Chapter 5

Hyperbolicity of Billiards

To tackle the problem of hyperbolicity, it is convenient to develop some further tools.

5.1 Collision map and Jacobi fields

To compute, in general, the collision map it is helpful to introduce appropriate coordinates in $\mathcal{T}X$. A very interesting choice is constitute by the *Jacobi fields*.[1] Let X_- be the set of configurations just before collision. For each $(x,\, v) \in X \backslash X_-$ there exists $\delta > 0$ such that

$$\phi_t(x,\, v) = (x + vt,\, v) \quad 0 \leq t \leq \delta.$$

Let us consider the curve in $\mathcal{X}$

$$\xi(\varepsilon) = (x(\varepsilon),\, v(\varepsilon)),$$

with $\xi(0) = (x,\, v)$ and $\|v(\varepsilon)\| = 1$.

For each t such that $\phi_t(\xi(0)) \notin X_-$, let

$$\xi(\varepsilon,\, t) = (x(\varepsilon,\, t),\, v(\varepsilon,\, t)) = \phi_t(\xi(\varepsilon)).$$

The Jacobi field $J(t)$ is defined by

$$J(t) \equiv \left.\frac{\partial x}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

[1]The Jacobi Fields are a widely used instrument in Riemannian geometry (see [25]) and have an important rôle in the study of Geodetic flows, although we will not insists on this aspect at present. Here they appear in a very simple form.

Note that, since $x(0, t) \notin X_-$, for $s < \delta$

$$\xi(\varepsilon, t + s) = \xi(\varepsilon, t) + (v(\varepsilon, t)s, 0),$$

so

$$J'(t) = \frac{dJ(t)}{dt} = \left.\frac{\partial v(\varepsilon, t)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

That is, $(J(t), J'(t)) = d\phi_t \xi'(0)$.

At each point $\xi = (x, v) \in X$ we choose the following base for $\mathcal{T}_\xi X$:[2]

$$\eta_0 = (v, 0); \quad \eta_1 = (v^\perp, 0); \quad \eta_2 = (0, v^\perp);$$

where $\|v^\perp\| = 1$, $\langle v, v^\perp \rangle = 0$.

The vector η_0 corresponds to a family of trajectories along the flow direction, and it is clearly invariant; η_1 to a family of parallel trajectories and η_2 to a family of trajectories just after focusing. It is very useful the following graphic representation. We represent a tangent vector by drawing a curve that is tangent to it. A curve in $\mathcal{T}X$ is given by a base curve that describes the variation of the x coordinate equipped with a direction at each point (specified by an arrow), which shows how the velocity varies (see Figure 5.1).

A direct check shows that each vector η perpendicular to the flow direction will stay so (see Lemma 4.1.1), i.e.

$$\langle d\phi_t \eta, (v_t, 0) \rangle = \langle d\phi_t \eta, d\phi_t(v, 0) \rangle = \langle \eta, (v, 0) \rangle = 0.$$

So the free flow is described by

$$d\phi^t \eta_0 = \eta_0; \quad d\phi^t \eta_1 = \eta_1; \quad d\phi^t \eta_2 = \eta_2 + t\eta_1,$$

that is, in the above coordinates

$$d\phi^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}. \tag{5.1.1}$$

Let us see now what happens at a collision.

Let $x_0 \in \partial\mathbb{B}$ be the collision point and let $\xi_c = (x_0, v)$ be the configuration at the collision. We want to compute $R_\varepsilon := d_{\phi^{-\varepsilon}\xi_c}\phi^{2\varepsilon}$, that is the

[2] Here $v^\perp = Jv$ with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

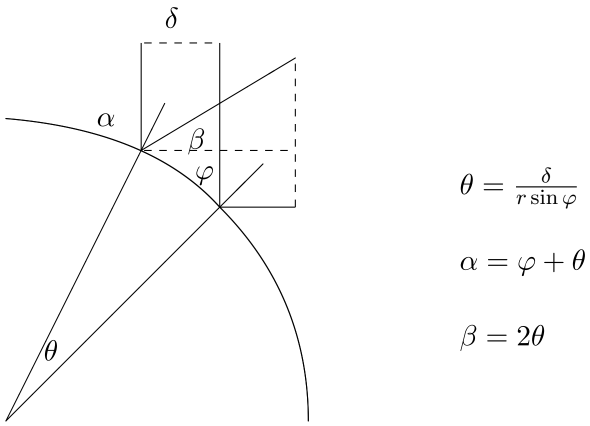

Figure 5.1: Collision

tangent map from just before to just after the collision. Clearly $R_\varepsilon\eta_0 = \eta_0$. From the Figure 5.1 follows that, if $\gamma(s)$ is the curve associated to η_1 at the point $\phi^{-\varepsilon}\xi_c$,

$$d\phi^{2\varepsilon}\gamma(s) = \left(v_+^\perp\left[s + \varepsilon\frac{2s}{r\sin\varphi}\right], \frac{2s}{r\sin\varphi}\right) + \mathcal{O}(s^2)$$

where r is the radius of the osculating circle (that is the circle tangent to the boundary up to second order) which is the inverse of the curvature $K(x_0)$ of the boundary at the collision point.

The above equation means that

$$J(\varepsilon) = (1 + \frac{2\varepsilon K(x_0)}{\sin\varphi})v_+^\perp.$$

Accordingly, calling $R = \lim_{\varepsilon\to 0} R_\varepsilon$ the collision map, we have

$$R\eta_1 = \eta_1 + \frac{2K}{\sin\varphi}\eta_2; \quad R\eta_2 = \eta_2.$$

Hence,

$$DR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{2K}{\sin\varphi} \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.1.2}$$

The above computations provide the following formula for the derivative of the Poincaré section from the boundary of the obstacle, just after collision, to the boundary of the obstacle just after the next collision

$$DT = \begin{pmatrix} 1 & \frac{2K}{\sin\varphi} \\ \tau & 1 + \frac{2\tau K}{\sin\varphi} \end{pmatrix}, \tag{5.1.3}$$

where τ is the flying time between the two collisions and φ the collision angle. Formula (5.1.3) is sometimes called *Benettin formula* (e.g., [36]).

5.2 Hyperbolicity of Sinai Billiard

As an example let us consider the Sinai Billiard depicted in Figure 4.4. Note that the system cannot be uniformly hyperbolic since there are trajectories that never hit the obstacle, and hence have clearly zero Lyapunov exponents. We define a cone family in the plane perpendicular to the flow direction $(v, 0)$, that is in the plane η_1, η_2, this plane is naturally isomorphic to the tangent space of $\mathcal{M}$ (just project along the flow direction) in each non-tangent point.

In the case in which no collision takes place, we have seen that the parallel family η_1 stays parallel, while the most divergent family (the vector η_2) becomes less divergent (a linear combination of η_1 and η_2 with positive coefficients). This means that the first quadrant (in the η_i coordinates goes into itself but the η_1 side stays put). Let us study what happens at a reflection. Any divergent family of trajectories will be divergent after the collision, and in particular, the parallel family will be strictly divergent. To be more precise the η_2 family will go into itself from just before to just after the collision, while the parallel one will be strictly divergent. Again the cone goes strictly inside itself but one side (the η_2 one this time) stays put. Nevertheless, the combination of free motion and reflection clearly sends the cone strictly inside itself.

Note that if a trajectory has a velocity with components with irrational ratios, then the flow without the obstacle is ergodic. This means that it is impossible that the trajectory does not hit the obstacle. Since the set of trajectories with velocities having components with rational ratios are of zero measure, it follows that almost all trajectories experience a collision. Hence, the billiard cocycle is eventually strictly monotone, and Wojtkowski's theorem applies. Accordingly, all the Lyapunov exponents are different from zero almost everywhere for the dynamical system $(\mathcal{M}, T, m)$.

5.3 Hyperbolicity of Bunimovich stadium

The naïve understanding of the previous example is that the obstacle acts as a defocusing mirror and thus makes the trajectories diverge, whereby creating instability. This idea was already present in Krylov work [39] and was considered the natural mechanism producing hyperbolicity. With this point of view in mind it seems that a table with convex boundaries (in which parallel trajectories are focused after reflections) is unlikely to yield hyperbolic behavior. This impression can be only confirmed by the presence of caustics in smooth convex billiards [40]. It came then as a surprise the discovery by Bunimovich that perturbations of the circle[3] could be hyperbolic.

The main intuition behind it is that, although the trajectories after reflection maybe focusing, after some time, they focus and then become divergent, so if there is enough time between two consecutive collisions, we can have divergent families going into divergent families, again (provided we look at the right place). Another equivalent point of view is that the instability is measured not just by the change in position but also, by the change in velocity, from this point of view, a very strong convergence is not so different from a strong divergence.

To find a new invariant family of cones, let us consider first a circular billiard. The collision angle is a conserved quantity of the motion. It is then natural to consider, at each point in phase space, the tangent vector η_3 associated to a family of trajectories that, at the next collision with one of the two half circles, will have the same collision angle.

We have defined η_3 in geometrical terms, clearly its expression in terms of η_1 and η_2 changes from point to point. Yet, there are special points (the middle of the cord between two consecutive collision with the same half-circle) in which η_3 coincides with η_2 (this is seen immediately by geometric considerations).

Clearly, in a sequence of collisions with the same circumference the vector η_3 is invariant. Also, from the above considerations follows that before collisions η_3 corresponds to a diverging family, while immediately after a collision it corresponds to a convergent family.

What happens to the parallel family η_1? Since divergent families becomes convergent it is obvious that the parallel family, after reflection, becomes even more convergent. Hence, it will focus before the middle point to the next collision (the point where the family η_3 focuses).

[3]Clearly non smooth perturbations such as the stadium, otherwise the KAM theorem would apply, see [29].

The previous considerations suggest to consider the cone $\mathcal{C}(x, v) = \{\xi \in \mathcal{TM} \mid \alpha\eta_1 + \beta\eta_3 \text{ with } \alpha\beta \geq 0\}$.

Hence, for a trajectory that collides only with a half circle the cone just defined is invariant but not strictly invariant. Since this would be true also for a billiard inside a circle it is clearly not sufficient (the billiard inside a circle has zero Lyapunov exponents, since, as we have already remarked, the motion is integrable).

Let us go back to the Bunimovich stadium. Clearly, it will behave as a circular billiard for trajectories colliding only with a half circle. So we need to see what happens if a trajectory goes from one circumference to the other (which will happen with probability one). In this case, the infinitesimal motion is the same that would happen if the straight line would be not present. In fact, if we reflect the billiard table along the straight lines we can imagine that the motion proceeds in a straight line.

Hence the family η_3 will first focus and than diverge for a longer time (and so get closer to the parallel family) than would happen if the collision would be in the same circle. This is exactly what we need to get strict invariance of the cone family.

In conclusion the cone family is strictly invariant each time that the trajectory goes from one half circle to the other. Since this happens almost surely, again we have proven hyperbolicity of the system.

It is interesting to notice that the cone family coincide with the one used in the Sinai Billiard (divergent trajectories) if one looks at it at the right point: the point laying in the intersection between the trajectory before collision and the circle of radius $r/2$ (if r is the radius of the half-circles forming the table) tangent to the the boundary at the next collision with a half-circle (but nowhere else).[4]

Problems

5.1 Let

$$L_i = \begin{pmatrix} 1 & 0 \\ t_i & 1 \end{pmatrix} \quad \text{and} \quad R_i = \begin{pmatrix} 1 & k_i \\ 0 & 1 \end{pmatrix},$$

for each $u \in \mathbb{R}^+$ write

$$\prod_{i=0}^{n} R_i L_i \begin{pmatrix} 1 \\ u \end{pmatrix} = \lambda_n \begin{pmatrix} 1 \\ u_n \end{pmatrix}.$$

[4] If the last collision was with a flat wall, then the point is obtained by reflecting the billiard so the trajectory backward looks straight, determining the point and then reflecting back to find the real point on the trajectory.

Show that

$$u_n = k_n + \cfrac{1}{t_n + \cfrac{1}{k_{n-1} + \cfrac{1}{t_{n-1} + \ddots + \cfrac{1}{t_1 + \cfrac{1}{k_1 + \frac{1}{u}}}}}}$$

And find conditions for the convergence of the continuous fraction. (Hint: see Problem 3.21)