the Liouville measure normalized so that $\mu(M) = 1$. The goal of this section is to prove the following theorem:

Theorem 3.5.1 ([63], or see [44] Sections 5, 6, or [43]) *If a flow on M_E is* eventually strictly monotone*, then all its Lyapunov exponents, apart from the one in the flow direction, are non-zero.*

By the results of section 3.2, we can restrict ourselves to a discrete-time analysis. We will consider the time one map $f = \phi_1$ with the differential acting on the quotient space there described; the study of the Poincarè map being similar. For $x \in M$, le $s(x) = \min\{k \, : \, Df^k \text{ is strictly monotone}\}$.

By *eventually strictly monotone*, we mean that, for almost all $x \in M$, $D_x f$ is monotone and $s(x) < \infty$.

Proof of Theorem 3.5.1. Let $A_m = \{x \in M \, : \, s(x) = m\}$. For such m we define the first hyperbolic return time to A_m as

$$n_m(x) = \begin{cases} 0 & \text{if } x \notin A_m \\ \min\{k \geq m \, : \, f^k(x) \in A_m\} & \text{otherwise.} \end{cases}$$

Lemma 3.5.2 (Kac' s theorem) *For each $m \in \mathbb{N}$, $n_m \in L^1$.*

Proof. If $\mu(A_m) = 0$, the statement is trivial. We can then limit ourselves to the case $\mu(A_m) > 0$. Let $A_{m,k} = \{x \in A_m \, : \, n_m(x) = k\}$. Note that f, being the time one map of a flow, is invertible, so f^{-1} is measurable and preserves the measure. Moreover, if $x \in f^j(A_{m,k}) \cap f^l(A_{m,k'})$ for some $j + m \leq l \leq k'$ and $j \leq k$, then, setting $y = f^{-l}(x) \in A_{m,k'}$ and $w = f^{-j}(x) \in A_{m,k}$ we have $f^j(w) = f^l(y)$, that is $f^{l-j}(y) = w \in A_m$ which contradicts the fact that $y \in A_{m,k'}$ since $m \leq l - j < l \leq k'$. It follows that, for $j < k, l < k'$, $f^j(A_{m,k}) \cap f^l(A_{m,k'}) \neq \emptyset$ implies $|j - l| \leq m$. That is a point of A_m can belong to no more than $2m+1$ sets $f^j(A_{m,k})$. Then

$$\int_{A_m} n_m(s)\mu(dx) = \sum_{k=1}^{\infty} k\mu(A_{m,k}) = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} \mu(f^j(A_{m,k}))$$
$$\leq (2m+1)\mu(\cup_{j=0}^{\infty} f^j(A_m)) \leq 2m+1.$$

The lemma follows since, by definition, $\int_M n_m(s)\mu(dx) = \int_{A_m} n_m(s)\mu(dx)$. □

If f is eventually strictly monotone, then $\sum_{m=1}^{\infty}\mu(A_m) = 1$. Hence, there exists $m > 0$ such that $\mu(A_m) > 0$. Define the return map $F(x) = f^{n_m(x)}(x)$, $x \in A_m$. For each $n \in \mathbb{N}$ let $k(x) = \min\{k \in \mathbb{N} \ : \ \sum_{j=0}^{k} n_m(F^j(x)) \geq n\}$. Then

$$\sigma(Df^n) \geq \sigma(D_{F^{k(x)-1}(x)} f^{n_m(F^{k(x)-1}(x))} \cdots D_x f^{n_m(x)})$$
$$\geq \prod_{j=0}^{k(x)-1} \sigma(D_{F^j(x)} f^{n_m(F^j(x))}).$$

Also, note that, by definition, it must be $n_m(s) \geq m$. So, by Lemma 3.4.7, we have, for each $y \in A_m$, $\sigma(D_y f^{n_m(y)}) \geq \sqrt{t}(x) + \sqrt{1+t(x)} =: e^{\alpha(x)}$ where $\alpha(x) > 0$. Since α could be unbounded it is convenient to set $\bar{\alpha}(x) = \min\{1, \alpha(x)\}$ and again $\sigma(D_y f^{n_m(y)}) \geq e^{\bar{\alpha}(x)}$. Accordingly,

$$\lim_{n\to\infty} \frac{1}{n} \ln \sigma(D_x f^n) \geq \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{k(x)-1} \ln \sigma(D_{F^j(x)} f^{n_m(F^j(x))})$$
$$\geq \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{k(x)-1} \bar{\alpha}(F^j(x)) \geq \frac{\lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} \bar{\alpha}(F^j(x))}{\lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k} n_m(F^j(x))}.$$

By Birkhoff's ergodic theorem, the limits exist almost surely and are L^1 functions. Hence, the limit can be zero on a positive measure set only if the numerator is. Also, the points for which the numerator is zero form an invariant set $B \subset A_m$. But if $\mu(B) > 0$, then we can restrict the above argument to B and we obtain, for almost al $x \in B$, the contradiction

$$0 = \int_B \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \bar{\alpha}(F^j(x)) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \int_B \bar{\alpha}(F^j(x)) = \int_B \bar{\alpha}(x) > 0.$$

The above implies that for each $v \in \mathcal{C}$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \ln \|Df^n v\| &= \lim_{n\to\infty} \frac{1}{2n} \ln \|Df^n v\|^2 \geq \lim_{n\to\infty} \frac{1}{2n} \ln Q(Df^n v) \\ &\geq \lim_{n\to\infty} \frac{1}{2n} \ln \sigma(Df^n) > 0.\end{aligned}$$

Since the Lagrangian space $\mathbb{W} = \{(w, w)\} \subset \mathcal{C} \cup \{0\}$, we have a d-dimensional subspace with strictly positive Lyapunov exponents. Hence, we have d strictly positive Lyapunov exponents. Let $\mathbb{V}_i$ the spaces in Oseldets' Theorem, so that $\dim(\mathbb{V}_i) = i$ (note that if the spectrum is not simple, there are many possible choices). Note that for each $i \in \{d+1, \dots 2d\}$, there must be $v \in \mathbb{V}_i$, and $w \in \mathbb{V}_{2d-i+1}$ such that $\omega(v, w) \neq 0$; otherwise the two spaces would be skew orthogonal which is impossible since the sum of their dimensions is $2d+1$. Thee, by continuity, we can find $v_i \in \mathbb{V}_i \setminus \mathbb{V}_{i-1}$ and $w_i \in \mathbb{V}_{2d-i+1} \setminus \mathbb{V}_{2d-i}$ such that $\omega(v_i, w_i) \neq 0$. By construction, λ_i is the Lyapunov exponent associated to v_i ad λ_{2d-i+1} the Lyapunov exponents associated with $\hat{w}_i$. Then

$$\begin{aligned}0 = \lim_{n\to\infty} \frac{1}{n} \ln \omega(v_i, w_i) &= \lim_{n\to\infty} \frac{1}{n} \ln \omega(Df^n v_i, Df^n w_i) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \ln \|Df^n v_i\| \|Df^n w_i\| = \lambda_i + \lambda_{2d-i+1}.\end{aligned}$$

On the other hand, by Oseledets Theorem 2.2.1,

$$\begin{aligned}0 &= \lim_{n\to\infty} \frac{1}{n} \ln |\wedge^d \omega(v_{d+1}, \dots v_{2d}, w_{d+1}, \dots, w_{2d})| \\ &= \lim_{n\to\infty} \frac{1}{n} \ln |\wedge^d \omega(Df^n v_{d+1}, \dots Df^n v_{2d}, Df^n w_{d+1}, \dots, Df^n w_{2d})| \\ &= \lim_{n\to\infty} \frac{1}{n} \ln |\det(Df^n)| \, |\wedge^d \omega(v_{d+1}, \dots v_{2d}, w_{d+1}, \dots, w_{2d})| \\ &= \sum_{i=1}^{2d} \lambda_i = \sum_{i=1}^{d} [\lambda_i + \lambda_{2d-i+1}] \geq 0\end{aligned}$$

which implies $\lambda_{2d-i+1} = -\lambda_i$, hence all the Lyapunov exponents are non zero. □

Problems

3.1 Construct a strictly invariant cone family for the irrational translation on $\mathbb{T}^2$ (see Examples 1.1.1) and show that it is not measurable.

(Hint: For each trajectory choose a point x. At such a point choose the standard cone $\mathcal{C}_+$, let $\mathcal{C}_n^- = \{(v_1, v_2) \in \mathbb{R}^2 \mid 1 + \frac{1}{n} \leq \frac{v_2}{v_1} \leq 2 + \frac{1}{n}\}$ and $\mathcal{C}_n^+ = \{(v_1, v_2) \in \mathbb{R}^2 \mid -2 - \frac{1}{n} \leq \frac{v_2}{v_1} \leq -1 - \frac{1}{n}\}^c$. Then set $\mathcal{C}(T^n x) = \mathcal{C}_n^+$ and $\mathcal{C}(T^{-n} x) = \mathcal{C}_n^-$. Such a cone family is strictly monotone by construction (since $D_x T = 1$), yet the system has obviously zero Lyapunov exponents. Since all the other hypothesis of Theorem 2.2.1 are satisfied, it follows that the above cone family cannot be measurable.)

3.2 Show that for two dimensional symplectic maps the sum of the Lyapunov exponent is zero (*pairing of the Lyapunov exponents*). (Hint: If $\omega(v, w) = 1$ then $1 = \omega(DT^n v, DT^n w) \sim \|DT^n v\| \|DT^n w\|$.)

3.3 Check that $\inf_{v \in \mathcal{C}_+} \sqrt{\frac{Q(Lv)}{Q(v)}} = \left[\inf_{v \in \mathcal{C}_-} \sqrt{\frac{Q(L^{-1}v)}{Q(v)}} \right]^{-1}$, remember that $\mathcal{C}_- = \overline{(\mathcal{C}_+)^c}$. (Hint: see [43])

3.4 Consider $\mathbb{R}^2$ endowed with the scalar product $\langle v, w \rangle_G := \langle v, Gw \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product and $G > 0$. Show that there exists a change of coordinates $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that, in the new coordinates $\langle \cdot, \cdot \rangle_G$ becomes the standard scalar product.

3.5 Consider the cone $\mathcal{C}$ defined by the two transversal vectors $v_1, v_2 \in \mathbb{R}^2$. This means that $v \in \mathbb{R}^2$ belongs to the cone iff $v = \alpha v_1 + \beta v_2$ with $\alpha\beta \geq 0$. Show that there is a linear change of coordinates $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that $M\mathcal{C} = \mathcal{C}_+$ and $\det M = 1$.

3.6 Show that, in a two dimensional area preserving systems, if the LE are different from zero then there exists and eventually strictly invariant cone family. (Hint: By Oseledets there exists the unstable distributions, then construct the cones around it.)

3.7 Prove that if M is the two by two matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

with $a, b, c \in \mathbb{Z}$, then $M > 0$ iff $a, c > 0$ and $c > \frac{b^2}{a}$.

3.8 Prove that is L is symplect then $\det L = 1$. (Hint: The determinant of a matrix is nothing else than the volume of the parallelepiped of sides $(Le_1, \ldots, Le_{2d})$ (where $e_1, \ldots, e_{2d}$ is the standard orthonormal basis of $\mathbb{R}^{2d}$). On the other hand the volume form can be written has $\wedge^d\omega$ (since that is a $2d$ form with the right normalization and the space of $2d$ forms is one dimensional). Thus $\det L = \wedge^d\omega(Le_1, \dots, Le_{2d}) = \wedge^d\omega(e_1, \dots, e_{2d}) = 1$ where we have used the fact that $\omega(Lv, Lu) = \omega(v, u)$. The reader that wants to appreciate the power of the above geometrical interpretation of the determinant and of the external forms can try to prove the statement by purely algebraic means.)

3.9 Show that all symplectic Q-isometrys L (that is $Q(Lv) = Q(v)$) have the form

$$L = \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}.$$

(Hint: Start by considering the vector $(0, u)$, $U \in \mathbb{R}^d$, clearly $Q((0, u)) = 0$ thus $Q(L(0, u)) = 0$ if L is a Q-isometry. But if

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

it follows $\langle Bu, Du \rangle = 0$ for each $u \in \mathbb{R}^d$, that is $B^*D = 0$. The same argument applied to the vector $(u, 0)$ yields $A^*C = 0$. Accordingly, by symplecticity

$$\begin{aligned} Q(L(v, u)) =& \langle Au + Bv, \, Cu + Dv \rangle = \langle u, (A^*D + C^*B)v \rangle \\ =& \langle u, (\mathbb{1} + 2C^*B)v \rangle \end{aligned}$$

thus $Q(L(v, u)) = Q(v, u)$ iff $C^*B = 0$ which implies $A^*D = \mathbb{1}$.)

3.10 Show that if the matrix

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is symplectic then

$$L^{-1} = \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix}$$

3.11 Show that the symplectic matrices form a multiplicative group. (Hint: Use the definition and the above problems.)

3.12 A symplectic map L is a Q-isometry iff $L\mathcal{C} = \mathcal{C}$. (Hint: One direction is trivial. On the other hand, if $L\mathcal{C} = \mathcal{C}$ it follows that L maps the boundary, of $\mathcal{C}$, to the boundary. Accordingly, if $\langle v, u \rangle = 0$ it must be

$$0 = \langle Av + bu, \, Cv + Du \rangle. \tag{3.5.8}$$

Choosing in 3.5.8 $u = 0$ yields $A^*C = 0$, choosing $v = 0$ shows that it must be $B^*D = 0$. Thus 3.5.8 yields

$$0 = \langle u,\, (A^*D + C^*B)v\rangle = 2\langle u,\, C^*Bv\rangle.$$

The above equality shows that C^*Bv is parallel to v for each $v \in R^d$, that is $C^*B = \alpha\mathbb{1}$ for some $\alpha \in \mathbb{R}$. If $\alpha = 0$, then $A^*D = \mathbb{1}$ and thus $C = 0$ which is the wanted result. If $\alpha \neq 0$, then B is invertible and $C = \alpha B^{*1}$. But this implies $A = 0$ and hence $-\mathbb{1} = C^*B = \alpha\mathbb{1}$, that is $\alpha = -1$. Accordingly the matrix would have the form

$$L = \begin{pmatrix} 0 & B \\ -B^{*-1} & 0 \end{pmatrix}$$

which sends $\mathcal{C}$ in its complement, contrary to our requirement.)

3.13 Show that a strictly monotone symplectic matrix can be put into the form

$$\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ M & \mathbb{1} + M \end{pmatrix}$$

by multiplying it by Q-isometries on the left and on the right.

3.14 Show that all the Lagrangian subspaces transversal to $V = \{(0, \eta) \in \mathbb{R}^{2d} \mid \eta \in \mathbb{R}^d\}$ can be represented as $\{(\xi,\, U\xi) \in \mathbb{R}^{2d} \mid \xi \in \mathbb{R}^d\}$ for some symmetric matrix U. (Hint: Let $V_U := \{(\xi,\, U\xi) \in \mathbb{R}^{2d} \mid \xi \in \mathbb{R}^d\}$, then $\omega((\xi,\, U\xi),\, (\zeta,\, U\zeta)) = 0$, thus V_U is Lagrangian. On the other hand, if $\tilde{V}$ is Lagrangian, then it is a d dimensional space. Let $\{(\xi_i, \eta_i)\}_{i=1}^d$ be a base for $\tilde{V}$, then $\xi_i \neq 0$ by the transversality assumption and we can define the matrix U via $U\xi := \eta_i$. It is immediate that $\tilde{V}$ Lagrangian implies $U = U^*$.)

3.15 Show that $V_U := \{(\xi,\, U\xi) \in \mathbb{R}^{2d} \mid \xi \in \mathbb{R}^d\}$, $U = U^*$, belongs to the standard cone iff $U \geq 0$.

3.16 Show that given any two transversal lagrangian subspaces V_1, V_2,[5] there exists a symplectic map L such that $LV_1 = \{(\xi, 0)\}$ and $LV_2 = \{(0, \eta)\}$. (Hint: choose coordinates in which V_i are transversal to $V = \{(0, \eta) \in \mathbb{R}^{2d} \mid \eta \in \mathbb{R}^d\}$, then we can write $V_i = \{(\xi, U_i\xi)\}$. Note that, since V_1 and V_2 are transversal, $U_1 - U_2$ must be invertible. The, e.g., set $D = \mathbb{1}$ and $B = (U_1 - U_2)^{-1}$ and check the algebra.)

[5]Recall that two space are transversal iff $V_1 \cap V_2 = \emptyset$.

3.17 Find a symplectic change of coordinates that transforms the standard form Q into the form Q_h defined by:

$$Q_h((x,y)) = \frac{1}{2}(\langle x, x\rangle - \langle y, y\rangle),$$

and draw the associate cone. (Hint: Consider

$$\begin{aligned} x &= \tfrac{x'-y'}{\sqrt{2}} \\ y &= \tfrac{x'+y'}{\sqrt{2}}.) \end{aligned}$$

3.18 Hilbert metric for a disc and the half plane–hyperbolic geometry.

3.19 Show that the Perron-Frobenius operator associated to a smooth expanding map of the circle has a spectral gap as an operator on $Lip(\mathbb{T}^2)$. (Hint: Check that there exists $b \in \mathbb{R}^+$ such that the norm

$$\|h\| := \|h\|_\infty + b\|h\|_{Lip}$$

is adapted to the cone. Define $\mathbb{V} = \{h \in Lip(\mathbb{T}^2) \mid \int h = 0\}$, notice that $\mathcal{L}\mathbb{V} = \mathbb{V}$. Then, for each $h \in \mathbb{V}$ there exists$\rho \in \mathbb{R}^+$ such that $h + \rho h_* \in \mathcal{C}_\alpha$, so

$$\|\mathcal{L}^n h\| = \|\mathcal{L}^n(h + \rho h_*) - \rho h_*\| \le K\Lambda^n \rho.$$

Thus the spectral radius of $\mathcal{L}|_{\mathbb{V}}$ is less than Λ.)

3.20 Estimate the rate of mixing for Lipschitz functions for a smooth expanding map of the circle (Hint: use the spectral gap of the previous Problem.)

3.21 Prove that any continuous fraction of the form

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

$a_i > 0$ is convergent provided the series $\sum_{n=1}^{\infty} a_n$ is divergent. (Hint: Let

$$\prod_{i=1}^{n} \begin{pmatrix} 1 & a_{2(n-i)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{2(n-1)+1} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} = \begin{pmatrix} \beta_n \\ \alpha_n \end{pmatrix}$$

and verify, by induction, that $\frac{\alpha_n}{\beta_n}$ is exactly the $2n$ truncation of the continuous fraction. Thus the continuous fraction is a projective coordinate for the vector (α_n, β_n). Consider the cone $\mathcal{C}_+ = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0; y \geq 0\}$. Then, for each $a, b \in \mathbb{R}^+$, holds

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{C}_+ \subset \mathcal{C}_+.$$

The result follows by computing the Hilbert metric contraction, see [20, Appendix D] for details on the Hilber metric and its properties.

For a different approach, see [62, Th14.1].)