Chernov Memorial Lectures

Hyperbolic Billiards, a personal outlook.

Lecture Two

The functional analytic approach to Billiards

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Geometrically constrained models

Recall the geometrically constrained models from the first lecture.

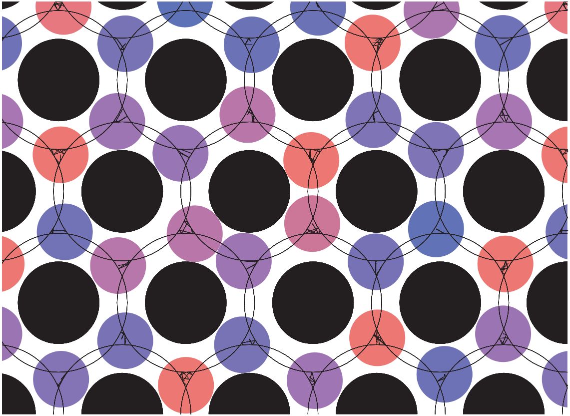

Figure: Obstacles in black, particles in colors, from
P. Gaspard and T. Gilbert Heat conduction and Fouriers law in a class of many particle dispersing billiards New J. Phys. 10 No 10 (2008).

Anosov Flow mixing

As illustrated by Szász, to investigate rarely interacting particles, the first thing that seems to be needed is a good quantitative understanding of the statistical properties of the Billiard flow.
Certainly we need decay of correlations, but also various refined limit theorems.
The development of the tools needed to tackle such problems has been rather slow.

Anosov Flow mixing

For the longest time no ideas were available on how to study decay of correlation for flows. The situation changed with the Chernov's 1998 paper, shortly followed by Dolgopyat revolutionary approach. This has started the study of Anosov flows.
The work of Chernov and Dolgopyat was based on the study of a Ruelle type transfer operator, but still required coding the system. A tool that has proven rather messy to adapt to Billiards.

Anosov Flow mixing

The situation has considerable improved after Lai-Sang Young tower construction (1998) that was refined in the billiard context by Chernov in a series of papers. Yet, in spite of several attempts, this strategy has failed to yield optimal results in the case of flows.

Anosov Flow mixing

An approach that bypassed completely the need of coding appeared in Dolgopyat (2005): standard pairs.
Again it was adapted to billiard by Chernov. Yet, while proving very effective to study the statistical properties of the Poincarè map, up to now the attempts to use it for flows have failed (but see Butterley-Eslami (2017)).
Finally, another approach independent of coding (now called the functional approach) was put forward in the work of Blank-Keller-L. (2002) and was successfully adapted to Anosov flows by L. (2004) and further developed in Giulietti-L.-Pollicott (2013).

Flow mixing

More recently, starting with the work of Baladi (2005), Baladi-Tsujii (2008) and Baladi-Gouezel (2009-2010) till the recent work of Tsujii-Faure (2012-2017) and Dyatlov-Zworski (2016-2017) has appeared the possibility to use semicalssical analysis to investigate directly the transfer operator associated to the time one map of an Anosov flow. Whereby obtaining much sharper spectral informations and opening the possibility to apply functional techniques not only to flows but more generally to partially hyperbolic systems.

Billiard Flow mixing

The problem with this more sophisticated approaches lays in the difficulty to adapt it to flows with low regularity (Billiards are well known to enjoy nasty discontinuities).
As a reaction there have been attempts to revert to older coding strategies: Melbourne (2007) proved rapid mixing and Chernov (2007) sub-exponential decay of correlations for the finite horizon billiard flow but further results have been lacking.

Billiard Flow mixing

In 2008 Demers-L. showed that the functional approach can be applied to maps with discontinuities. Then Demers-Zhang (2011-2013-2014) showed that such an approach can be applied to Billiards and other singular systems. At the same time Baladi-L. (2012) showed how also flows with discontinuities can be treated. Finally, Baladi, Demers and Liverani (preprint 2015) established the exponential decay of correlations by integrating all the above ideas.

The Banach Space

Let $\mathcal{W}^s$ to be the set of homogeneous connected lagrangian curves with tangent in the stable cone and uniformly bounded curvature. It is possible to define a "distance" $d_{\mathcal{W}^s}$ among elements of $\mathcal{W}^s$ that corresponds, roughly, to a distance in the "unstable" direction.

The weak norm

Fix $0 < \alpha \leq 1/3$

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\alpha(W) \\ |\psi|_{\mathcal{C}^\alpha(W)} \leq 1}} \int_W f \psi \, dm_W \,,$$

where dm_W denotes arclength along W. We define then the weak Banach space as the closure of $\mathcal{C}^1$ with respect to $|\cdot|_w$.

We define the neutral norm of f by

$$\begin{aligned}\|f\|_0 &= \sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in\mathcal{C}^\alpha(W)\\ |\psi|_{\mathcal{C}^\alpha(W)}\le 1}} \int_W \partial_t(f\circ\Phi_t)|_{t=0}\,\psi\, dm_W\\ &= |X^* f|_w.\end{aligned}$$

The stable and unstable norms

Now choose $1 < q < \infty$ and $0 < \beta < \min\{\alpha, 1/q\}$.

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\beta(W) \\ |W|^{1/q} |\psi|_{\mathcal{C}^\beta(W)} \leq 1}} \int_W f \psi \, dm_W \, .$$

Choose $0 < \gamma \leq \min\{1/q, \alpha - \beta\} < 1/3$.

$$\|f\|_u = \sup_{\varepsilon > 0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1, W_2) \leq \varepsilon}} \sup_{\substack{\psi_i \in \mathcal{C}^\alpha(W_i) \\ |\psi_i|_{\mathcal{C}^\alpha(W_i)} \leq 1 \\ d(\psi_1, \psi_2) = 0}} \varepsilon^{-\gamma} \left| \int_{W_1} f \psi_1 \, dm_{W_1} - \int_{W_2} f \psi_2 \, dm_{W_2} \right| .$$

The strong norm

We define the strong norm of f by

$$\|f\|_{\mathcal{B}} = \|f\|_s + c_u\|f\|_u + \|f\|_0 \, .$$

The Banach space $\mathcal{B}$ is the closure of $\mathcal{C}^2_*$ in the norm $\|f\|_{\mathcal{B}}$.

The Transfer operator

The transfer operator is given by $\mathcal{L}_t f = f \circ \Phi_{-t}$. Note that

$$\int f \cdot g \circ \Phi_t = \int g \mathcal{L}_t f.$$

Thus the behaviour of $\mathcal{L}_t$ for large t determines the correlations.

Properties

- for all $t \geq 0$, $\mathcal{L}_t$ extends continuously to $\mathcal{B}_w$ and $\mathcal{B}$.
- for all $t \geq 0$, $\mathcal{L}_t : \mathcal{B} \to \mathcal{B}_w$ is a compact operator.
- $\mathcal{L}_t$ forms a strongly continuous semigroup on $\mathcal{B}$. Let X be its generator.
- $\sup_{t\geq 0} \|\mathcal{L}_t\|_{\mathcal{B}} \leq C$, hence $\sigma(X) \subset \{z \in \mathbb{C} : \Re(z) \leq 0\}$.
- $0 \in \sigma(X)$ with eigenvector 1 (Lebesgue).

Resolvent

It is then convenient to study the resolvent

$$\mathcal{R}(z) = (z\,Id - X)^{-1}.$$

By the above properties for each $\Re(z) > 0$ we can write

$$\mathcal{R}(z)f = \int_0^\infty e^{-zt}\mathcal{L}_t f\,dt$$

Lasota-Yorke for the Resolvent

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) = a > 0$, all $f \in \mathcal{B}$ and all $n \geq 0$,

$$\begin{aligned}
|\mathcal{R}(z)^n f|_w &\leq Ca^{-n}|f|_w \\
\|\mathcal{R}(z)^n f\|_s &\leq C(a - \ln\tilde{\lambda})^{-n}\|f\|_s + Ca^{-n}|f|_w \\
\|\mathcal{R}(z)^n f\|_u &\leq C(a - \ln\tilde{\lambda})^{-n}\|f\|_u + Ca^{-n}\|f\|_s + Ca^{-n}\|f\|_0 \\
\|\mathcal{R}(z)^n f\|_0 &\leq Ca^{2-n}(1 + a^{-1}|z|)|f|_w \,.
\end{aligned}$$

Which implies that, for each $a > 0$, exists $\nu_a < 1$ such that

$$a^n\|\mathcal{R}(z)^n f\|_{\mathcal{B}} \leq \nu_a^n\|f\|_{\mathcal{B}} + C(1 + |z|a + a^2)|f|_w$$

Quasi compactness of the Resolvent

The Lasota-Yorke inequalities and Hennion theorem implies that there exists $\tilde{\lambda} \in (0,1)$ such that, for all $z \in \mathbb{C}$, $\Re(z) = a$, the essential spectral radius of $\mathcal{R}(z)$ on $\mathcal{B}$ is bounded by $(a - \ln \tilde{\lambda})^{-1}$, while the spectral radius is bounded by a^{-1}.

The spectrum of X

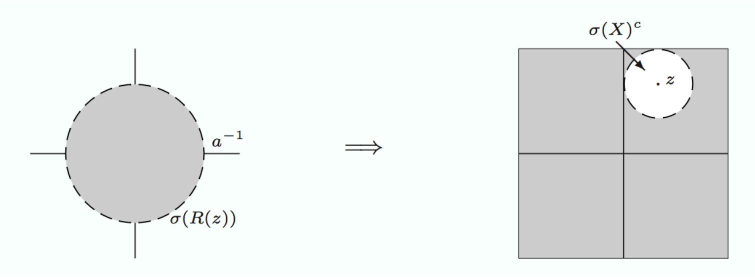

Figure: Conformal mapping of $\sigma(R(z))$ to $\sigma(X)$.

The spectrum of X

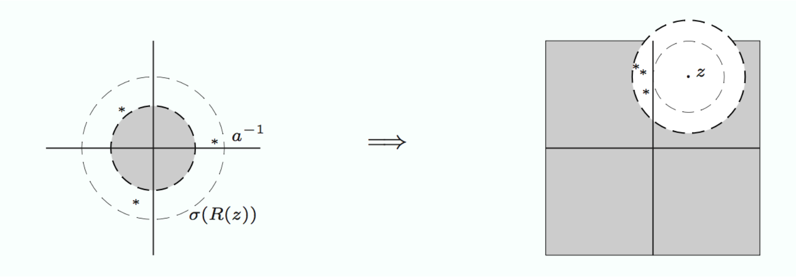

Figure: Conformal mapping of $\sigma_{\rm ess}(R(z))$ to $\sigma(X)$.

The spectrum of X

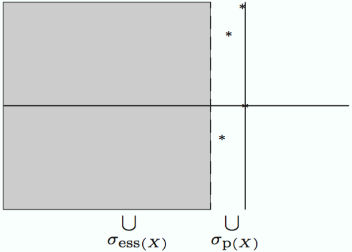

Figure: $\sigma(X)$.

Spectral gap

To establish a spectral gap we need a Dolgopyat estimate, that is to show that there are cancellations so, for $z = a + ib$, $\|R(z)\| < a^{-1}$ provided b is large enough.
Such estimate have first been obtained for smooth flows when the unstable foliation is $\mathcal{C}^1$ (Dolgopyat 1998) and then when the flow is contact (L. 2004). The latter result follows by combining Dolgopyat strategy with a quantitive version of an argument due to Katok-Burns (1994) to prove the join non integrability of the stable and unstable foliations for contact flows.

Spectral gap

Such an argument has been extend to contact flows with (mild) discontinuities (Baladi-L. 2012).
However to implement the Dolgopyat estimate some precise control on the invariant foliations is needed.
In essence, one wants to show that it is possible to discard a small portion of the phase space and in the rest the unstable foliation can be morally thought as the foliation of a smooth Anosov system.

The foliation

Foliation regularity (Baladi-Demers-L. (2017))

For each small enough $\eta > 0$, there exists a set of Lebesgue measure at most $C\eta^{\frac{4}{5}}$, the complement of which is foliated by leaves of the stable (or unstable) foliation with length at least η such that the foliation is Whitney–Lipschitz, with Lipschitz constant not larger than $C\eta^{-\frac{4}{5}}$.

Spectral gap

For any $0 < \gamma < \alpha \leq 1/3$, if $\beta > 0$ and $1/q < 1$ are such that $\max\{\beta, 1 - 1/q\}$ is small enough, then there exist $1 \leq a_0 \leq a_1 \leq b_0$, $0 < c_1 < c_2$, and $\nu > 0$, so that

$$\|\mathcal{R}(a + ib)^n\|_{\mathcal{B}} \leq \left(\frac{1}{a + \nu}\right)^n$$

$c_2 \leq (\ln(1 + a^{-1}\nu))^{-1}$ and for all $|b| \geq b_0$, $a \in [a_0, a_1]$, and $n \in [c_1 \ln |b|, c_2 \ln |b|]$.

The spectrum of X

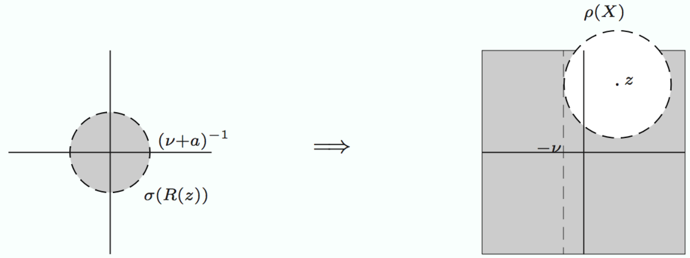

Figure: Spectral gap

Given the spectral gap the exponential decay of correlations (and much more) follows.

Decay of correlations

More precisely one can prove that there exists $C, \gamma > 0$ such that

$$\left| \mathcal{L}_t f - \int f \right|_{\mathcal{B}_w} \leq C e^{-\gamma t} \| X f \|_{\mathcal{B}}.$$

This implies rather strong consequences on the correlation decay.

For example, for each smooth f, g such that $\int g = 0$, we have

$$\left|\int g \circ \phi_t \cdot f\right| = \left|\int g \mathcal{L}_t f\right| \leq C e^{-\gamma t} |g|_{\mathcal{B}'_w} \|Xf\|_{\mathcal{B}}$$

The above can be adapted to various situations. For example, if $g \in \mathcal{C}^\alpha$, then $|g|_{\mathcal{B}'_w} \leq C\|g\|_{\mathcal{C}^\alpha}$ and for any mollifier j_ε

$$\left|\int g \circ \phi_t j_\varepsilon(t) dt - g\right| \leq C\varepsilon^\alpha \|g\|_{\mathcal{C}^\alpha}.$$

Decay of correlations

Thus, setting $f_\varepsilon = \int f \circ \phi_t j_\varepsilon(t)dt$,

$$\left|\int g \circ \phi_t \cdot f\right| \leq C\varepsilon^\alpha \|g\|_{\mathcal{C}^\alpha} \|f\|_{TV} + \left|\int g\mathcal{L}_t f_\varepsilon\right|$$

$$\leq C\varepsilon^\alpha \|g\|_{\mathcal{C}^\alpha} \|f\|_{TV} + Ce^{-\gamma t} \|g\|_{\mathcal{C}^\alpha} \left[\varepsilon^{-1}\|f\|_u + \varepsilon^{-2}\|f\|_s\right]$$

So, choosing $\varepsilon = e^{-\frac{\gamma}{2+\alpha}t}$, we have

$$\left|\int g \circ \phi_t \cdot f\right| \leq +Ce^{-\frac{\alpha\gamma}{2+\alpha}t} \|g\|_{\mathcal{C}^\alpha} \left[\|f\|_{TV} + \|f\|_u + \|f\|_s\right]$$

Decay of correlations

Accordingly, if we have a probability measure μ that can be approximated by $d\nu_\varepsilon = f_\varepsilon d\mathrm{Leb}$ so that $\|\mu - \nu_\varepsilon\|_{TV} \leq \varepsilon$ and $\|f_\varepsilon\|_s + \|f_\varepsilon\|_u \leq \varepsilon^{-\beta} C_\mu$, then there exist $\gamma_{\alpha,\beta} \geq 0$ such that

$$|\mu(g \circ \phi_t)| \leq e^{-\gamma_{\alpha,\beta} t} \|g\|_{\mathcal{C}^\alpha} C_\mu$$

This is just an example, but shows that one can obtain exponential decay of correlation also for rather singular initial conditions.

Let g, f_1, f_2 be smooth functions of zero average and $t, s > 0$, then

$$\left|\int g \circ \phi_{t+s} \cdot f_1 \circ \phi_s \cdot f_2\right| = \left|\int g \cdot \mathcal{L}_t \left(f_1 \cdot \mathcal{L}_{t+s} f_2\right)\right|$$
$$\leq Ce^{-\gamma t} \|g\|_{\mathcal{B}'_w} \|X \left(f_1 \cdot \mathcal{L}_{t+s} f_2\right)\|_{\mathcal{B}} \leq C_{g,f_1,f_2} e^{-\gamma t}$$

But also

$$\left|\int g \circ \phi_{t+s} \cdot f_1 \circ \phi_s \cdot f_2\right| = \left|\int g \circ \phi_t f_1 \cdot \mathcal{L}_s f_2\right|$$
$$\leq Ce^{-\gamma s} \|g \circ \phi_t f_1\|_{\mathcal{B}'_w} \|X f_2\|_{\mathcal{B}} \leq C_{g,f_1,f_2} e^{-\gamma s}.$$

That is: it is possible to control multiple correlations.