Coupled Markov chains and coupled map lattices

Carlangelo Liverani

Department of Mathematics, University of Rome II

The problem

In the study of extended system one is often confronted with the study of many simple systems weakly coupled.
For example:

- Coupled Markov Chains
- Coupled Map Lattices
- Random Walks in Random Environments
  weakly dependent on environment

How to handle such situations?

Coupled Markov Chains

Consider a probability space $\Omega := \Omega_0^{\mathbb{Z}^d}$ and a Markov process with the transition operator

$$\mathcal{K}'\varphi(\omega) := \int \varphi(y) \prod_{p\in\mathbb{Z}^d} K_p(\omega, y_p)dy$$

Weak coupling: Assume that for each $q \neq p$,

$$\int |K_p(\omega, y) - K_p(\omega^q, y)|dy \leq \delta_{q-p}.$$

where the norm refers to the total variation of measures and $\omega_p = \omega_p^q$ for each $p \neq q$.

Single site:

$$\int |K_p(\omega_p, y) - K_p(\omega_p^p, y)|dy \leq 2\delta_0$$

Which implies that, fixed $\omega_{\neq p}$, there exists $\nu_{p,\omega_{\neq p}}$ such that

$$|K^n_{p,\omega_{\neq p}}\varphi - \nu_{p,\omega_{\neq p}}(\varphi)|_\infty \leq \delta_0^n|\varphi|_\infty.$$

Thus, we have a spectral gap. In the uncoupled case this yields a spectral gap for $\mathcal{K}'$ when seen as a tensor product.

Why not do perturbation theory?

A difficulty

Consider $K_p^0(\omega, y) := K_p(\overline{\omega}^p, y)$, where $\overline{\omega}_q^p = a$ for each $q \neq p$ and $\overline{\omega}_p^p = \omega_p$. Then

$$\int |K_p^0(\omega, y) - K_p(\omega, y)| dy \leq \sum_{q \neq p} \delta_{p-q}.$$

this can be assumed small, but if we do a tensor product on a region Λ, then the difference between the uncoupled and the coupled operators is of order $|\Lambda|$.
Maybe something can be done for $d = 1$, but $d > 1$ looks bad.

Just another stupid idea?

A little twist

The uncoupled case revisited:
Let $\mu \in \mathcal{M}^0(\Omega) := \{\mu \ : \ \mu(\Omega) = 0\}$.

- Decompose $\mu = \sum_{q \in \mathbb{Z}^d} \mu_q$ where $\mu_q \in \mathcal{M}_q(\Omega) := \{\mu \ : \ \mu(\varphi) = 0, \ \varphi$ independent on $\omega_q\}$.
- Than $\mathcal{K}\mathcal{M}_p \subset \mathcal{M}_p$ and, for each, $\mu \in \mathcal{M}_p$

$$|\mathcal{K}\mu| \leq \delta_0 |\mu|$$

- Represent, $\mu \in \mathcal{M}^0$ as the vector $\bar{\mu} := (\mu_p)$ with norm $|\bar{\mu}| := \sup_p |\mu_p|$. Call $\overline{\mathcal{M}}$ the resulting Banach space.
- Defines the covering dynamics $\bar{\mathcal{K}}\bar{\mu} := (\mathcal{K}\mu_p)$, then

$$|\bar{\mathcal{K}}\bar{\mu}| \leq \delta_0 |\bar{\mu}|$$

Covering Dynamics

Pictorially we have the following commuting diagram

$$\begin{tikzcd} \overline{\mathcal{M}} \arrow[r, "\overline{\mathcal{K}}^n"] & \overline{\mathcal{M}} \arrow[d, "P"] \\ \mathcal{M}^0 \arrow[u, "\Psi"] \arrow[r, "\mathcal{K}^n"'] & \mathcal{M}^0 \end{tikzcd}$$

for all $n \geq 1.$

where $\Psi\mu := \bar{\mu}$ and $P\bar{\mu} := \sum_p \mu_p$

What have we achieved ?

- We have a spectral gap (although in a strange space) without using tensor spaces but using a local and robust property
- In the coupled case $\mathcal{K}\mathcal{M}_p \not\subset \mathcal{M}_p$. Yet, define

$$\mathcal{K}'_p\varphi(\omega) := \int \varphi(y) \prod_{q\neq p} K_q(\omega^p, y_q) \cdot K_p(\omega, y_p) dy,$$

where $\omega^p_q = \omega_q$ for each $q \neq p$ and $\omega^p_p = a$. Then $\mathcal{K}_p\mathcal{M}_p \subset \mathcal{M}_p$, for each $\mu \in \mathcal{M}_p$ one has $|\mathcal{K}_p\mu| \leq \delta_0|\mu|$, and $|\mathcal{K} - \mathcal{K}_p| \leq \sum_{q\neq 0} \delta_q$.

Maybe perturbation theory is possible after all

Perturbation Theory

For each local function φ and $p \in \mathbb{Z}^d$, it is possible to write

$$\mathcal{K}'\varphi := \mathcal{K}'_p\varphi + \sum_{q\in\mathbb{Z}_d\setminus\{p\}} \mathcal{K}'_{p,q}\varphi,$$

where $\mathcal{K}_{p,q}\mathcal{M}^0 \subset \mathcal{M}_q$, $|\mathcal{K}_{p,q}| \leq \delta_{p-q}$.
We can then define a covering dynamics

$$(\overline{\mathcal{K}}\bar{\mu})_p := \mathcal{K}_p\mu_p + \sum_{q\neq p} \mathcal{K}_{q,p}\mu_q$$

Moreover, our hypotheses imply $|\overline{\mathcal{K}}| \leq \sum_q \delta_q$, thus
if $\sum_q \delta_q < 1$ the spectral gap follows. (mixing)

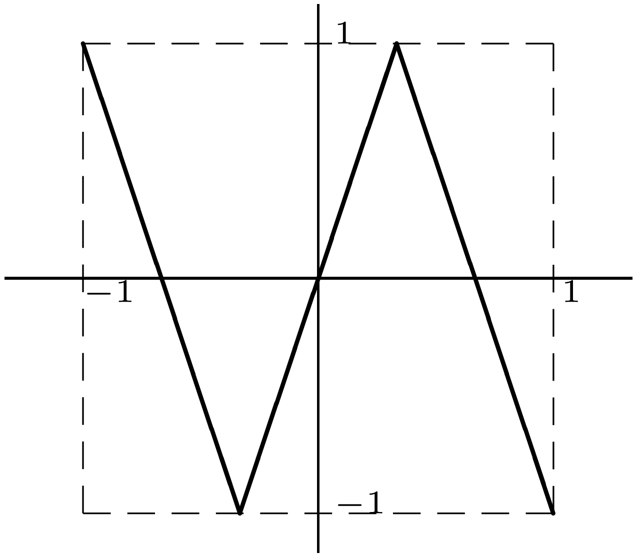
Theorem 0.1 *Any Markov chain defined as above has a unique invariant measure, provided*

$$\sum_{q \in \mathbb{Z}^d} \delta_q < 1.$$

In addition, such a measure is exponentially mixing in time. Moreover, form the space translation invariance of the dynamic, follows the spatial mixing as well.

The above result is essentially the Dobrushin criterion (see the work of C.Maes). The alternative proof just outlined is joint work with G.Keller.

CML: Miller-Huse's experiment (1993)



Let $\tau : [-1,1] \to [-1,1]$, $|\tau'| > 1$.
Set $\Omega := I^{\mathbb{Z}^2}$, define $T_0 : \Omega \to \Omega$ by
$T_0(x)_{\boldsymbol{i}} := \tau(x_{\boldsymbol{i}})$ for each $\boldsymbol{i} \in \mathbb{Z}^2$.
$\forall \epsilon \in (0,1)$ define $\Phi_\epsilon : \Omega \to \Omega$ by
$\Phi_\epsilon(x)_{\boldsymbol{i}} := x_{\boldsymbol{i}} + \frac{\epsilon}{4}\sum_{|\boldsymbol{j}-\boldsymbol{i}|=1}(x_{\boldsymbol{j}} - x_{j\boldsymbol{i}}$
Finally the coupled dynamics
$T_\epsilon := \Phi_\epsilon \circ T_0$

The single site map τ

Compute the average $\langle x_{\boldsymbol{i}} x_{\boldsymbol{k}} \rangle$ with respect to a steady state, for $\epsilon \lesssim .82$ holds $\lim_{|\boldsymbol{i}-\boldsymbol{k}|\to\infty} \langle x_{\boldsymbol{i}} x_{\boldsymbol{k}} \rangle = 0$, while for $\epsilon \gtrsim .82$ holds $\lim_{|\boldsymbol{i}-\boldsymbol{k}|\to\infty} \langle x_{\boldsymbol{i}} x_{\boldsymbol{k}} \rangle > 0$: *a Phase Transition*.

It would be nice to have a proof.

Single phase–the easy part

A lot of work has been done to investigate the single phase regime of CML, starting with
L.A. Bunimovich and Ya.G. Sinai (1988)
Typical assumptions:

- τ smooth
- Φ_ε diffeomorphic
- ϵ small

Can we apply the strategy used for the Markov Chains in order to prove the existence of only one phase for weak coupling?

A difficulty

The operator $\mathcal{K}'$ is now simply the composition $\mathcal{K}'\varphi := \varphi \circ T_\epsilon$ and $\mathcal{K}\mu(\varphi) := \mu(\mathcal{K}'\varphi)$.

Similarly to before we can define the operator $\mathcal{K}_p'$, for which at the site p the map τ acts independently on the rest.

Unfortunately, in any reasonable norm,

$$|\mathcal{K} - \mathcal{K}_p| = 2$$

Not all is lost

The work on one dimensional maps tells us what to do:

- Define the norms

$$|\mu| := \sup_{|\varphi|_{\mathcal{C}^0(\Omega)} \leq 1} \mu(\varphi) \qquad (L^1 \text{ like})$$

$$\operatorname{Var} \mu := \sup_{p} \sup_{|\varphi|_{\mathcal{C}^0(\Omega)} \leq 1} \mu(\partial_p \varphi) \quad (\text{BV like})$$

- Then (Lasota-Yorke type inequalities)

$$|\mathcal{K}\mu| \leq |\mu|$$

$$\operatorname{Var}(\mathcal{K}^n \mu) \leq A\lambda^{-n} \operatorname{Var} \mu + B|\mu|$$

- Moreover

$$|\mathcal{K}\mu - \mathcal{K}_p\mu| \le \epsilon \operatorname{Var}\mu.$$

- This means that

$$\operatorname{Var}(\mathcal{K}^n(\mathcal{K} - \mathcal{K}_p)\mu) \le \max\{A\lambda^{-n}, B\epsilon\} \operatorname{Var}\mu$$

The above suffices to mimic what we have done for the Markov Chains yielding

Theorem 0.2 (Joint with G.Keller) *In the Miller-Huse example, for small coupling, there exists a unique SRB measure. In addition, such a measure is space-time exponentially mixing.*

Random Walk

- The environment: $\Omega = \mathbb{T}^{\mathbb{Z}}$
- Its evolution: $(T\theta)_i = \tau\theta_i$, where τ is an expanding map.
- The random walk: $X_n \in \mathbb{Z}$, with transition probabilities

$$\mathbb{P}(X_{n+1} = i \mid X_n = j, \{\theta_i\}) := \begin{cases} \alpha(T^n\theta_j) =: \alpha_n & i = j-1 \\ \beta(T^n\theta_j) =: \beta_n & i = j+1 \\ 1 - \alpha_n - \beta_n & i = j \\ 0 & \text{otherwise} \end{cases}$$

What has this to do with our previous discussion?

Environment seen by the particle

Let $\omega(n)$ be the environment as seen by the particle at time n, we have $\mathbb{E}(\phi(\omega(n+1))|\omega(n)) =: \mathcal{K}'\phi(\omega(n))$ where

$$\begin{aligned}\mathcal{K}'\phi(\omega) =& \alpha(\omega_0)\phi(\sigma T(\omega)) + \beta(\omega_0)\phi(\sigma^{-1}T(\omega)) \\ &+ (1-\alpha(\omega_0)-\beta(\omega_0))\phi(T(\omega))\end{aligned}$$

$(\sigma\omega)_i := \omega_{i-1}$ being the shift.

If α and β are smooth and close to constant (the Boldrighini-Minlos-Pellegrinotti case), then the above operator can be treated using the previous ideas by studying some covering dynamics $\overline{\mathcal{K}}$.

- The operator $\overline{\mathcal{K}}$ has one as a simple eigenvalue, the eigenvector yields the invariant measure.
- The operator $\overline{\mathcal{K}}$ has a spectral gap.
- $\mathbb{E}(e^{izX_n} \mid \omega(0)) = (\mathcal{K}'_z)^n 1(\omega(0))$ where

$$\mathcal{K}'_z\phi(\omega) := \alpha(\omega_0)e^{-iz}\phi(\sigma T\omega) + \beta(\omega_0)e^{iz}\phi(\sigma^{-1}T\omega) \\ + [1 - \alpha(\omega_0) - \beta(\omega_0)]\phi(T\omega).$$

- Standard perturbation theory yields the CLT (averaged over the environment) Joint work in progress with D.Dolgopyat and S.Olla.