

# APPENDIX B

## Implicit function theorem (a quantitative version)

In this appendix we recall the implicit function Theorem. We provide an explicit proof because we use in the text a quantitative version of the theorem so it is important to keep track of the various constants.

### B.1 The theorem

Let  $n, m \in \mathbb{N}$  and  $F \in \mathcal{C}^1(\mathbb{R}^{m+n}, \mathbb{R}^m)$  and let  $(x_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}^m$  such that  $F(x_0, \lambda_0) = 0$ . For each  $\delta > 0$  let  $V_\delta = \{(x, \lambda) \in \mathbb{R}^{n+m} : \|x - x_0\| \leq \delta, \|\lambda - \lambda_0\| \leq \delta\}$ .

**Theorem B.1.1** *Assume that  $\partial_x F(x_0, \lambda_0)$  is invertible and choose  $\delta > 0$  such that  $\sup_{(x, \lambda) \in V_\delta} \|\mathbb{1} - [\partial_x F(x_0, \lambda_0)]^{-1} \partial_x F(x, \lambda)\| \leq \frac{1}{2}$ . Let  $B_\delta = \sup_{(x, \lambda) \in V_\delta} \|\partial_\lambda F(x, \lambda)\|$  and  $M = \|\partial_x F(x_0, \lambda_0)^{-1}\|$ . Set  $\delta_1 = (2MB_\delta)^{-1}\delta$  and  $\Lambda_{\delta_1} := \{\lambda \in \mathbb{R}^m : \|\lambda - \lambda_0\| < \delta_1\}$ . Then there exists  $g \in \mathcal{C}^1(\Lambda_{\delta_1}, \mathbb{R}^n)$  such that all the solutions of the equation  $F(x, \lambda) = 0$  in the set  $\{(x, \lambda) \in \mathcal{B}_1 \times \mathcal{B}_2 : \|\lambda - \lambda_0\| < \delta_1, \|x - x_0\| < \delta\}$  are given by  $(g(\lambda), \lambda)$ . In addition,*

$$\partial_\lambda g(\lambda) = -(\partial_x F(g(\lambda), \lambda))^{-1} \partial_\lambda F(g(\lambda), \lambda).$$

We will do the proof in several steps.

#### B.1.1 Existence of the solution

Let  $A(x, \lambda) = \partial_x F(x, \lambda)$ ,  $M = \|A(x_0, \lambda_0)^{-1}\|$ .

We want to solve the equation  $F(x, \lambda) = 0$ , various approaches are possible. Here we will use a simplification of Newton method, made possible by the fact that we already know a good approximation of the zero we are looking for. Let  $\lambda$  be such that  $\|\lambda - \lambda_0\| < \delta_1 \leq \delta$ . Consider  $U_\delta = \{x \in \mathbb{R}^n : \|x - x_0\| \leq \delta\}$

and the function  $\Theta_\lambda : U_\delta \rightarrow \mathbb{R}^n$  defined by<sup>1</sup>

$$\Theta_\lambda(x) = x - A(x_0, \lambda_0)^{-1}F(x, \lambda). \quad (\text{B.1.1})$$

**Problem B.1** *Prove that, for  $x \in U(\lambda)$ ,  $F(x, \lambda) = 0$  is equivalent to  $x = \Theta_\lambda(x)$ .*

Next,

$$\|\Theta_\lambda(x_0) - \Theta_{\lambda_0}(x_0)\| \leq M\|F(x_0, \lambda)\| \leq MB_\delta\delta_1.$$

In addition,  $\|\partial_x \Theta_\lambda\| = \|\mathbb{1} - A(x_0, \lambda_0)^{-1}A(x, \lambda)\| \leq \frac{1}{2}$ . Thus,

$$\|\Theta_\lambda(x) - x_0\| \leq \frac{1}{2}\|x - x_0\| + \|\Theta_\lambda(x_0) - x_0\| \leq \frac{1}{2}\|x - x_0\| + MB_\delta\delta_1 \leq \delta.$$

The existence of  $x \in U_\delta$  such that  $\Theta_\lambda(x) = x$  follows then by the standard fixed point Theorem A.1.1. We have so obtained a function  $g : \{\lambda : \|\lambda - \lambda_0\| \leq \delta_1\} = \Lambda_{\delta_1} \rightarrow \mathbb{R}^n$  such that  $F(g(\lambda), \lambda) = 0$ . it remains the question of the regularity.

### B.1.2 Lipschitz continuity and Differentiability

Let  $\lambda, \lambda' \in \Lambda_{\delta_1}$ . By (B.1.1)

$$\|g(\lambda) - g(\lambda')\| \leq \frac{1}{2}\|g(\lambda) - g(\lambda')\| + MB_\delta|\lambda - \lambda'|$$

This yields the Lipschitz continuity of the function  $g$ . To obtain the differentiability we note that, by the differentiability of  $F$  and the above Lipschitz continuity of  $g$ , for  $h \in \mathbb{R}^m$  small enough,

$$\|F(g(\lambda + h), \lambda + h) - F(g(\lambda), \lambda) + \partial_x F[g(\lambda + h) - g(\lambda)] + \partial_\lambda Fh\| = o(\|h\|).$$

Since  $F(g(\lambda + h), \lambda + h) = F(g(\lambda), \lambda) = 0$ , we have that

$$\lim_{h \rightarrow 0} \|h\|^{-1}\|g(\lambda + h) - g(\lambda) + [\partial_x F]^{-1}\partial_\lambda Fh\| = 0$$

which concludes the proof of the Theorem, the continuity of the derivative being obvious by the obtained explicit formula.

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<sup>1</sup>The Newton method would consist in finding a fixed point for the function  $x - A(x, \lambda)^{-1}F(x, \lambda)$ . This gives a much faster convergence and hence is preferable in applications, yet here it would make the estimates a bit more complicated.

## B.2 Generalization

First of all note that the above theorem implies the inverse function theorem. Indeed if  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a function such that  $\partial_x f$  is invertible at some point  $x_0$ , then one can consider the function  $F(x, y) = f(x) - y$ . Applying the implicit function theorem to the equation  $F(x, y) = 0$  it follows that  $y = f(x)$  are the only solution, hence the function is locally invertible.

The above theorem can be generalized in several ways.

**Problem B.2** Show that if  $F$  in Theorem [B.1.1](#) is  $\mathcal{C}^r$ , then also  $g$  is  $\mathcal{C}^r$ .

**Problem B.3** Verify that if  $\mathcal{B}_1, \mathcal{B}_2$  are two Banach spaces and in Theorem [B.1.1](#) we have  $\mathcal{B}_1$  instead of  $\mathbb{R}^n$  and  $\mathcal{B}_2$  instead of  $\mathbb{R}^m$  the Theorem remains true and the proof remains exactly the same.

As I mentioned the statement of Theorem [B.1.1](#) is suitable for quantitative applications.

**Problem B.4** Suppose that in Theorem [B.1.1](#) we have  $F \in \mathcal{C}^2$ , then show that we can chose

$$\delta = [2\|D\partial_x F\|_\infty]^{-1}.$$