Chapter 6

Hyperbolicity of hard spheres

For hard balls or radius $\frac{1}{2}$, and mass one, in dimension d, the flow is given by $\phi_t(q,p) = q + tp$ if no collision occurs. If the ball i collides with the ball j, then let p_i^-, p_j^- and p_i^+, p_j^+ be the velocities just before and after the collision, respectively. Note that for the balls to collide is must be that before the collision

$$0 > \frac{d}{dt}\|q_i - q_j\|^2 = \langle q_i - q_j, p_i - p_j \rangle.$$

Thus, at collision, $\langle q_i - q_j, p_i - p_j \rangle \leq 0$. Let $n = q_i - q_j$, then

$$\begin{aligned} p_i^+ &= p_i^- - \langle n, p_i^- - p_j^- \rangle n \\ p_j^+ &= p_j^- + \langle n, p_i^- - p_j^- \rangle n. \end{aligned} \tag{6.0.1}$$

Let us compute $d_{(q,p)}\phi_t(\delta q, \delta p)$ across a collision. If τ is the collision time of the trajectory then $\|q_i(\tau) - q_j(\tau)\| = 1$. If we consider the trajectories $\phi_t((q,p) + s(\delta q, \delta p))$, then the collision time $\tau(s)$ satisfies

$$\langle q_i(\tau) - q_j(\tau), \delta q_i(\tau) - \delta q_j(\tau) \rangle + \langle q_i(\tau) - q_j(\tau), p_i(\tau) - p_j(\tau) \rangle \tau'(0) = 0.$$

If the collision is non tangent (i.e. $\langle n, p_i(\tau) - p_j(\tau) \rangle \neq 0$), then,

$$\tau'(0) = -\frac{\langle n, \delta q_i(\tau) - \delta q_j(\tau) \rangle}{\langle n, p_i(\tau) - p_j(\tau) \rangle}.$$

To compute $d\phi_t$ is then convenient to shift along the flow direction by τ so all the trajectories $(q,p) + s(\delta q(\tau(s)), \delta p) + \tau(s)(p, 0)$ collide simultaneously.

Let us call $(\tilde{\delta q}^-, \tilde{\delta p}^-) = (\delta q(\tau(0)) + \tau'(0)p_i, \delta p_i)$, the shifted tangent vector. For such a tangent vector, we have that (6.0.1) yields

$$
\begin{aligned}
\tilde{\delta q}_i^+ =& \tilde{\delta q}_i^- \\
\tilde{\delta q}_j^+ =& \tilde{\delta q}_j^- \\
\delta p_i^+ =& \delta p_i^- - \langle \tilde{\delta q}_i^- - \tilde{\delta q}_j^-, p_i^- - p_j^- \rangle n - \langle n, p_i^- - p_j^- \rangle (\tilde{\delta q}_i^- - \tilde{\delta q}_j^-) \\
& - \langle n, \delta p_i^- - \delta p_j^- \rangle n \\
\delta p_j^+ =& \delta p_j^- + \langle \tilde{\delta q}_i^- - \tilde{\delta q}_j^-, p_i^- - p_j^- \rangle n + \langle n, p_i^- - p_j^- \rangle (\tilde{\delta q}_i^- - \tilde{\delta q}_j^-) \\
& + \langle n, \delta p_i^- - \delta p_j^- \rangle n.
\end{aligned}
\tag{6.0.2}
$$

The derivative is then obtained by shifting back along the flow direction, by $-\tau'(0)$. By taking the case $\tau(0) = 0$ we can thus obtain the behavior of a tangent from just before to just after a collision:

$$
\begin{aligned}
\delta q_i^+ =& \delta q_i + \langle p_i - p_j, \delta q_i - \delta q_j \rangle n \\
\delta q_j^+ =& \delta q_j - \langle p_i - p_j, \delta q_i - \delta q_j \rangle n \\
\delta p_i^+ =& \delta p_i - \langle \delta q_i - \delta q_j, p_i^- - p_j^- \rangle n - \tau'(0) \| p_i^- - p_j^- \|^2 n \\
& - \langle n, p_i^- - p_j^- \rangle (\delta q_i - \delta q_j) - \tau'(0) \langle n, p_i^- - p_j^- \rangle (p_i^- - p_j^-) \\
& - \langle n, \delta p_i - \delta p_j \rangle n \\
\delta p_j^+ =& \delta p_j + \langle \delta q_i - \delta q_j, p_i^- - p_j^- \rangle n + \tau'(0) \| p_i^- - p_j^- \|^2 n \\
& + \langle n, p_i^- - p_j^- \rangle (\delta q_i - \delta q_j) + \tau'(0) \langle n, p_i^- - p_j^- \rangle (p_i^- - p_j^-) \\
& + \langle n, \delta p_i - \delta p_j \rangle n.
\end{aligned}
\tag{6.0.3}
$$

To apply Theorem 3.5.1 we have thus to construct the quadratic form Q. We choose the lagrangian spaces $\mathbb{V}_1 = \{\delta q = 0\}$ and $\mathbb{V}_2 = \{\delta p = 0\}$. The energy is only kinetic energy, then the vectors tangent to the constant energy are $\langle p, \delta p \rangle = 0$. This yields the form $Q(\delta q, \delta p) = \langle \delta q, \delta p \rangle$. The vector field is $(p, 0)$, and $Q(\delta q + \alpha p, \delta p) = Q(\delta q, \delta p)$, so Q is well defined on the quotient and we can restrict ourselves to the vectors $\{(\delta q, \delta p) \ : \ \langle p, \delta p \rangle = \langle p, \delta q \rangle = 0\}$. Note that

$$
Q((\delta q + t\delta p, \delta p)) = Q(\delta q, \delta p) + t \| \delta p \|^2 \geq 0.
$$

It remains to compute the change in the quadratic form from just before to just after a collision. Since Q is invariant along the flow direction, we can use formula (6.0.2) and, since by construction $\langle \tilde{\delta q}_i^- - \tilde{\delta q}_j^-, n \rangle = 0$, obtain

$$
\begin{aligned}
Q(\delta q^+, \delta p^+) = Q(\tilde{\delta q}^+, \tilde{\delta p}^+) &= Q(\tilde{\delta q}^-, \tilde{\delta p}^-) - \langle n, p_i^- - p_j^- \rangle \| \tilde{\delta q}_i^- - \tilde{\delta q}_j^- \|^2 \\
&= Q(\delta q, \delta p) - \langle n, p_i^- - p_j^- \rangle \| \tilde{\delta q}_i^- - \tilde{\delta q}_j^- \|^2 \geq 0.
\end{aligned}
$$

The invariance of the cone follows.

Note that we have strict invariance if $\delta p \neq 0$. If $\delta p = 0$, then we have the strict invariance if $\tilde{\delta} q_i^- \neq \tilde{\delta} q_j^-$. This fails only if

$$\delta q_i^- - \frac{\langle n, \delta q_i - \delta q_j \rangle}{\langle n, p_i(\tau) - p_j(\tau) \rangle} p_i = \delta q_j^- - \frac{\langle n, \delta q_i - \delta q_j \rangle}{\langle n, p_i(\tau) - p_j(\tau) \rangle} p_j,$$

i.e. there exists $z \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} \delta q_i &= z + \lambda p_i \\ \delta q_j &= z + \lambda p_j. \end{aligned} \tag{6.0.4}$$

To see how to use the above facts, it is convenient to introduce a bit of notation.

6.1 Collision graphs and their decorations

First of all, I will introduce a *collision graph* to describe pictorially the relevant features of a trajectory, it will be a directed graph (the direction being given by time). The graph starts with n roots (each one representing one ball), from each root starts an edge (representing the path of a ball). A collision is represented by a vertex in the graph (I will indicate it pictorially by a star, so as not to confuse it with edges that cross due to the two-dimensional representation). If the collision involves k balls, then the vertex will have degree $2k$ with k entering edges–representing the incoming particles–and k exiting edges–representing the outcoming particles. We arrange the height of the vertices vertically proportionally to the time.

Lemma 6.1.1 *The graphs with a degree higher than two or with two vertices at the same height happen only on a set of codimension* 2.

PROOF. By the implicit function theorem, the collision of two balls is a codimension one condition, while of three balls is of codimension two. The same holds for the case of two collisions that happen at the same time. □

See figure 6.1 for the case of four balls in which number one collides with two, then two with four, and finally two with three.[1]

[1] The rule for tracing the graph is that the order of the balls is not changed at collision, so the line on the left represents the particles entering the collision vertex from the left. Remark that the collision graph is only a symbolic device and does not respect the geometry of the actual collisions, so the ordering of the balls is only a device to tell them apart and has no relation with the actual geometry of the associated configuration. Keeping this in mind, in figure 6.1 the final disposition of the balls is: one, four, two, three.

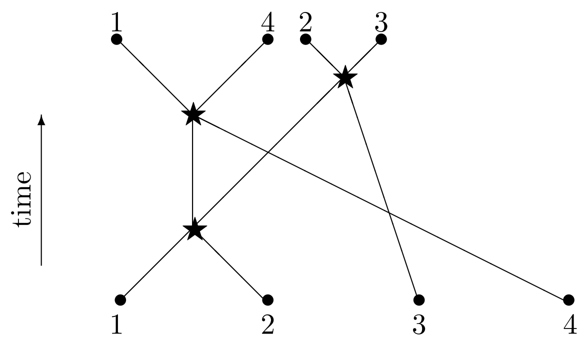

Figure 6.1: A simple collision graph (the stars are the collisions)

Next, let us call $\mathcal{G}$ a collision graph and let $V(\mathcal{G})$ be the collection of its vertexes, $\widetilde{E}(\mathcal{G})$ the collection of its edges and $E(\mathcal{G})$ the collection of edges that connect starred vertices. In addition for each edge $b \in E(\mathcal{G})$ let $\nu(b), \nu_+(b)$ be the two vertices joined by the edge.[2]

Definition 15 *Define $\pi : \widetilde{E}(\mathcal{G}) \to \{0, \dots, n\}$ so that $\pi(b)$ is the particle associated to the vertex $b \in \widetilde{E}(\mathcal{G})$. Also, $q^{\pm}_{\pi(b)}$, $p^{\pm}_{\pi(b)}$, $\delta q^{\pm}_{\pi(b)}$, $\delta p^{\pm}_{\pi(b)}$ are the position, velocity, and components of the tangent vector just after the collision represented by $\nu_-(b)$ and just before the collision represented by $\nu_+(b)$, respectively.*

To follow the history of a vector of type $(\delta q, 0)$ that stubbornly refuses to enter strictly in the cone it is convenient to specify at each vertex the values (λ_ν, z_ν) appearing in the associated equation (6.0.4). Of course, to recover the tangent vectors from the $\{(\lambda_\nu, z_\nu)\}_{\nu \in V(\mathcal{G})}$, it is necessary to specify the velocities. To this end we specify for each edge the velocity $v(b)$ of the particle associated to such a line. We can then decorate a graph with the above informations and we obtain a full description of the history of a tangent vector that keeps being not increased by the dynamics in the trajectory piece described by the graph (of course provided such a vector exists at all).

Now consider a edge $b \in E(\mathcal{G})$, if it represents the trajectory of the particle j between the collision corresponding to the the vertex $\nu(b)$ and the one corresponding to the vertex $\nu_+(b)$, then the corresponding component of the tangent vector at such times can be written both as $\delta q_j = z(\nu(b)) + \lambda(\nu(b))v(b)$ and $\delta q_j = z(\nu_+(b)) + \lambda(\nu_+(b))v(b)$. Accordingly, the following compatibility condition must be satisfied:

$$z(\nu(b)) - z(\nu_+(b)) = [\lambda(\nu_+(b)) - \lambda(\nu(b))]v(b). \tag{6.1.5}$$

[2]By convention $\nu(b)$ corresponds to the lower collision and $\nu_+(b)$ to the upper.

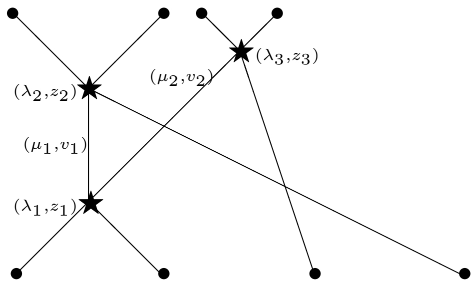

Figure 6.2: A decorated collision graph

It is then natural to define another decoration, this time associated to edges that connect two collision vertexes,

$$\mu(b) := \lambda(\nu_+(b)) - \lambda(\nu(b)). \tag{6.1.6}$$

By decorated collision graph, we will mean a graph with $(\lambda(\nu), z(\nu))$ attached to each vertex and $\mu(b), v(b)$ to each edge connecting two collisions, with a mild abuse of notations we will call such decorated graph $\mathcal{G}$ as well, see figure 6.2.[3]

6.2 Close path formula and cycles

As time progresses, the graph will grow more complex, in particular it may develop *cycles*.

Definition 16 *Let $m > 1$. A* path *of lenght m is a sequence of edges $\{b_1, \cdots, b_m\}$ and vertices $\{\nu_0, \cdots, \nu_m\}$ such that $\{\nu_{i-1}, \nu_i\} = \{\nu_\pm(b_i)\}$. A path is* simple *if $i, j \in \{1, \dots, m\}$, with $i \neq j$, implies $b_i \neq b_j$. A* closed path *of lenght m is a path such that $\nu_0 = \nu_m$. A* cycle *of length m, is a closed path of length $m > 2$ for which if $i, j \in \{0, \dots, m-1\}$ with $i \neq j$, then $\nu_i \neq \nu_j$, or of lenght $m = 2$ for which $b_1 \neq b_2$. (e.g. the thick edges in the graph of figure 6.3). We call close paths of type $\{b, b\}$* null-path.

Note that if we erase on edge from a cycle, then we do not have a cycle anymore. Next, we show that cycles capture all the closed path.

Lemma 6.2.1 *The edges of any closed path can be seen as the edges of the union of cycles and null path.*

[3] Note that the above description is quite redundant due to (6.1.5), yet we will see in the following that such a description is quite convenient.

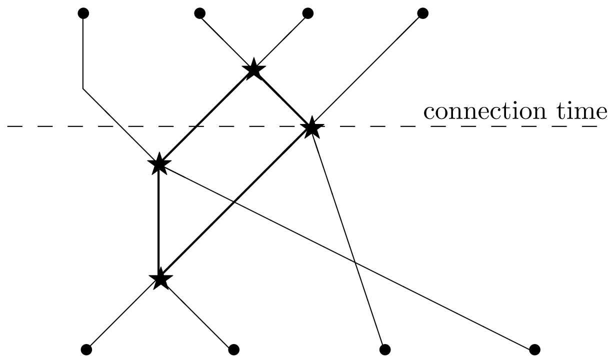

Figure 6.3: A cycle

PROOF. Let $\{b_1, \cdots, b_m\}$ be a closed path which is not a cycle. This implies that, for some $m \geq i > j > 0$, $\nu_j = \nu_i$. This implies that the edges can be seen as the edges of the two closed paths $\{b_{i+1}, \ldots, b_m, b_1, \ldots b_j\}$ and $\{b_{j+1}, \cdots, b_{i-1}\}$ We can then continue the decomposition until we are left only with cycles or null-paths. Note that the same cycle or null-path can appear several time, so it is enough to consider each only once. □

Given a closed path a remarkable compatibility condition can be derived. In fact, let $C \subset \mathcal{G}$ be a cycle, let us run it counterclockwise and define, for each edge $b \in C$, $\varepsilon_C(b) = 1$ if the edge is run from bottom to top and $\varepsilon_C(b) = -1$ if it is run from top to bottom. We have, by definition (6.1.6), $\sum_{b \in E(C)} \varepsilon_C(b)\mu(b) = 0$. In addition, we can sum equation (6.1.5) for each edge in the cycle and obtain

$$\begin{aligned} \sum_{b \in E(C)} \mu(b)\varepsilon_C(b)v(b) = 0 \\ \sum_{b \in E(C)} \varepsilon_C(b)\mu(b) = 0. \end{aligned} \tag{6.2.7}$$

The above formula is essentially the *closed path formula* introduced by Simanyi in [54]. Notice that, given a graph, the set of closed paths contains many more objects than we are interested in. In fact, a closed path can contain an edge that is run consecutively in opposite directions, in particular, a null-path. This adds zero to the constraint (6.2.7). Hence, by Lemma 6.2.1, we can consider only cycles. Also, a cycle can be run in opposite directions, which gives rise to the same closed path formula; hence, we will always use the counterclockwise orientation.[4]

[4]Since, by definition, a cycle has no self-intersections, it is orientable, and, by Jordan's

The formula 6.2.7 expresses a compatibility condition that puts a clear restriction on the possible existence of the decorated collision graph, and hence of the corresponding nonincreasing vector. Studying the combinatorics of such collisions, it is possible to establish the hyperbolicity and ergodicity of a gas of n particles. This has been done first in a series of papers by the *Hungarian team* [38, 54, 55, 56, 57]. In the following we will enter in some of the related details.

6.2.1 A cohomological point of view

Althogh not strictly necessary in the following, it is amusing to note that the above structure is very reminiscent of cohomology over a decorated graph as I am going to briefly explain.
Let $\mathcal{C}(\mathcal{G})$ be the set of cycles of the graph $\mathcal{G}$. Consider, the following set of fucntions on a decorated graph $\mathcal{G}$: $V_v(\mathcal{G}) = \{f : V(\mathcal{G}) \to \mathbb{R}^{d+1}\}$; $V_e = \{f : E(\mathcal{G}) \to \mathbb{R}^{d+1}\}$; $V_c = \{f : \mathcal{G}(\mathcal{G}) \to \mathbb{R}^{d+1}\}$. We can then define the operator $d : V_v(\mathcal{G}) \to V_e(\mathcal{G})$ and $d : V_e(\mathcal{G}) \to V_c(\mathcal{G})$ as

$$(df)(b) = f(\nu_+(b)) - f(\nu_-(b))$$
$$(df)(C) = \sum_{b \in C} \varepsilon_C(b) f(b).$$

Note that $d^2 = 0$, so we can define a cohomology on decorated graphs. Note that if $f \in V_e(\mathcal{G})$ is closed, that is $df = 0$, then it is exact, that is there exists $g \in V_v(\mathcal{G})$ such that $dg = f$.

With a small twist with respect to the usual convention we define the restricted space

$$V_e^*(\mathcal{G}) = \{f \in V_e(\mathcal{G}) \ : \ f(b) = \mu(b)\bar{v}(b), \mu(b) \in \mathbb{R}\},$$

and say that a function $f \in V_v(\mathcal{G})$ is closed iff $df(b) \in V_e^*(\mathcal{G})$. While, as usual and already stated earlier, $f \in V_e(\mathcal{G})$ is closed iff $df = 0$. Also, note that for a function in $V_e^*(\mathcal{G})$ being closed means exaclty that it satisfies the cycle contraints (6.2.7).

With such a notation, for a trajectory to be hyperbolic (i.e. with all the Lyapunov exponent non zero) is equivalent to saying that there exists a time such that the associated graph $\mathcal{G}$ has the following property: the only closed functions in $V_v(\mathcal{G})$ are the constant ones.

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theorem, it has an interior. By counterclockwise orientation we mean that it is run so that the interior lies on the left.

6.2.2 Basic cycles

For the following, it is convenient to write (6.2.7) more compactly. To this end define, for each $b \in E(\mathcal{G})$, $\bar{v}(b) \in \mathbb{R}^{d+1}$ by

$$\bar{v}(b) := \begin{pmatrix} 1 \\ v(b) \end{pmatrix}.$$

We can then write

$$\sum_{b \in E(C)} \varepsilon_C(b)\mu(b)\bar{v}(b) = 0. \tag{6.2.8}$$

As already said, the basic idea of the approach is to follow the graph in time, collect the various equations (6.2.8) after the formation of new cycles and then try to analyze such a set of equations to show that, at a certain time, they do admit only the trivial solution. To do that, it is First, it is necessary to understand the process of cycle formation a bit better.

First of all, one must notice that it suffices to consider connected graphs.[5] Indeed, if there is no chain of collision whatsoever connecting two given balls, then there exists an obvious non increasing vector. Given a graph there will exist a well defined *connecting time*, that is the first time for which the graph is connected, (see figure 6.3). Before such a time the graph can be decomposed into connected components. Let us call *collision clusters* the set of balls associated to a connected components of the graph.

Let $C(\mathcal{G})$ be the set of cycles of the graph $\mathcal{G}$.

Note that if two cycles have a nonempty intersection, then there exists a cycle obtained by joining them; yet the equations (6.2.8) associated with such "new" cycle do not carry any new information. Hence, we want to describe how to partition cycles (that we will call *basic cycles*) that keep track of all the constraints on the neutral tangent vectors.

Definition 17 *Given a cycle C, we call $\nu_+(C)$ its* top vertex *and $\nu_-(C)$ its* bottom vertex.

Definition 18 *We will use the following rule for constructing* basic cycles*: when a collision creates new cycles*[6] *then we choose a new basic cycle that has the highest possible lower vertex. In this way, we create a collection of basic cycles associated with the graph, and we will order them according to the height of their top vertices. Hence, given a graph $\mathcal{G}$ we define a set of basic cycles $\mathcal{C}(\mathcal{G}) = \{C_1, \dots, C_m\}$.*

[5] That is graphs in which there are paths that join any two given roots.

[6] That is, cycles that were not present before such a collision.

Note that the above construction is not unique, yet once the choice is made, it provides a prescription on how to usefully write the equations (6.2.7).

6.2.3 Graphs revisited: cycle formation an bookkeeping

Let us start by substantiating the claim that to analyze (6.2.7) it suffice to consider basic cycles.

Lemma 6.2.2 *All the cycles present in a graph $\mathcal{G}$ can be obtained by composing the cycles in $\mathcal{C}(\mathcal{G})$.*[7]

PROOF. The proof is obtained by induction on the number of cycles. Clearly, the first time a cycle is created, there is only one cycle to choose from. Next, suppose we have already gathered m basic cycles, and a new cycle is created. Such a cycle contains the two edges that enter the new vertex and cannot be contained in any previous cycle. Moreover, if one of them appears twice in the cycle, so must the other. Suppose this is the case, let us follow the path starting from the top vertex till it comes back to the top vertex, this must give rise to two cycles. If they are the same, then we can erase one. If they are different, then we can erase the common edges starting from the top vertex, and we still have a cycle. By the induction hypothesis, such a cycle is generated by the old basic cycles, so we can erase one of the two cycles without loss of generality. We can continue in such a way till the top vertices appear only in one cycle, call it C_*. It follows that all the other cycles are generated by the already existing basic cycles, and, adding to the list C_*, we can generate the new cycles as well. □

Next, let us see how cycles are created.

Lemma 6.2.3 *At each new collision, either the number of collision clusters decreases by one or a new basic cycle is formed.*

PROOF. The two colliding balls can either belong to different clusters or to the same. In the first case, clearly the number of clusters decreases by one. In the second case, notice that the two edges that join to create the new vertex cannot belong to any existing cycle since they did not lead anywhere before the last collision. On the other hand, such edges either come from the same vertex, and then we have a new two-cycle, or come from different vertices. In the latter case, each of such vertices will be connected to some root of the graph. On the other hand, since the graphs is connected, there

[7] This means that each other minimal cycle is a subset of the union of the chosen ones.

will be a path connecting two such roots. Joining such paths and eliminating iteratively the edges that are run consecutively in opposite directions, we obtain a closed path. If an edge appears twice in such a path, then there are two possibilities: either the two occurrences follow exactly the same path (hence, the path is run more than one time, and we can disregard the repetitions), or the two paths diverge. But this means that the path has two different ways to go back to the same edge, hence it contains two closed paths. We can then keep the closed path that contains the new vertex and has the highest lower vertex. Doing this iteratively, we obtain a closed path that contains the new vertex and in which all the edges are crossed only one, that is a new basic cycle. □

Let us see how we can conveniently gather the information contained in $\mathcal{C}(\mathcal{G})$. Let $N(\mathcal{G})$ be the number of edges in $\cup_{C\in\mathcal{C}(\mathcal{G})}E(C) =: E(\mathcal{C}(\mathcal{G}))$ and let $N_C(\mathcal{G})$ be the number of elements of $\mathcal{C}(\mathcal{G})$. Write $\mathcal{C}(\mathcal{G}) = \{C_1, \ldots, C_{N_C(\mathcal{G})}\}$. For each $b \in E(\mathcal{C}(\mathcal{G}))$ we define the $\mathbb{R}^{(d+1)N_C(\mathcal{G})}$ vector

$$V_{\mathcal{G}}(b)_i = \begin{cases} \varepsilon_{C_k}(b)\bar{v}(b)_{i-(k-1)(d+1)+1} & \text{if } i \in \{(k-1)(d+1), k(d+1)d-1\} \\ & \text{and } b \in C_k \\ 0 & \text{otherwse.} \end{cases}$$

We can then write all the constraints (6.2.8) together as

$$\sum_{b\in E(\mathcal{C}(\mathcal{G}))} \mu(b)V_{\mathcal{G}}(b) = 0. \tag{6.2.9}$$

Let $\mathcal{G}_t$, $t \in \mathbb{N}$, be the collision graph after t collisions. Let $\tau_* \in \mathbb{R}$ be the first time in which the graph is connected. By Lemma 6.2.3 after collision τ_* at each collision, a new cycle is formed. It follows that at each new collision, the total number of edges grows by two while the dimension of the $V_{\mathcal{G}}$ grows by $d+1$. Let us call $E_0(\mathcal{G})$ the edges $b \in E(\mathcal{G})$ such that equation (6.2.9) implies $\mu(b) = 0$. Let C the the new basic cycle. If there exists $\{b_1, b_2, b_3\} \subset E(C) \setminus E_0(\mathcal{G})$, such that $\{v(b_i)\}_{i=1}^3$ are linearly independent, then the dimension of $\dim\{V_{\mathcal{G}}(b) \ : \ b \in C(\mathcal{G})\}$ must increase by at least one.

To continue, we must show that the equations (6.2.9) are generic, and hence after enough cycles are formed, the only solution is $\mu \equiv 0$ with probability one. This analysis is outside the scope of the present notes and can be found in the original papers, e.g., [56]; here, we will content ourselves with some simple examples.

6.3 Hyperbolicity: some examples

Having developed a general theory to describe the evolution of tangent vectors that fail to become hyperbolic, we are going to check the effectiveness of the theory by analyzing a series of examples of increasing complexity.

6.3.1 2 balls in dimension $d \geq 2$

First of all, recall that there are always zero Lyapunov exponents connected with the flow direction and momenta conservation. To avoid that, we consider only the situation where the center of mass is at rest. This implies that $\sum_i \delta q_i = 0$.

Next, notice that the situation in which the two balls never collide is of zero measure: if there is no collision, the two balls just perform a translation on the torus. One can then see it as a translation on $\mathbb{T}^{2d}$, which is ergodic if the velocities have entries that are not rational among them, which is a zero-measure condition. Hence, the ball will collide with probability one.

Once the two balls collide for the second time, we have a close cycle made of just two bonds $\{b_1, b_2\}$. In this case (6.2.7) implies

$$\mu(b_1)(v(b_1) - v(b_2)) = 0,$$

which has solutions either $\mu(b_1) = 0$ or $v(b_1) = v(b_2)$. The latter condition is a codimension $d - 1$ condition, hence with probability one $\mu(b_1) = 0$. But then the second of (6.2.7) implies $\mu(b_2) = 0$. It follows that $\delta q = (z, z)$, which implies $z = 0$, hence eventual strict monotonicity and hence hyperbolicity.

6.3.2 3 balls in dimension $d \geq 2$

Again we can assume $\sum_i \delta q_i = 0$. Also, we would like to know that the situation in which a ball does not collide with the other two has zero measure. This is more complex and needs the mixing of the two ball systems. Let us assume it and proceed.

After the first collision (say of particles $1, 2$), we wait until a collision involving particle 3, say a collision with particle 1. The next collision will close a cycle. If the cycle involves particles $1, 3$, then the previous discussion implies $\mu(b_1) = \mu(b_3) = 0$. This situation will persist until there is a collision with particle 2, say with particle 3. At that point (6.2.7) implies $\mu(b_2) = 0$. It follows that, almost surely $\delta q = (z, z, z)$ and hence $z = 0$.

The case in which the cycle involves all the particles remains to be analyzed, say particle 3 collides with particle 2. In this case (6.2.7) implies

$$\mu(b_1)\varepsilon_C(b_1)(v(b_1) - v(b_2)) + \mu(b_3)\varepsilon_C(b_3)(v(b_3) - v(b_2)) = 0$$

where b_1 is the edge associated to the particle 1 before the collisione with 3, b_3 is the edge associated to the partile 3 after the collision with 1 and b_2 the edge associated to the particle 2 after the collision with 1. On the other had, calling b_3^- the edge associated to 3 before the collision with 1, by (6.0.1) we have

$$v(b_3) = v(b_3^-) - \langle n, v(b_3^-) - v(b_1)\rangle n.$$

Note that $R := \langle n, v(b_3^-) - v(b_1)\rangle = 0$ is a codimension one condition hence it happens on a zero measure set. Accordingly, just before the collision of 1 and 3 we have

$$\mu(b_1)\varepsilon_C(b_1)(v(b_1) - v(b_2)) - \mu(b_3)\varepsilon_C(b_3)(v(b_1) - v(b_3^-) - Rn) = 0.$$

Again, for $v(b_1) - v(b_2)$ and $v(b_1) - v(b_3^-) - Rn$ to be linearly dependent is a codimension one condition. It follows $\mu(b_1) = \mu(b_3) = 0$ and then (6.2.7) implies $\mu(b_2) = 0$. So $\delta q_i = 0$, thus the form is eventually strictly increasing.