Chernov Memorial Lectures

Hyperbolic Billiards, a personal outlook.

Lecture One

The Lorentz gas: where we stand and where I'd like to go

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The Lorentz gas

The Lorentz gas was proposed by H. A. Lorentz in 1905 to model thermal and electrical conductivity in metals. It consists of

- a periodic array of fixed obstacles with finite free path
- a gas of non–interacting particles (electrons)
- particles colliding elastically with the obstacles

Since the particles are non interacting it suffices to understand the beahviour of just one particle.

The Lorentz gas–example

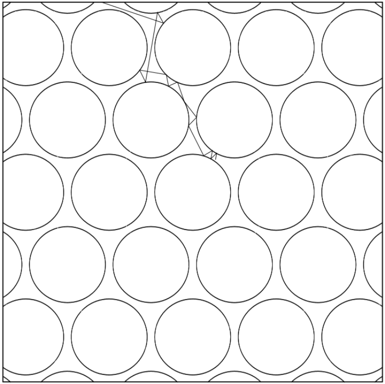

Figure: A finite horizon Lorentz gas on a triangular array. Picture taken from *Diffusion in the Lorentz gas*, Dettmann, 2014

Lorentz gas–another picture

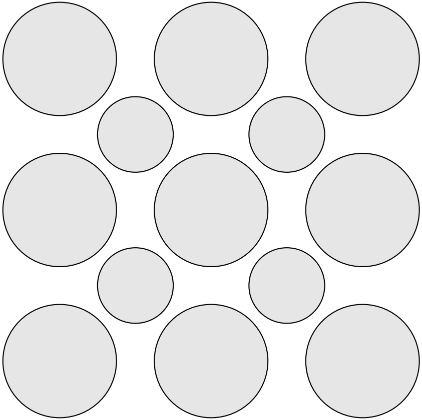

Figure: A periodic obstacle configuration in a square array for the finite horizon Lorentz gas

If the square cell has size ℓ and $q \in \mathbb{R}^2$ is the position of a particle, then we can write it as $q = x + K\ell$, where $x \in \mathbb{T}^2 = \mathbb{R}^2/\ell\mathbb{Z}^2$ and $K \in \mathbb{Z}^2$. In other word we can consider only the motion in a cell with periodic boundary conditions and recover the position on $\mathbb{R}^2$ by considering a $\mathbb{Z}^2$ cocycle over the single cell.

Compactification

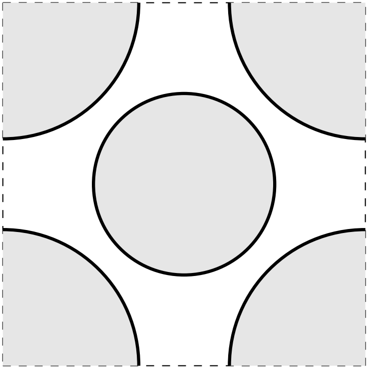

Figure: One cell of the Lorentz gas. That is: a periodic Billiard

Billiards

In his 1970 paper, *Dynamical systems with elastic reflections*, Sinai founded the rigorous study of the statistical properties of Billiards systems. In particular he established the ergodicity of what is today called a two dimensional dispersing Sinai billiard.

Ergodicity

Since then there has been a tremendous (ongoing) effort to extend Sinai's ergodicity results to a more general setting, starting with the Sinai, Chernov (1987) paper, which first considered a gas of interacting balls (in two and three dimensions).
To date the most notable success is the proof of ergodicity of n hard spheres on a torus colliding elastically. This result has a very long history (to which Chernov greatly contributed) but the proof was finally completed by Simanyi in 2013.

Mixing

Already in the '50 Krylov, in his amazing book on the foundations of statistical mechanics, pointed out that ergodicity does not suffices for many important physical applications. In particular, if one wants to explain how a system reaches equilibrium, then some quantitative form of mixing is necessary.

Mixing

Given a dynamical system (X, ϕ_t) and a reference measure m, we say that the system (X, ϕ_t, m), X compact, ϕ_t measurable, is mixing if there exists a probability measure μ such that for each $h \in L^1(X, m)$ and $\varphi \in \mathcal{C}^0(X, m)$ we have

$$\lim_{t\to\infty} m(h\varphi \circ \phi_t) = m(h)\mu(\varphi).$$

Quantitative Mixing

By quantitative mixing I mean that there exists a Banach space $\mathcal{B}$, such that $\mathcal{C}^\infty(X,\mathbb{R})' \supset \mathcal{B}, \mathcal{B}' \supset \mathcal{C}^\infty(X,\mathbb{R})$, $m \in \mathcal{B}$, and a known function $\alpha : \mathbb{R} \to \mathbb{R}$ such that, for all $h, \varphi \in \mathcal{C}^\infty(X,\mathbb{R})$, we have

$$|h(\varphi \circ \phi_t) - h(1)\mu(\varphi)| \le \alpha(t)\|h\|_{\mathcal{B}}\|\varphi\|_{\mathcal{B}'}$$

where we use the notation $h(\varphi) = m(h\varphi) = \varphi(hm)$.
If α is exponential we have exponential mixing.
If $1/\alpha$ is polynomial, then we have polynomial mixing
and so on.

Mixing (Poincarè map)

Sinai was aware of the need to study mixing and pioneered the attempts to get quantitative mixing results in his 1981 paper together with Bunimovich. In such a paper it is proven the sub-exponential mixing of the Poincarè map of a two dimensional dispersing billiard with finite horizon.

Poincarè map

The billiard Poincarè map consists in looking at the system only at collisions (e.g., just after a collision). Hence the phase space is given by the boundary in configuration space (at which collisions take place) and the directions of the velocity, to which the dynamics (Poincarè map) associates the position and direction of the velocity just after the next collision.

Central Limit Theorem

In the same 1981 paper Bunimovich and Sinai proved a central limit theorem for the periodic Lorentz gas.
Such results were clarified and substantially improved in Bunimovich, Sinai and Chernov (1991).

Central Limit Theorem

In particular, if $q(0)$ belongs to the zero cell and has a smooth initial distribution, then for each $\varphi \in \mathcal{C}_0^0(\mathbb{R}^2, \mathbb{R})$ and $t > 0$,

$$\lim_{L \to \infty} \mathbb{E}(\varphi(L^{-1} q(L^2 t))) = \int_{\mathbb{R}^2} \varphi(y) \rho(y, t) dy$$

where the particle density ρ satisfies the heat equation:

$$\partial_t \rho(y, t) = \sigma \Delta_y \rho(y, t)$$
$$\rho(y, 0) = \delta(y).$$

for some $\sigma \neq 0$ given by a Green-Kubo formula.

Limit Laws

The Bunimovich-Sinai-Chernov paper has been the template for a huge research field aiming at generalising such type of results:
Lai-Sang Young (1998) exponential mixing for the billiard map.
Szász and Varjú (2007) CLT for the case with infinite horizon (which was conjectured by Bleher in 1992).
A manifold of finer results for the finite horizon case have been established starting with Pène (2002) and ending with Dolgopyat, Szász and Varjú (2008, 2009) in which it is treated also the locally perturbed (hence non periodic) case.

Random Lorentz gas

The non periodic case is physically very relevant since real materials inevitably do have defects. Yet, exactly for the same reason, there is no reason why the defects should be localised. Unfortunately, very little is known for non local perturbations, i.e. perturbations described by a translation invariant probability measure. Such models are called random Lorentz gases.

Random Lorentz gas

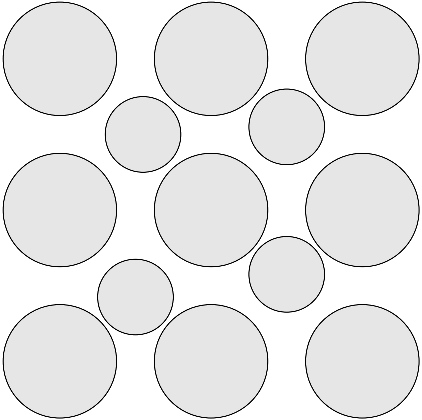

Figure: An obstacle configuration for the random Lorentz gas

Random Lorentz gas

Naively one could think that the extra randomness helps, but a little though shows that in fact it make things much more complex: all the probabilistic difficulties connected to random walk in random environment are present, but compounded with the peculiar difficulties coming from the fact that we are dealing with a deterministic walk in random environment.

Hopes

One possible approach to this problem is to try to separate the dynamical difficulties from the probabilistic ones.
The hope being that one can reduce deterministic walks in random environment to a pure probabilistic model:
a random walk in random environment with short memory.

Short memory

The environments are given by a translation invariant probability distribution $\mathbb{P}$ on $\Omega = \mathcal{A}^{\mathbb{Z}^d}$, $\mathcal{A}$ a finite set.
For each $\omega \in \Omega$ let $\mathbb{P}_\omega$ be a measure on the space of paths $\mathbb{N}^{\mathbb{Z}^d}$, $\mathbb{P}_\omega(z(0) = 0) = 1$.
By short memory I mean: for each ω and path $z(0), \dots, z(n)$, setting $\xi(k) = z(k+1) - z(k)$,

$$\left|\mathbb{P}_\omega(\xi(n) \mid z(0), \dots, z(n)) - \mathbb{P}_\omega(\xi(n) \mid z(n-m), \dots z(n))\right| \le C\nu^m.$$

Such a reduction holds in simplified models (Aimino, L. w.i.p.) but for the Lorentz gas it is an open problem.

Energy transport

All the above results (a part from ergodicity) pertain independent particles. Hence the mass transport is the only relevant quantity to study.
If however the particles can interact among themselves (i.e. they are disks which can collide), then they can exchange energy, and hence the study of the energy transport becomes of paramount importance.
However, since the mass transport and energy transport influence each other, this is an extremely hard problem to study.

Interacting particels

The first real attempt to study quantitatively the statistical properties of interacting particles is contained in the monumental *Brownian Brownian motion. I* by Chernov and Dolgopyat (2009) where is considered a heavy particle interacting with a light one.
The end game is to understand the real Brownian motion where the erratic behaviour of a heavy particle is due to the interaction with **many** light particles.
As remarkable as the paper is, it also shows the limits of present days techniques.

Geometrically constrained models

To simplify the problem one can consider geometrically constrained models. That is billiards tables in which the particles are geometrically confined in certain regions and yet can interact with each others. This does not allow mass transport, hence separating the convection from the conduction and substantially simplifying the problem.

Such models were first introduced in 1992 by Bunimovich, Liverani, Pellegrinotti and Suhov, where the ergodicity was proven, and later proposed by Gaspard and Gilbert (2008) a models to study heat conduction.

Geometrically constrained models

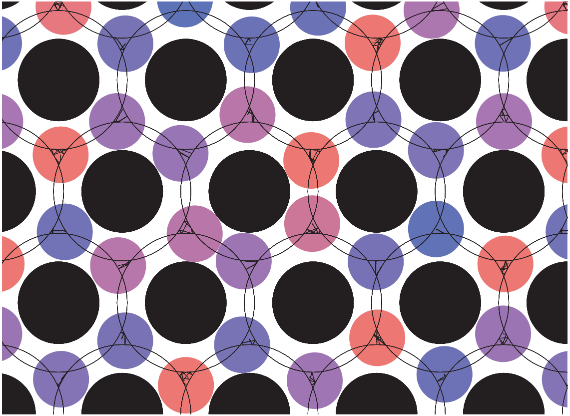

Figure: Obstacles in black, particles in colors, from
P. Gaspard and T. Gilbert Heat conduction and Fouriers law in a class of many particle dispersing billiards New J. Phys. 10 No 10 (2008).

Geometrically constrained models

After the Gaspard-Gilbert paper a substantial amount of work has been dedicated to trying to rigorously derive the heat equation for such models. Yet, it has clashed against substantial technical difficulties that are currently object of active research.
More satisfactory results have been obtained for the (different, but related) case of weakly interacting geodesic flows in negative curvature Dolgopyat, Liverani (2011).

Flow mixing

One of the first obstacles in the above line of research is to effectively estimate the probability of two balls colliding. To do so the decay of correlation of the Poincarè map does not suffice, one needs the decay of correlation of the billiard flow. This seems a small difference but it is instead a very hard technical and conceptual problem.

Flow mixing

For the longest time no ideas were available on how to study decay of correlation for flows. The situation totally changed with the Chernov's 1998 paper, shortly followed by Dolgopyat substantial improvement. This has started the study of billiard flows: Melbourne (2007) proved rapid mixing and Chernov (2007) sub-exponential decay of correlations for the finite horizon flow. Finally, Baladi, Demers and Liverani (2017) established the exponential decay of correlations in the same setting.

Future

Even though considerable progress has been made in the study of billiards, the field is still full of open problems.
I discussed just two, but I believe they suffice to show that our achievements are dwarf by what it remains to do.
Also, it is a fact that the size of the articles dealing with such problems is growing significantly in time, I believe this shows that some new ideas are needed to advance further,
we badly need a new Chernov.