Chapter 7

Geometry of foliations and ergodicity (very few words)

We have seen conditions that imply hyperbolicity. Once the map is hyperbolic Pesin Theory (e.g. see [37]) implies that there exists stable and unstable manifolds. However, the objects constructed in Pesin's theory have very poor regularity properties. For applications, a more explicit construction can be essential, especially if it provides extra information on the properties of the manifolds. For this, the mere existence of an eventually strictly invariant cone field is not enough; from now on, we will assume that the cone field is continuous.

7.1 Cones and invariant distributions

We have seen that the growth of an appropriate quadratic form implies the contraction of a cone. A natural question is if such a contraction can be described in a more quantitative way. This is possible, a general theory can be found in [45], here we give only a quick overview.

Definition 20 *The symplectic angle between two vectors* $u, w \in \operatorname{int}(\mathcal{C})$ *is the real number* $\Theta(u, w)$ *defined by*

$$\omega(u, v) = \sqrt{Q(u)}\sqrt{Q(v)} \sinh \Theta(u, v)$$

Definition 21 *The distance* $s(U, W)$ *of two Lagrangian subspaces* $U, V \subset \operatorname{int}(\mathcal{C})$ *is equal to the supremum of absolute values of symplectic angles be-*

tween nonzero vectors from the two Lagrangian subspaces i.e.

$$s(U,V) = \sup_{\substack{0 \neq u \in U \\ 0 \neq v \in V}} |\Theta(u,v)|.$$

It turns out that s is a metric on the set of Lagrangian subspaces contained in the cone (e.g. see [45]). Here, we just note that if $s(U,V) = 0$ it follows, by definition, that $\omega(v,u) = 0$ for all $u \in U$ and $v \in V$, but this implies that $V = U$.
Note that if $s(U,V) < \infty$ and L is a monotone symplectic matrix, then

$$\begin{aligned}\sqrt{Q(u)}\sqrt{Q(v)}\sinh\Theta(u,v) &= \omega(u,v) = \omega(Lu,Lv) \\ &= \sqrt{Q(Lu)}\sqrt{Q(Lv)}\sinh\Theta(Lu,Lv)\end{aligned}$$

Thus,

$$\begin{aligned}\sinh s(LU,LV) &= \sup_{\substack{0 \neq u \in U \\ 0 \neq v \in V}} \sinh\Theta(u,v)\sqrt{\frac{Q(u)}{Q(Lu)}}\sqrt{\frac{Q(v)}{Q(Lv)}} \\ &\leq \sigma(L)^{-2}\sinh s(U,V).\end{aligned} \tag{7.1.1}$$

The following Theorem gives a criterion for $s(U,V)$ to be finite.

Theorem 7.1.1 (Theorem 2 [45]) *For a strictly monotone map L the diameter of $L\mathcal{C}$, where $\mathcal{C}$ is determined by the transversal lagrangian spaces V_1, V_2, is equal to the s distance of LV_1 and LV_2, Moreover, for each Lagrangian spaces $V, U \subset \mathcal{C}$*

$$\tanh\left(\frac{s(LV,LU)}{2}\right) = \sigma(L)^{-2}.$$

Lemma 7.1.2 *Given a smooth Symplectic Dynamical Systems with syngularities (X, T, μ), X a symplectic two dimensional manifold, μ the symplectic volume, if the systems is eventually strictly monotone, then $\{E^u(x)\}$ is almost everywhere well defined. Moreover, if $\mathcal{C}(x)$ is continuous, then $\{E^u(x)\}$ is continuous (where it is defined). In addition, if the cone family is strictly monotone, then $\{E^u(x)\}$ is everywhere defined.*

PROOF. Let $\mathcal{C}_n(x) := D_{T^{-n}x}T^n\mathcal{C}(T^{-n}x)$ and $\Delta_n(x) := \operatorname{diam}(\mathcal{C}_n(x))$, then Δ_n is decreasing, thus we can define

$$\Delta_\infty(x) := \lim_{n\to\infty}\Delta_n(x).$$

By Theorem 7.1.1

$$\begin{aligned}\Delta_\infty(T^m x) &= \lim_{n\to\infty} \text{diam}(D_{T^{m-n}x} T^n \mathcal{C}(T^{-n+m}x)) \\ &= \lim_{n\to\infty} \text{diam}(D_x T^m D_{T^{-n}x} T^n \mathcal{C}(T^{-n}x)) \\ &\leq \frac{1}{\sigma(D_x T^m)^2} \Delta_\infty(x).\end{aligned}$$

Next, let $\Omega = \{x \in X \mid \Delta_\infty(x) = \infty\}$, we claim that $\mu(\Omega) = 0$. In fact, let $B_m = \{x \in X \mid \sigma(D_x T^m) \geq 2\}$, by eventual strict monotonicity of the cone field it follows $\mu(\cup_{m\in\mathbb{N}} B_m) = \mu(X)$, recall (3.5.8). In addition, $B_m \supset B_{m_0}$ for all $m > m_0$. Moreover, if $x \in B_m$, then $\Delta_\infty(T^m x) < \infty$ (see Theorem 7.1.1). Thus $T^{-n}\Omega \cap B_m = \emptyset$ for all $n \geq m$, and

$$\mu(\Omega) = \lim_{n\to\infty} \mu(T^{-n}\Omega) \leq \lim_{n\to\infty} \mu(X \setminus \cup_{m\leq n} B_m) = 0.$$

Finally, let $\Omega_L = \{x \in X \mid \frac{L}{2} \leq \Delta_\infty(x) \leq L\}$ and suppose $\mu(\Omega_L) > 0$. Then, there exists $m \in \mathbb{N}$ such that $\mu(\Omega_L \cap B_m) > 0$. Consequently, for almost all $x \in \Omega_L \cap B_m$ there exists a return time $\bar{n}m \in \mathbb{N}$ in the past (that is $T^{-m\bar{n}}x \in \Omega_L \cap B_m$). Accordingly,

$$\frac{L}{2} \leq \Delta_\infty(x) \leq \frac{1}{\sigma(D_x T^m)^2} \Delta_\infty(T^{-m\bar{n}}x) \leq \frac{L}{4},$$

which is a contradiction unless $L = 0$. We have so proven that $\mu(\Omega_0) = \mu(X)$. In other words the cones $\mathcal{C}_\infty = \cap_{n\geq 0} \mathcal{C}_n(x)$ is almost everywhere degenerate since, having zero diameter, means that the cone is a Lagrangian suspace which is precisely the unstable direction.

To prove the continuity of the above distribution note that the cone family $\mathcal{C}_n(x)$ is continuous. Let x be such that $\Delta_\infty(x) = 0$, then, for each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\Delta_m(x) < \frac{\varepsilon}{2}$. Then one can chose δ such that the edges of $\mathcal{C}_m(y)$ vary by an amount less than $\frac{\varepsilon}{2}$ if $d(x, y) < \delta$. The result follows then taking into account that the Hilbert metric bounds the angle and that the unstable distribution is contained in $\mathcal{C}_n$ for each $n \in \mathbb{N}$.

The proof of the last fact is obvious: just a simplification of the above arguments. □

Let us conclude with an interesting simple fact.

Lemma 7.1.3 *A smooth two-dimensional Symplectic Dynamical System* (X, T, μ) *is Anosov iff it admits a strictly monotone continuous cone family.*

PROOF. By Lemma 7.1.2 it follows that the stable and unstable distribution are continuous. But then, by continuity, there exists $\alpha > 0$ and $\sigma > 1$ such that

$$\alpha\sqrt{Q(v)} \leq \|v\| \leq \alpha^{-1}\sqrt{Q(v)} \quad \forall x, \in X \text{ and } v \in E^u(x)$$
$$\sigma(D_xT) \geq \sigma \quad \forall x \in X.$$

Thus,

$$\|D_xT^n v\| \geq \alpha\sqrt{Q(D_xT^n v)} \geq \alpha\sigma^n\sqrt{Q(v)} \geq \alpha^2\sigma^n\|v\|.$$

Analogously one can obtain the statement for the stable direction by using the cone family given by the complementary cones (see Problem 3.3).

The proof that an Anosov systems admit a continuous, strictly invariant cone family is obvious and it is left to the reader. □

7.2 Invariant foliations

Once we know that the system is hyperbolic, we can try to take advantage of hyperbolicity: the first step is to construct stable and unstable manifolds.

The strategy is the usual one: e.g., to construct the unstable manifold at x, consider the trajectory $f^{-n}(x)$ (for simplicity, we consider the Poincarè map). If the trajectory does not meet a discontinuity, then we can consider a manifold W, with tangent space in the unstable cone, centered at $f^{-n}(x)$ and push it forward with the dynamics. In this way, we obtain a sequence of manifolds $W_n = f^n(W)$ that we expect to converge to a limit object. Yet, one has to take into account that the manifold can be cut by singularities, and this could be a serious problem.

In the uniformly hyperbolic case, the analysis is especially simple: since the manifold W expands exponentially ($|W_n| \geq e^{\lambda n}|W|$), we have that the manifolds are cut at a distance shorter than δ only if the distance of $f^{-n}(x)$ from the singularities is less than $\delta e^{-\lambda n}$. This means that the manifold is cut short only if $f^{-n}(x)$ belongs to a neighborhood $\mathcal{S}_n$ of measure $\delta e^{-\lambda n}$. But since the measure is preserved, we have

$$\text{Leb}\left(\cup_{n=0}^{\infty} f^n(\mathcal{S}_n)\right) \leq \sum_{n=0}^{\infty} e^{-\lambda n}\delta \leq C\delta.$$

It follows that there exists a set of measure $1 - C\delta$ in which the unstable manifold has a length larger than δ.

Implementing the above basic idea can be technically challenging, especially since the formula (5.1.3) shows that the derivative blows up near tangencies. Yet, it can be done, for details, see [38, 17]. A technical tool used to deal with the blow-up of the differential at tangent collisions is the introduction, by Sinai, of homogeneity strips. See [17] for details.

The above construction provides a stable foliation, yet the foliation has very poor regularity properties, and this makes it very hard to use it; in general, it is only measurable. Luckily, the holonomy is absolutely continuous. Moreover, it turns out that it can be approximated by a foliation with much better properties that can be conveniently used, see [6, Section 6] for details.

The next step is to prove ergodicity. Once we have an absolutely continuous foliation, you can try to copy Hopf's argument. Such an argument is based on the observation that the ergodic averages of continuous functions are constant along stable and unstable manifolds. This was achieved by Sinai [61]. But see [45] for a more general version. In addition, [45] discusses a piecewise linear example in which the technical difficulties are reduced to a bare minimum, and hence Sinai's argument can be easily understood. The idea is to prove local ergodicity, and then a global argument can be employed to prove ergodicity. The same argument proves that all the powers of the Poincare maps are ergodic, which implies mixing.

It remains the problem of flows. Since the flow can be seen as a suspension over the Poicnarè map, the ergodicity of the flow follows from the ergodicity of the map. Not so for mixing: think of a suspension with a constant ceiling. Mixing for the flows follows from the contact structure. Forgetting for one second the discontinuities, the fact that the flow is contact implies that is we do a cycle stable, unstable, stable, unstable, we move in the flow direction, see Figure 7.1.

Indeed, let α be the contact form, then if v is a strong unstable or a strong stable vector, then $\alpha(v) = 0$, while $\alpha((p, 0)) = 1$, where $(p, 0)$ is the flow direction, it follows that if the cycle in bold in figure 7.1, call it γ, has sides of length δ, then

$$\delta^2 = \int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha = \int_\gamma \alpha$$

which equals exactly the displacement in the flow direction, which is then non-zero. It follows that the stable and unstable foliations are not *jointly integrable*, and this property shows that the flow cannot be reduced to a constant flow suspension by a change of coordinates (since, in such a case, the foliations would indeed be jointly integrable). This suffices to prove the mixing of the flow, eventhough the argument is a bit more technical than this.

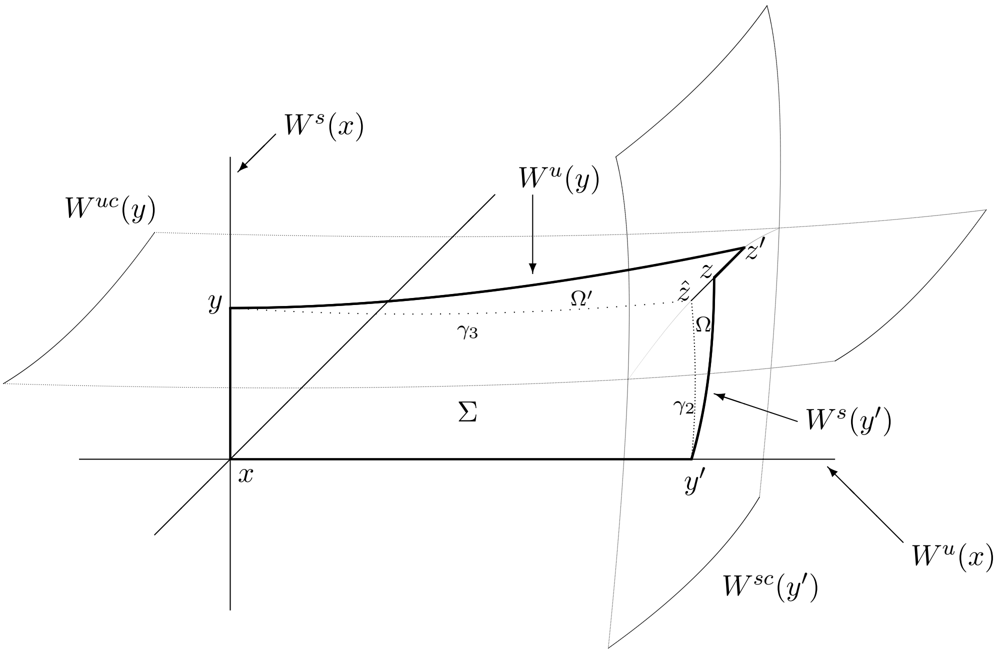

Figure 7.1: Definition of the *temporal function* $\Delta(y, y')$ and related quantities

Chapter 8

Statistical Properties

8.1 The problem and a brief overview

Given a topological Dynamical System we would like first to characterize the invariant measures in order to have a clearer picture of which measurable Dynamical Systems can be associated with them. This is still at the qualitative level. In addition, we would like to have tools to actually compute such invariant measures with a given precision, and this is a first quantitative issue.

Next, we would like to study in-depth statistical properties for some measures that we deem interesting. The type of questions we would like to address are

If we make repeated finite time and precision measurements, what do we observe?

Remember that a measurement is represented by the evaluation of a function. The fact that the measurement has a finite precision corresponds to the fact that the function has some uniform regularity (otherwise, we could identify the point with an arbitrary precision). The fact that the measure is made for finite time means that we are able only to measure finite time averages. In other words, we would like to understand the behavior of

$$\sum_{k=0}^{N-1} f \circ T^k$$

for large but finite N.

We will see that to achieve this, it is necessary, first and foremost, an estimate of the speed of mixing. In the case of two-dimensional hyperbolic billiards, Bunimovich and Sinai first achieved this [9] for the Poincarè map, while the result for the flow is due to Demers, Baladi Liverani [19], almost forty years later (not for lack of trying). For higher dimensional billiards, the problem is still open.

Several techniques have been developed to study the speed of decay of correlations, the main one are

1. coding the system via Markov Partitions (Bunimovich and Sinai [9])
2. coding the systems via towers (Lai-Sang Young [68, 69])
3. standard pairs and coupling (Lai-Sang Young [69], Dolgopyat [28])
4. operator renewal theory (Sarig [55])
5. Functional spaces adapted to the transfer operator (Blank, Keller, Liverani [8]; Liverani, Gouezel [36]; Baladi, Tsujii [5]; Demers, Liverani [20]; Demers, Zhang [24])
6. Hibert metric (Ferrero, Schmitt [34], Liverani [47]; Demers, Liverani [25])
7. Random perturbations (Liverani, Saussol, Vaienti [48])

The most powerful techniques are probably (5, 6), but they can work only if the decay of correlations is exponential. For polynomial decay of correlations (2, 4) or even the rougher (7) are the way to go. While (3) is unquestionably the more versatile technique.

For an introduction to (3,5,6) see [21].

To conclude, let me recap part of the state of the art, giving a, idiosyncratic, list of results.

The ergodicity of various billiard tables was established in many papers, e.g., [66, 14]. Ergodicity results also exist for billiards in which the particle is subject to a soft potential, rather than a hard core one, e.g. [39, 30]. The ergodicity of a gas of hard spheres was established, building on a rather long string of papers, in [59]. The statistical properties of billiards with finite and infinite horizon can be found in [16, 29] where the standard pair technology is put to work. The functional analytic approach has been developed in [22, 24]; such an approach also allows establishing how the statistical properties depend on the billiard shape [23]. In addition, the functional approach has proven instrumental in the proof of exponential mixing for two dimensional uniformly hyperbolic billiard flow [6]. Many limit theorems have been obtained for billiard systems for which mixing properties have been established. Notable results are the polynomial decay of correlations in the Bunimovich stadium [7] and the monumental study of one massive particle interacting with a light one in a box [18].

All the previous papers deal with isolated systems, if the system changes in time (e.g. a time-dependent billiard table), then the simple study of the spectral properties of the transfer operator does not suffice; one has to deal with the product of different operators. This can be done using perturbation theory if the change in time is very slow [63]. However, if the change in time is more violent, perturbation theory fails, and a new approach is needed. This has been recently achieved in [25] using Hilbert metrics on invariant cones of densities.

Even though the above list of results is very partial, I hope it gives an idea of the breadth of the field and of the many directions along which the research is developing.

Given a Dynamical System, it is, in general, very hard to study its ergodic properties, especially if the goal is to have a *quantitative* understanding. To make clear what is meant by a *quantitative understanding* and which type of obstacles may prevent it, I refer the reader to the first chapters of the book [21], available online.

8.2 A machine to construct invariant measures

Dynamical systems typically have many invariant measures, exceptions like minimal or unique ergodic systems are rare but important.

In particular, given any periodic orbit, one can construct an invariant measure supported on such an orbit. These are not so interesting measures, even though the asymptotic growth of the periodic orbits contains an incredible amount of information about the system, see the book [27] to have an idea of what I am talking about.

In this section we describe how to construct manifold invariant measures for the simplest possible system: a smooth expanding map of the cirlce.

Let $f \in \mathcal{C}^2(\mathbb{T}, \mathbb{T})$ with $f' \geq \lambda > 1$ and $g \in \mathcal{C}^1(\mathbb{T}, \mathbb{R})$. We define the operator

$$\mathcal{L}_g h(x) = \sum_{y \in f^{-1}(x)} e^{g(y)} h(y).$$

Problem 8.2.1 *Show that, for all $n \in \mathbb{N}$,*

$$\mathcal{L}_g^n h(x) = \sum_{y \in f^{-n}(x)} e^{g_n(y)} h(y),$$
$$g_n(y) = \sum_{k=0}^{n-1} g \circ f^k(y).$$

Problem 8.2.2 (Lasota-Yorke) *Show that, for all $n \in \mathbb{N}$, and $h \in \mathcal{C}^1(\mathbb{T}, \mathbb{C})$,*

$$\|\mathcal{L}_g^n h\|_\infty \leq \|\mathcal{L}_g^n 1\|_\infty \|h\|_\infty$$
$$\left\| \frac{d}{dx} \mathcal{L}_g^n h \right\|_\infty \leq \|\mathcal{L}_g^n 1\|_\infty \left[\lambda^{-n} \|h'\|_\infty + \frac{\|g'\|_\infty}{1 - \lambda^{-1}} \|h\|_\infty \right]$$

Note that Problem 8.2.2 implies that $\mathcal{L}_g \in L(\mathcal{C}^1, \mathcal{C}^1)$, that is a bounded operator on $\mathcal{C}^1$ functions.

Problem 8.2.3 *Use subjectivity of $a_n = \ln \|\mathcal{L}_g^n 1\|_\infty$ to show that it exists*

$$\nu = \lim_{n \to \infty} \|\mathcal{L}_g^n 1\|_\infty^{\frac{1}{n}}.$$

Then use Problem 8.2.2 to show that

$$\nu = \lim_{n \to \infty} \|\mathcal{L}_g^n\|^{\frac{1}{n}}$$

where the norm is the usual operator norm that makes $L(\mathcal{C}^1, \mathcal{C}^1)$ a multiplicative algebra.

Thus $\sup\{|z| \ : \ z \in \sigma(\mathcal{L}_g)\} = \nu$. Also, since the unit ball of $\mathcal{C}^1$ is compact in $\mathcal{C}^0$ by Ascoli-Arzela, it follows that the Hennion Theorem applies (see [21, Appendix B] for a precise statement and a proof). As a consequence, we know that the essential spectral radius of $\mathcal{L}_g$ is bounded by $\lambda^{-1}\nu$. We then have the following spectral decomposition

$$\mathcal{L}_g = \sum_{\ell=1}^{M} \left[e^{i\theta_\ell} \nu \Pi_\ell + N_\ell \right] + R$$

where $\theta_\ell \in [0, 2\pi)$, $d_\ell \in \mathbb{N} \cup \{0\}$, with the convention $N_\ell^0 = \Pi_\ell$, the spectral radius of R is strictly samller than $\nu_1 < \nu$, and $P_\ell P_{\ell'} = \delta_{\ell,\ell'}$, $P_\ell N_{\ell'} = N_{\ell'} \Pi_\ell = \delta_{\ell,\ell'} N_\ell$, $N_\ell^{d_\ell} = 0$ and $N_\ell^{d_\ell - 1} \neq 0$, and $P_\ell R = N_\ell R = R \Pi_\ell = R N_\ell = 0$.

Problem 8.2.4 *Show that*

$$\mathcal{L}_g^n = \sum_{\ell=1}^{M} \left[e^{i\theta_\ell}\nu\Pi_\ell + N_\ell\right]^n + R^n$$

Let $d = \sup d_\ell$. For the following, it turns out to be useful to do the following computation, for all $\theta \in [0, 2\pi)$,

$$\frac{1}{n^d}\sum_{k=0}^{n-1} e^{-i\theta k}\nu^{-k}\mathcal{L}_g^k = \frac{1}{n^d}\sum_{k=0}^{n-1}\sum_{\ell=1}^{M}\left[\sum_{j=0}^{d-1} e^{-i\theta k}\binom{k}{j}\nu^{-j}e^{i\theta_\ell(k-j)}\Pi_\ell N_\ell^j\right] + e^{-i\theta k}\nu^{-k}R^k$$
$$= \sum_{\ell=1}^{M} e^{-i\theta_\ell(d-1)}\nu^{-d+1}\frac{1}{n^d}\sum_{k=0}^{n-1} e^{i(\theta_\ell-\theta)k}\frac{k\cdots(k-d+2)}{(d-1)!}N_\ell^{d-1} + \mathcal{O}\left(\frac{1}{n}\right)$$

Problem 8.2.5 *Show that if* $\theta \notin \{\theta_\ell\}$*, then*

$$\lim_{n\to\infty}\frac{1}{n^d}\sum_{k=0}^{n-1} e^{-i\theta k}\nu^{-k}\mathcal{L}_g^k = 0.$$

It remains to consider the case when $\theta = \theta_{\bar\ell}$ for some $\theta_{\bar\ell} \in \{\theta_\ell\}$. Then

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n^d}\sum_{k=0}^{n-1} e^{-i\theta_{\bar\ell}k}\nu^{-k}\mathcal{L}_g^k &= \lim_{n\to\infty} e^{-i\theta_{\bar\ell}(d-1)}\frac{\nu^{-d+1}}{n^d}\sum_{k=0}^{n-1}\frac{k\cdots(k-d+2)}{(d-1)!}N_{\bar\ell}^{d-1}\\ &= e^{-i\theta_{\bar\ell}(d-1)}\frac{\nu^{-d+1}}{(d-1)!}\left[\lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^{d-1}\frac{1}{n}\right]N_{\bar\ell}^{d-1}\\ &= e^{-i\theta_{\bar\ell}(d-1)}\frac{\nu^{-d+1}}{(d-1)!}\left[\int_0^1 x^{d-1}dx\right]N_{\bar\ell}^{d-1}\\ &= e^{-i\theta_{\bar\ell}(d-1)}\frac{\nu^{-d+1}}{d!}N_{\bar\ell}^{d-1}.\end{aligned} \tag{8.2.1}$$

We can now start to harvest the above facts. First of all, if $0 \notin \{\theta_\ell\}$, then

$$\lim_{n\to\infty}\frac{1}{n^d}\sum_{k=0}^{n-1}\nu^{-k}\|\mathcal{L}_g^k 1\|_\infty = 0.$$

But by definition there exists $\bar\ell$ and $\bar\psi \in \mathcal{C}^1$ such that $N_{\bar\ell}\bar\psi = \tilde\psi \neq 0$, hence equation (8.2.1) implies

$$\frac{\nu^{-d+1}}{d!}\|\tilde{\Psi}\|_\infty = \lim_{n\to\infty}\left\|\frac{1}{n^d}\sum_{k=0}^{n-1} e^{-i\theta k}\nu^{-k}\mathcal{L}_g^k\bar{\psi}\right\|_\infty$$
$$\leq \lim_{n\to\infty}\frac{1}{n^d}\sum_{k=0}^{n-1}\nu^{-k}\|\mathcal{L}_g^k 1\|_\infty\|\bar{\psi}\|_\infty = 0$$

yielding a contradiction. Hence, $0 \in \{\theta_\ell\}$. Then let $\psi = N_0^{d-1}1$, it follows that $\mathcal{L}_g\psi = \nu\psi$. Also equation (8.2.1) implies $\psi \geq 0$.

Problem 8.2.6 *Show that for all $x \in \mathbb{T}$ the preimmages $\bigcup_{n=1}^{\infty} f^{-n}(x)$ form a dense set.*

Suppose now there exist $\bar{x}$ such that $\psi(\bar{x}) = 0$, then

$$0 = \sum_{y\in f^{-n}(\bar{x})} e^{g_n(y)}\psi(y)$$

hence ψ is zero on the preimmags of $\bar{x}$ and, by Problem 8.2.6, this implies that $\psi \equiv 0$. Hence, $c = \inf_x \psi(x) > 0$ and, reacalling Problem 8.2.4, for each $n \in \mathbb{N}$,

$$\nu^n\|\psi\|_\infty = \|\mathcal{L}_g^n\psi\|_\infty \geq c\|\mathcal{L}_g^n 1\|_\infty \geq c\|\mathcal{L}_g\|^n \geq C_\#\nu^n n^{d-1}$$

which implies $d = 1$; that is, the peripheral spectrum has no Jordan blocks, and then neither has the rest of the peripheral spectrum.

Next, suppose there is another eigenvector $\bar{\psi} \neq \psi$ such that $\mathcal{L}_g\bar{\psi} = \nu\bar{\psi}$. Note that the real and imaginary parts of $\bar{\psi}$ must be eigenvectors as well. Hence, we can consider $\bar{\psi}$ real without loss of generality. Let $a \in \mathbb{R}$ be such that $\inf_x \psi(x) - a\bar{\psi}(x) = 0$. But then, since also $\psi - a\bar{\psi}$ is an eigenvector of $\mathcal{L}_g$, we can argue as before and conclude that $\psi \equiv a\bar{\psi}$. This implies that ν is a simple eigenvalue of $\mathcal{L}_g$.

At the moment, we have thus established the following decomposition

$$\mathcal{L}_g = \nu\Pi_0 + \sum_{\ell=1}^{M-1} e^{i\theta_\ell}\nu\Pi_\ell + R$$

where Π_0 is a rank one operator, that is, there exists $\ell \in (\mathcal{C}^1)'$ such that $\ell(\psi) = 1$ and $\Pi_0 h = \psi \cdot \ell(h)$, or, alternatively, $\Pi_0 = \psi \otimes \ell$. On the other hand (8.2.1), for $\theta_{\bar{\ell}} = 0$ implies that $\ell(\varphi) \geq 0$ for all $\varphi \geq 0$. In turn, this implies $\ell \in (\mathcal{C}^0)'$, that is ℓ is a measure.

Since $\Pi_0\mathcal{L}_g = \mathcal{L}_g\Pi_0 = \nu\Pi_0$, we have $\ell(\mathcal{L}_g\varphi) = \nu\ell(\varphi)$ for all $\varphi \in \mathcal{C}^0$.

Problem 8.2.7 *Show that the support of ℓ is all $\mathbb{T}$.*

We can now suppose that there exists $\theta \in \{\theta_1, \dots, \theta_M\}$ and h such that

$$\mathcal{L}_g h = e^{i\theta} \nu h.$$

Then

$$\nu|h| \leq \mathcal{L}_g|h|$$

but then $0 \leq \ell(\mathcal{L}_g|h| - \nu|h|) = 0$ implies $\nu|h| = \mathcal{L}_g|h|$ ℓ-almost-surely, which implies $\nu|h| = \mathcal{L}_g|h|$ by Problem 8.2.7. By yhe simplicity of ν the above implies $|h| = \psi$. Accordingly, it must be $h = e^{i\vartheta}\psi$, for some $\vartheta \in \mathcal{C}^1(\mathbb{T}, \mathbb{R})$. We can then write

$$\mathcal{L}_g(e^{i\vartheta - i(\vartheta\circ f + \theta)}\psi) = e^{-i(\vartheta+\theta)}\mathcal{L}_g(e^{i\vartheta}\psi) = \nu\psi = \mathcal{L}_g\psi$$

which, applying ℓ and taking the real part implies

$$\ell\left((1 - \cos(\vartheta - \vartheta \circ f - \theta))\psi\right) = 0$$

which is possible only if $\vartheta - \vartheta \circ f - \theta = 2\pi k$ for some $k \in \mathbb{N}$. Since the map is expanding, it must have at least a fixed point; let us call it p. Then

$$2\pi k = \vartheta(p) - \vartheta \circ f(p) - \theta = -\theta$$

which means $\theta = 0$, contrary to the hypotheis. We have finally obtained the spectral decomposition

$$\mathcal{L}_g = \nu\psi \otimes \ell + R.$$

What does this tell about invariant measures?

Problem 8.2.8 *Prove that $\mu(\varphi) := \ell(\varphi \cdot \psi)$ is a probability measure.*

Here is the punchline

$$\mu(\varphi \circ f) = \ell(\varphi \circ f\psi) = \nu^{-1}\ell(\mathcal{L}_g(\varphi \circ f\psi)) = \nu^{-1}\ell(\varphi\mathcal{L}_g\psi) = \ell(\varphi\psi) = \mu(\varphi)$$

that is we have a invariant measure of full support.