STATISTICAL PROPERTIES OF HYPERBOLIC BILLIARDS

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Abstract. I will discuss some recent results and open problems concerning the statistical properties of hyperbolic billiards. First I will discuss how to establish hyperbolicity, and then I will discuss the statistical properties. I will put the emphasis on some mathematical techniques useful to tackle such problems (e.g. standard pairs, dynamical functional spaces and transfer operators, strictly invariant cones, and Hilber metric).

Contents

1. Hyperbolic Billiards

The study of billiards has a double parallel history. On the one hand, starting at least with G. Birkhoff, they are seen as simple examples of dynamical systems and a tool to understand issues of integrability (billiard in an ellipse, polygonal billiards) and tool to understand strongly irregular motion (Sinai and Bunimovich Billiards). We here will concentrate on the second class of models.

1.1. **Billiard tables.**

The genesis of the study of the latter type of billiards goes back at least to Boltzmann who proposed to study the properties of a gas imagining that it consists of balls colliding elastically.

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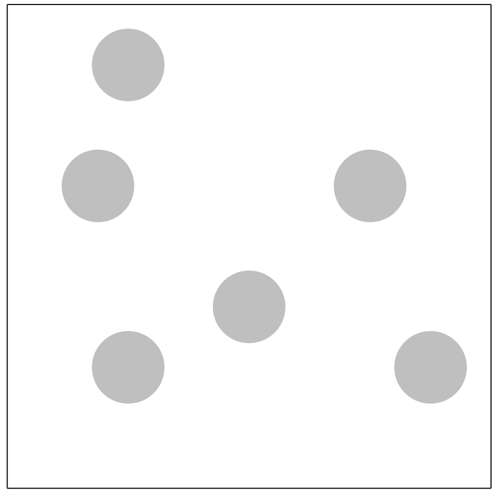


A two dimensional gas of particles in a box

The (seemingly ridiculous) simplest case is a gas of two particles in two dimensions. For simplicity, let us consider two particles of radius $r < \frac{1}{2}$ in a torus of size one. Let $x_1, x_2 \in \mathbb{T}^2$ be the coordinate of the center of the disks, the velocity changes at collision according to the law

$$\begin{cases} v_1^+ = v_1^- - \langle n, v_2^- - v_1^- \rangle n \\ v_2^+ = v_2^- + \langle n, v_2^- - v_1^- \rangle n \end{cases} \tag{1.1}$$

where n is a unit vector in the direction $x_2 - x_1$.[1]

Here there are three integral of motion: the energy $E = \frac{1}{2}(\|v_1\|^2 + \|v_2\|^2)$ and the total momentum $P = v_1 + v_2$. Thus, if we want to obtain an ergodic systems, we have to reduce the system. We will then consider that phase spaces

$$X_{E,P} = \left\{ (x_1, x_2, v_1, v_2) \in X \mid \frac{1}{2}(\|v_1\|^2 + \|v_2\|^2) = E;\ v_1 + v_2 = P \right\}.$$

Since, in the velocity space, the previous conditions correspond to the intersection between the surface of a four dimensional sphere (S^3) and a two dimensional linear space, the velocity vectors $(v_1 + v_2)$ is contained in a one dimensional circle. Thus, topologically, $X_{E,P} = \mathbb{T}^4 \times S^1$.[2] It is then natural to choose an angle θ as coordinate on S^1, moreover, since

$$2E = \|v_1\|^2 + \|v_2\|^2 = \frac{1}{2}\|v_1 - v_2\|^2 + \frac{1}{2}\|P\|^2,$$

it is hard to resist setting $v_2 - v_1 = v(\theta)$.[3] Hence,

$$\begin{cases} v_1 = \frac{1}{2}(P - v(\theta)) \\ v_2 = \frac{1}{2}(P + v(\theta)). \end{cases}$$

[1]To be precise $x_2 - x_1$ has no meaning since $\mathbb{T}^2$ it is not a linear space. Yet, at collision, the distance between the two disks is $2r$, so the global structure of $\mathbb{T}^2$ is irrelevant and we can safely confuse it with a piece of $\mathbb{R}^2$.

[2]Of course, we are considering only the cases $E \neq 0$.

[3]As usual $v(\theta) = (\sin\theta, \cos\theta)$.

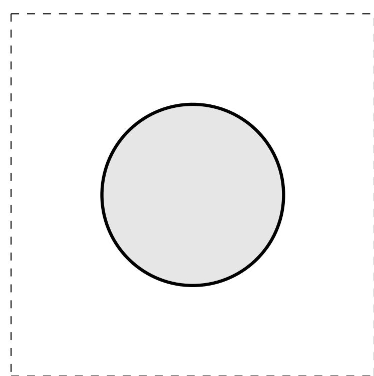
The free motion is then given by

$$\begin{cases} x_1(t) = x_1(0) + \frac{1}{2}(P - v(\theta))t \\ x_2(t) = x_2(0) + \frac{1}{2}(P + v(\theta))t. \end{cases}$$

Accordingly,

$$\begin{cases} x_1(t) + x_2(t) = x_1(0) + x_2(0) + Pt \mod 1 \\ x_2(t) - x_1(t) = x_2(0) - x_1(0) + v(\theta)t \mod 1. \end{cases}$$

It is then clear the need to introduce the two new variables $Q = x_1 + x_2$ and $\xi = x_2 - x_1$. The variable Q performs a translation on the torus, such a motions is completely understood and we can then disregard it. The only relevant motion is the one in the variables (ξ, θ). The reduced phase space is then $\mathcal{B} \times S^1$ where $\mathcal{B} = \mathbb{T}^2 \backslash \{\|\xi\| \leq 2r\}$, that is the torus minus a disk of radius $2r$. The domain $\mathcal{B}$ is represented in the next Figure and, apart the different choice of the fundamental domain, it corresponds exactly to simplest Sinai billiard.



The Siani billiard $\mathcal{B}$

The free motion corresponds to the free motion of a point as well, while at collision, from (1.1), we have

$$v(\theta^+) = v(\theta^-) - 2\left\langle \frac{\xi}{2r}, v(\theta^-) \right\rangle v(\theta^-)$$

that is exactly the elastic reflection from the disk!

It is the natural to consider the general problem of a particle moving in a region with reflecting boundary conditions. Let $\mathcal{B} \subset \mathbb{R}^d$ (or $\mathcal{B} \subset \mathbb{T}^d$) be the region and suppose that the boundary $\partial\mathcal{B}$ is made of finitely many smooth manifolds. Calling $(x, v) \in \mathcal{B} \times \mathbb{R}^d$ the position and the velocity, respectively, the motion inside $\mathcal{B}$ is described by a free flow

$$\phi_t(x, y) = (x + vt, v), \tag{1.2}$$

When $x \in \partial\mathcal{B}$ a collision takes place. If $n \in \mathbb{R}^d$, $\|n\| = 1$, is the nomal to $\partial\mathcal{B}$ at x, then, calling v_- and v_+ the velocities before and after collsion, respectively, the elastic collision is described by

$$v_+ = v_- - 2\langle v_-, n\rangle n.$$

1.2. **Hyperbolicity and how to establish it.**

Since we will discuss *hyperbolic billiards*, we must say exactly what we mean and how to see if a billiard is hyperbolic

First of all, recall Oseledec [32] (see [42] for a nice introduction and [23] for a generalization and more recent bibliography). We content ourselves with the following version.

Theorem 1.1 (Wojtkowski [40]). *Let (X, μ) be a probability space and $f : X \to X$ a measure-preserving transformation. Let $A : X \to GL(n, \mathbb{R})$ be a measurable mapping to $n \times n$ matrices such that $\log_+ \|A(\cdot)\| \in L^1(X, \mu)$. Then for μ-almost all $x \in X$ there are subspaces $\{0\} = V_x^0 \subset V_x^1 \subset \cdots \subset V_x^n = \mathbb{R}^n$ and numbers $\lambda_1(x) \le \cdots \le \lambda^n(x)$ such that, for all $i \in \{1, \dots, n\}$.*

$$\lim_{k \to \infty} \frac{1}{k} \ln \|A(f^{k-1}(x)) \cdots A(\tau(x))A(x)v\| = \chi_i(x)$$

if $v \in V_x^i \setminus V_x^{i-1}$.

We are interested in the case $A(x) = D_x\phi_1$, where Φ_t is the billiard flow. Of course, the flow will have a zero Lyapunov exponent (the flow direction).

Definition 1. *A Billiard is hyperbolic if the only zero Lyapunov exponent is the one associated with the flow direction. Equivalently, a Billiard is hyperbolic if the Poincarè map has no zero Lyapunov exponent.*

The problem is to have a tool to establish hyperbolicity. The following theorem provides a very efficient tool.

Theorem 1.2 (Wojtkowski [40]). *Let X be a Riemannian manifold, possibly with boundaries, $\{\mathcal{C}(x) \subset \mathcal{T}_x X \ : \ x \in X\}$ a family of closed cones in the tangent space. Let $f : X \to X$ and $A : X \to SL(n, \mathbb{R})$ as in Theorem 1.1. If for μ almost $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $A(f^{n(x)-1}) \cdots A(x)\mathcal{C}(x) \subset \operatorname{int}(\mathcal{C}(f^{n(x)}(x)))$, then the maximal Lyapunov exponent is strictly positive.*

The above theorem suffices for planar billiards, where there are two Lyapunov exponents λ_i and, by volume conservation $\lambda_1 = -\lambda_2$. For higher dimensional billiard, it does not control all the Lyapunov exponents. To achieve this, we have to use explicitely the fact that the Billiards flows are Hamiltonian, and hence symplectic. In addition, while a two dimensional cone is simply a sector, a higher dimensional cone can have many different shapes and it is not obvious what is a natural cone shape.

Given a symplectic form ω left invariant by ma $f : M \to M$, we have a symplectic flow. If $\mathcal{T}M = \mathbb{R}^{2d}$, then a d-dimensional subspace $V \subset \mathbb{R}^{2d}$ is called *Lagrangian* if $\omega|_V \equiv 0$. Given two transversal Lagrangian subspaces V_1, V_2, we can write uniquely $v \in \mathbb{R}^{2n}$ ad $v = w_1 + w_2$, with $w_i \in V_i$. we can then define the quadratic function

$$Q(v) = \omega(v_1, v_2).$$

This allows us to define special cones with remarkable properties:

$$\mathcal{C} = \{v \in \mathbb{R}^{2n} \ : \ Q(v) > 0\}.$$

Accordingly, if we specify a field of transversal Lagrangian subspace, we have the quadratic functions Q_x and the cone field $\mathcal{C}_x$.

Obviously, if $Q_{f(x)}(d_x f v) \ge Q_x(v)$, then $d_x f \mathcal{C}_x \subset \mathcal{C}_{f(x)}$, hence we have cone invarince. Such maps are called *monotone*.

Lemma 1.3 ([29], Sections 6)**.** *A map is monotone if and only if the cone field is invariant. The same is true for strict monotonicity.*

Theorem 1.4 ([29] Sections 5, 6, or [28])**.** *If a map is eventually strictly monotone, then all its Lyapunov exponents are non-zero.*

The above also has a continuous version: a Hamiltonian flow in a $2n+2$ dimensional manifold, is determined by a Hamiltonian H and the corresponding vector field X_H defined by

$$dH(v) = \omega(X_H, v)$$

for all tangent vectors v. If v is tangent to a constant energy surface $H = c$, we have $dH(v) = 0$, it follows that $\omega(X_H, v) = 0$ and then $\omega(X_H, \nabla H) \neq 0$, otherwise ω would be degenerate. We can then define the spaces, on the surface $H = c$,

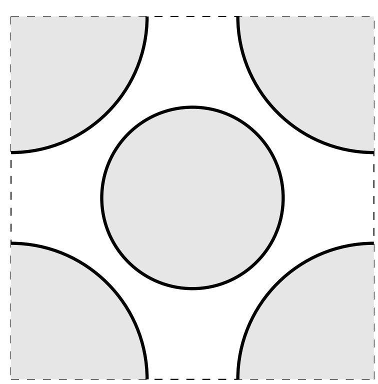
$$\mathbb{V}(x) = \{v \in \mathbb{R}^{2(n+1)} \ : \ dH(v) = 0\}/X_H,$$

that is, given two vectors v, w tangent to the energy surface we consider them equal if $v = w + \alpha X_H(x)$ for some $\alpha \in \mathbb{R}$. Then $\mathbb{V}(x)$ have dimension $2n$, moreover $\omega(v + \alpha X_H, w + \beta X_H) = \omega(v, w)$, so we can quotient the simplectic form on $\mathbb{V}(x)$. Moreover, $d_x\phi_t(v + \alpha X_H(x)) = d_x\phi_t(v) + \alpha X_H(\phi_t(x))$. Thus also $d\phi_t$ can be quotiented, and we are are reduced to the discrete case.

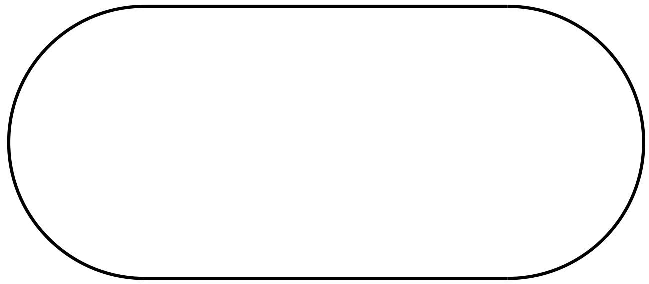
In alternative, one can consider the Poincarè map.

1.3. **Some Billiard tables.**

In the two dimensional case there are many possible billiards table that have been studied. The most famous two are the Sinai billiard and the Bunimovich stadium.



Sinai Billiard with finite horizon



Bunimovich stadium

Further interesting billiard tables can be found in [41, 7, 9] and references therein.

1.4. **Collision map and Jacobi fields.**

To compute, in general, the collision map it is helpful to introduce appropriate coordinates in $\mathcal{T}X$. A very interesting choice is constituted by the *Jacobi fields*.[4] Let X_- be the set of configurations just before collision. For each $(x, v) \in X \backslash X_-$ there exists $\delta > 0$ such that

$$\phi_t(x, v) = (x + vt, v) \quad 0 \le t \le \delta.$$

Let us consider the curve in $\mathcal{X}$

$$\xi(\varepsilon) = (x(\varepsilon), v(\varepsilon)),$$

with $\xi(0) = (x, v)$ and $\|v(\varepsilon)\| = 1$.

For each t such that $\phi_t(\xi(0)) \notin X_-$, let

$$\xi(\varepsilon, t) = (x(\varepsilon, t), v(\varepsilon, t)) = \phi_t(\xi(\varepsilon)).$$

The Jacobi field $J(t)$ is defined by

$$J(t) \equiv \left.\frac{\partial x}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

Note that, since $x(0, t) \notin X_-$, for $s < \delta$

$$\xi(\varepsilon, t + s) = \xi(\varepsilon, t) + (v(\varepsilon, t)s, 0),$$

so

$$J'(t) = \frac{dJ(t)}{dt} = \left.\frac{\partial v(\varepsilon, t)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

That is, $(J(t), J'(t)) = d\phi_t \xi'(0)$.

At each point $\xi = (x, v) \in X$ we choose the following base for $\mathcal{T}_\xi X$:[5]

$$\eta_0 = (v, 0); \quad \eta_1 = (v^\perp, 0); \quad \eta_2 = (0, v^\perp);$$

where $\|v^\perp\| = 1$, $\langle v, v^\perp \rangle = 0$.

The vector η_0 corresponds to a family of trajectories along the the flow direction and it is clearly invariant; η_1 to a family of parallel trajectories and η_2 to a family of

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[4]The Jacobi Fields are a widely used instrument in Riemannian geometry (see [18]) and have an important rôle in the study of Geodetic flows, although we will not insists on this aspect at present. Here they appear in a very simple form.

[5]Here $v^\perp = Jv$ with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

trajectories just after focusing. It is very useful the following graphic representation. We represent a tangent vector by drawing a curve that it is tangent to it. A curve in $\mathcal{T}X$ is given by a base curve that describes the variation of the x coordinate equipped with a direction at each point (specified by an arrow) which show how varies the velocity.

A direct check shows that each vector η perpendicular to the flow direction will stay so i.e.

$$\langle d\phi_t\eta,\, (v_t,\, 0)\rangle = \langle d\phi_t\eta,\, d\phi_t(v,\, 0)\rangle = \langle \eta,\, (v,\, 0)\rangle = 0.$$

So the free flow is described by

$$d\phi^t\eta_0 = \eta_0; \quad d\phi^t\eta_1 = \eta_1; \quad d\phi^t\eta_2 = \eta_2 + t\eta_1,$$

that is, in the above coordinates

$$d\phi^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}. \tag{1.3}$$

Let us see now what happens at a collision.

Let $x_0 \in \partial\mathcal{B}$ be the collision point and let $\xi_c = (x_0, v)$ be the configuration at the collision. We want to compute $R_\varepsilon := d_{\phi^{-\varepsilon}\xi_c}\phi^{2\varepsilon}$, that is the tangent map from just before to just after the collision. Clearly $R_\varepsilon\eta_0 = \eta_0$. If $\gamma(s)$ is the curve associated to η_1 at the point $\phi^{-\varepsilon}\xi_c$,

$$d\phi^{2\varepsilon}\gamma(s) = \left(v_+^\perp\left[s + \varepsilon\frac{2s}{r\sin\varphi}\right], \frac{2s}{r\sin\varphi}\right) + \mathcal{O}(s^2)$$

where r is the radius of the osculating circle (that is the circle tangent to the boundary up to second order) which is the inverse of the curvature $K(x_0)$ of the boundary at the collision point.

The above equation means that

$$J(\varepsilon) = (1 + \frac{2\varepsilon K(x_0)}{\sin\varphi})v_+^\perp.$$

Accordingly, calling $R = \lim_{\varepsilon\to 0} R_\varepsilon$ the collision map, we have

$$R\eta_1 = \eta_1 + \frac{2K}{\sin\varphi}\eta_2; \quad R\eta_2 = \eta_2.$$

Hence,

$$DR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{2K}{\sin\varphi} \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.4}$$

The above computations provide the following formula for the derivative of the Poincaré section from the boundary of the obstacle, just after collision, to the boundary of the obstacle just after the next collision

$$DT = \begin{pmatrix} 1 & \frac{2K}{\sin\varphi} \\ \tau & 1 + \frac{2\tau K}{\sin\varphi} \end{pmatrix}, \tag{1.5}$$

where τ is the flying time between the two collisions and φ the collision angle.

Formula (1.5) is sometimes called *Benettin formula* (e.g., [25]).

1.5. **Hard spheres.** For hard balls or radius $\frac{1}{2}$, and mass one, in dimension d, the flow is given by $\phi_t(q,p) = q + tp$ if no collision occurs. If the ball i collides with the ball j, then let p_i^-, p_j^- and p_i^+, p_j^+ be the velocities just before and after the collision, respectively. Note that for the balls to collide is must be that before the collision

$$0 > \frac{d}{dt}\|q_i - q_j\|^2 = \langle q_i - q_j, p_i - p_j\rangle.$$

Thus, at collision, $\langle q_i - q_j, p_i - p_j\rangle \leq 0$. Let $n = q_i - q_j$, then

$$\begin{aligned} p_i^+ &= p_i^- - \langle n, p_i^- - p_j^-\rangle n \\ p_j^+ &= p_j^- + \langle n, p_i^- - p_j^-\rangle n. \end{aligned} \tag{1.6}$$

Let us $d_{(q,p)}\phi_t(\delta q, \delta p)$ across a collision. If τ is the collision time of the trajectory then $\|q_i(\tau) - q_j(\tau)\| = 1$. If we consider the trajectories $\phi_t((q,p) + s(\delta q, \delta p))$, then the collision time $\tau(s)$ satisfies

$$\langle q_i(\tau) - q_j(\tau), \delta q_i - \delta q_j\rangle + \langle q_i(\tau) - q_j(\tau), p_i(\tau) - p_j(\tau)\rangle\tau'(0) = 0.$$

If the collision is non tangent (i.e. $\langle n, p_i(\tau) - p_j(\tau)\rangle \neq 0$), then,

$$\tau'(0) = -\frac{\langle n, \delta q_i - \delta q_j\rangle}{\langle n, p_i(\tau) - p_j(\tau)\rangle}.$$

To compute $d\phi_t$ is then convenient to shift along the flow direction by τ so all the trajectories $(q,p) + s(\delta q, \delta p)$ collide simultaneously. Let us call $(\tilde{\delta} q, \tilde{\delta} p)$, the shifted tangent vector. For such a tangent vector, we have that (1.6) yields

$$\begin{aligned} \tilde{\delta} q_i^+ =& \tilde{\delta} q_i^- \\ \tilde{\delta} q_j^+ =& \tilde{\delta} q_j^- \\ \tilde{\delta} p_i^+ =& \tilde{\delta} p_i^- - \langle \tilde{\delta} q_i^- - \tilde{\delta} q_j^-, p_i^- - p_j^-\rangle n - \langle n, p_i^- - p_j^-\rangle(\tilde{\delta} q_i^- - \tilde{\delta} q_j^-) \\ &- \langle n, \tilde{\delta} p_i^- - \tilde{\delta} p_j^-\rangle n \\ \tilde{\delta} p_j^+ =& \tilde{\delta} p_j^- + \langle \tilde{\delta} q_i^- - \tilde{\delta} q_j^-, p_i^- - p_j^-\rangle n + \langle n, p_i^- - p_j^-\rangle(\tilde{\delta} q_i^- - \tilde{\delta} q_j^-) \\ &+ \langle n, \tilde{\delta} p_i^- - \tilde{\delta} p_j^-\rangle n. \end{aligned} \tag{1.7}$$

And the derivative is then obtained shifting back along the flow direction. Note that, by construction $\langle \tilde{\delta} q_i^- - \tilde{\delta} q_j^-, n\rangle = 0$.

To apply Theorem 1.4 we have thus to construct the quadratic form Q. We choose the lagrangian spaces $\mathbb{V}_1 = \{\delta q = 0\}$ and $\mathbb{V}_2 = \{\delta p = 0\}$. The energy is only kinetic energy, then the vectors tangent to the constant energy are $\langle p, \delta p\rangle = 0$. This yields the form $Q(\delta q, \delta p) = \langle \delta q, \delta p\rangle$. The vector field is $(p, 0)$, and $Q(\delta q + \alpha p, \delta p) = Q(\delta q, \delta p)$, so Q is well defined on the quotient and we can restrict ourselves to the vectors $\{(\delta q, \delta p) \;:\; \langle p, \delta p\rangle = \langle p, \delta q\rangle = 0\}$. Note that

$$Q((\delta q + t\delta p, \delta p)) = Q(\delta q, \delta p) + t\|\delta p\|^2 \geq 0.$$

and if just a collision takes place in the interval $[0, t]$, then

$$Q((\delta q + tp, \delta p)) = Q(\tilde{\delta} q, \tilde{\delta} p) = Q(\delta q, \delta p) - \langle n, p_i^- - p_j^-\rangle\|\tilde{\delta} q_i^- - \tilde{\delta} q_j^-\|^2 \geq 0.$$

The invariance of the cone follows.

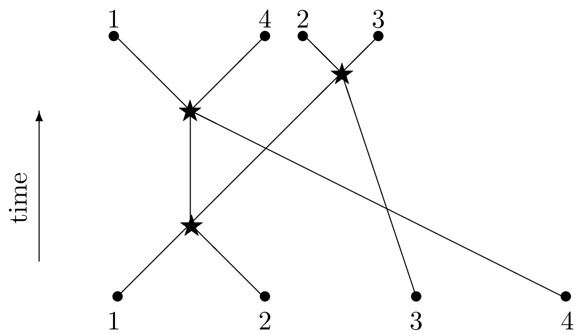

FIGURE 1. A simple collision graph (the stars are the collisions)

Note that we have strict invariance if $\delta p \neq 0$. If $\delta p = 0$, then we have the strict invariance if $\tilde{\delta} q_i^- \neq \tilde{\delta} q_j^-$. This fails only if

$$\delta q_i^- - \frac{\langle n, \delta q_i - \delta q_j \rangle}{\langle n, p_i(\tau) - p_j(\tau) \rangle} p_i = \delta q_j^- - \frac{\langle n, \delta q_i - \delta q_j \rangle}{\langle n, p_i(\tau) - p_j(\tau) \rangle} p_j,$$

i.e. there exists $z \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} \delta q_i &= z + \lambda p_i \\ \delta q_j &= z + \lambda p_j. \end{aligned} \tag{1.8}$$

To see how to use the above facts, it is convenient to introduce a bit of notation.

1.6. Collision graphs.

First of all I will introduce a *collision graph* to describe pictorially the relevant features of a trajectory, it will be a directed graph (the direction being given by time). The graph starts with n roots (each one representing one ball), from each root starts an edge (representing the path of a ball). A collision is represented by a vertex in the graph (I will idicate it pictorially by a star not to confuse it with edges that crosses on due to the two dimensional reprentation). If the collision involves k balls, then the vertex will have degree $2k$ with k entering edges–representing the incoming particles–and k exiting edges–representing the outcoming particles. Note that typically each vertex will have degree four, yet in the following we will generalize the meaning of a vertex and vertex of higher degree will play an important role.

See figure 1 for the case of four balls in which number one collides with two, then two with four, and finally two with three.[6]

Next, let us call $\mathcal{G}$ a collision graph and let $V(\mathcal{G})$ be the collection of its vertexes, $\tilde{B}(\mathcal{G})$ the collection of its edges and $B(\mathcal{G})$ the collection of edges that connect starred veteces. In addition for each edge $b \in B(\mathcal{G})$ let $\nu(b), \nu_+(b)$ be the two vertices joined by the edge.[7]

[6]The rule for tracing the graph is that the order of the balls is not changed at collision, so the line on the left represents the particles entering the collision vertex from the left. Remark that the collision graph is only a symbolic device and does not respect the geometry of the actual collisions, so the ordering of the balls is only a device to tell them apart and has no relation with the actual geometry of the associated configuration. Keeping this in mind, in figure 1 the final disposition of the balls is: one, four, two, three.

[7]By convention $\nu(b)$ corresponds to the lower collision and $\nu_+(b)$ to the upper.

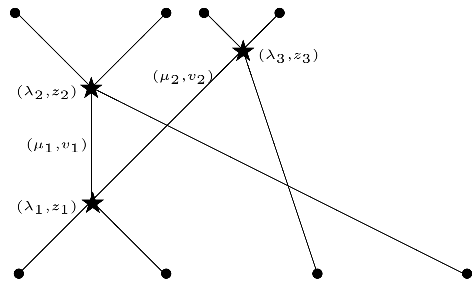

FIGURE 2. A decorated collision graph

To follow the history of a vector of type $(\delta q, 0)$ that stubbornly refuses to enter strictly in the cone it is convenient to specify at each vertex the values (λ_ν, z_ν) appearing in the associated equation (1.8). Of course, to recover the tangent vectors from the $\{(\lambda_\nu, z_\nu)\}_{\nu\in V(\mathcal{G})}$, it is necessary to specify the velocities. To this end we specify for each edge the velocity $v(b)$ of the particle associated to such a line. We can then decorate a graph with the above informations and we obtain a full description of the history of a tangent vector that keeps being not increased by the dynamics in the trajectory piece described by the graph (of course provided such a vector exists at all).

Now consider a edge $b \in B(\mathcal{G})$, if it represents the trajectory of the particle j between the collision corresponding to the the vertex $\nu(b)$ and the one corresponding to the vertex $\nu_+(b)$, then the corresponding component of the tangent vector at such times can be written both as $\delta q_j = z(\nu(b)) + \lambda(\nu(b))v(b)$ and $\delta q_j = z(\nu_+(b)) + \lambda(\nu_+(b))v(b)$. Accordingly, the following compatibility condition must be satisfied:

$$z(\nu(b)) - z(\nu_+(b)) = [\lambda(\nu_+(b)) - \lambda(\nu(b))]v(b). \tag{1.9}$$

It is then natural to define another decoration, this time associated to edges that connect two collision vertexes,

$$\mu(b) := \lambda(\nu_+(b)) - \lambda(\nu(b)). \tag{1.10}$$

By decorated collision graph, we will mean a graph with $(\lambda(\nu), z(\nu))$ attached to each vertex and $\mu(b), v(b)$ to each edge connecting two collisions, with a mild abuse of notations we will call such decorated graph $\mathcal{G}$ as well, see figure 2.[8]

1.7. **Cycles.** As time progress the graph will grow more complex, in particular it may develop *cycles*. By a cycle I mean a connected path of edges that leave a vertex and go back to it, e.g. the thick edges in the graph of figure 3.

Once a cycle is formed a remarkable compatibility condition can be derived. In fact, let $C \subset \mathcal{G}$ be a cycle, let us run it counterclockwise and define, for each edge $b \in C$, $\varepsilon_C(b) = 1$ if the edge is run from bottom to top and $\varepsilon_C(b) = -1$ if it is run from top to bottom. We have, by definition (1.10), $\sum_{b\in B(C)} \varepsilon_C(b)\mu(b) = 0$. In

[8]Note that the above description is quite redundant due to (1.9), yet we will see in the following that such a description is quite convenient.

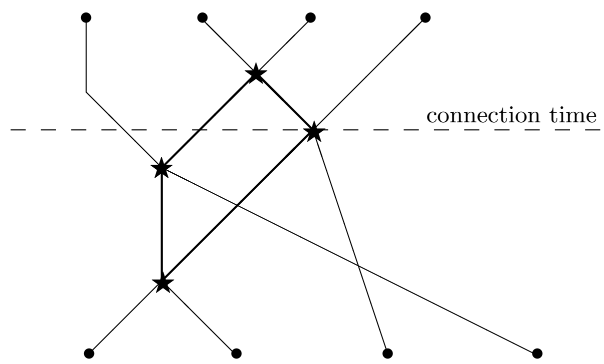

FIGURE 3. A cycle

addition, we can sum equation (1.9) for each edge in the cycle and obtain

$$\sum_{b\in B(C)} \mu(b)\varepsilon_C(b)v(b) = 0$$
$$\sum_{b\in B(C)} \varepsilon_C(b)\mu(b) = 0. \tag{1.11}$$

The above formula is essentially the *closed path formula* introduced by Simanyi in [34]. Such a formula expresses a compatibility condition that puts a clear restriction on the possible existence of the decorated collision graph, and hence of the corresponding nonincreasing vector. Studying the combinatorics of such collisions, it is possible to establish the hyperbolicity, and ergodicity, of a gas of n particles. This has been done in a series of papers of the *Hungarian team* [27, 34, 35, 36, 37].

2. Geometry of foliations

Once we know that the system is hyperbolic, we can try to take advantage of hyperbolicity: the first step is to construct stable and unstable manifolds. The strategy is the usual one: e.g., to construct the unstable manifold at x, consider the trajectory $f^{-n}(x)$ (for simplicity, we consider the Poincarè map). If the trajectory does not meet a discontinuity, then we can consider a manifold W, with tangent space in the unstable cone, centered at $f^{-n}(x)$ and push it forward with the dynamics. In this way, we obtain a sequence of manifolds $W_n = f^n(W)$ that we expect to converge to a limit object. Yet, one has to take into account that the manifold can be cut by singularities, and this could be a serious problem.

In the uniformly hyperbolic case, the analysis is especially simple: since the manifold W expands exponentially ($|W_n| \geq e^{\lambda n}|W|$), we have that the manifolds are cut at a distance shorter than δ only if the distance of $f^{-n}(x)$ from the singularities is less than $\delta e^{-\lambda n}$. This means that the manifold is cut short only if $f^{-n}(x)$ belongs to a neighborhood $\mathcal{S}_n$ of measure $\delta e^{-\lambda n}$. But since the measure is preserved, we have

$$\text{Leb}\left(\cup_{n=0}^{\infty} f^n(\mathcal{S}_n)\right) \leq \sum_{n=0}^{\infty} e^{-\lambda n}\delta \leq C\delta.$$

It follows that there exists a set of measure $1 - C\delta$ in which the unstable manifold has a length larger than δ.

Implementing the above basic idea can be technically challenging, especially since the formula (1.5) shows that the derivative blows up near tangencies. Yet, it can be done, for details, see [25, 9]. A technical tool used to deal with the blow-up of the differential at tangent collisions is the introduction, by Sinai, of homogeneity strips. See [9] for details.

The above construction provides a stable foliation, yet the foliation has very poor regularity properties, and this makes it very hard to use it; in general, it is only measurable. Luckily, the holonomy is absolutely continuous. Moreover, it turns out that it can be approximated by a foliation with much better properties that can be conveniently used, see [2, Section 6] for details.

The next step is to prove ergodicity. Once we have an absolutely continuous foliation, you can try to copy Hopf's argument. Such an argument is based on the observation that the ergodic averages of continuous functions are constant along stable and unstable manifolds. This was achieved by Sinai [38]. But see [28] for a more general version. In addition, [28] discusses a piecewise linear example in which the technical difficulties are reduced to a bare minimum, and hence Sinai's argument can be easily understood. The idea is to prove local ergodicity, and then a global argument can be employed to prove ergodicity. The same argument proves that all the powers of the Poincare maps are ergodic, which implies mixing.

It remains the problem of flows. Since the flow can be seen as a suspension over the Poicnarè map, the ergodicity of the flow follows from the ergodicity of the map. Not so for mixing: think of a suspension with a constant ceiling. Mixing for the flows follows from the contact structure. Forgetting for one second the discontinuities, the fact that the flow is contact implies that is we do a cycle stable, unstable, stable, unstable, we move in the flow direction, see Figure 4.

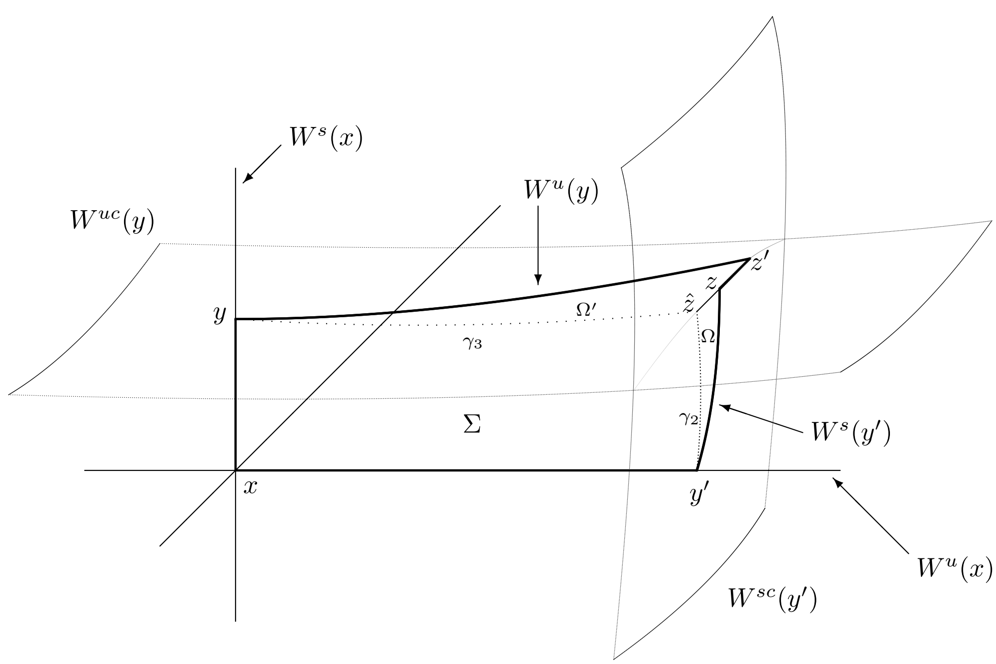

FIGURE 4. Definition of the *temporal function* $\Delta(y, y')$ and related quantities

Indeed, let α be the contact form, then if v is a strong unstable or a strong stable vector, then $\alpha(v) = 0$, while $\alpha((p, 0)) = 1$, where $(p, 0)$ is the flow direction, it follows that if the cycle in bold in figure 4, call it γ, has sides of length δ, then

$$\delta^2 = \int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha = \int_\gamma \alpha$$

which equals exactly the displacement in the flow direction, which is then non-zero. It follows that the stable and unstable foliations are not *jointly integrable*, and this property shows that the flow cannot be reduced to a constant flow suspension by a change of coordinates (since, in such a case, the foliations would indeed be jointly integrable). This suffices to prove mixing of the flow.

3. Statistical Properties

All the properties discussed so far are of a qualitative nature, yet to obtain physically relevant facts it is necessary to have quantitive information. First and foremost, an estimate of the speed of mixing. Bunimovich and Sinai first achieved this [5] for the Poincarè map, while the result for the flow is due to Demers, Baladi Liverani [11], almost forty years later (not for lack of trying).

Several techniques have been developed to study the speed of decay of correlations, the main one are

(1) coding the system via Markov Partitions (Bunimovich and Sinai [5])
(2) coding the systems via towers (Lai-Sang Young [43, 44])
(3) standard pairs and coupling (Lai-Sang Young [44], Dolgopyat [19])
(4) operator renewal theory (Sarig [33])
(5) Functional spaces adapted to the transfer operator (Blank, Keller, Liverani [4]; Liverani, Gouezel [24]; Baladi, Tsujii [1]; Demers, Liverani [12]; Demers, Zhang [16])
(6) Hibert metric (Ferrero, Schmitt [22], Liverani [30]; Demers, Liverani [17])
(7) Random perturbations (Liverani, Saussol, Vaienti [31])

The most powerful techniques are probably (5, 6), but they can work only if the decay of correlations is exponential. For polynomial decay of correlations (2, 4) or even the rougher (7) are the way to go. While (3) is unquestionably the more versatile technique.

For an introduction to (3,5,6) see [13].

To conclude, let me recap part of the state of the art, giving a, idiosyncratic, list of results.

The ergodicity of various billiard tables was established in many papers, e.g., [41, 7]. Ergodicity results also exist for billiards in which the particle is subject to a soft potential, rather than a hard core one, e.g. [26, 21]. The ergodicity of a gas of hard spheres was established, building on a rather long string of papers, in [37]. The statistical properties of billiards with finite and infinite horizon can be found in [8, 20] where the standard pair technology is put to work. The functional analytic approach has been developed in [14, 16]; such an approach also allows establishing how the statistical properties depend on the billiard shape [15]. In addition, the functional approach has proven instrumental in the proof of exponential mixing for two dimensional uniformly hyperbolic billiard flow [2]. Many limit theorems have been obtained for billiard systems for which mixing properties have been established. Notable results are the polynomial decay of correlations in the Bunimovich stadium [3] and the monumental study of one massive particle interacting with a light one in a box [10].

All the previous papers deal with isolated systems, if the system changes in time (e.g. a time-dependent billiard table), then the simple study of the spectral properties of the transfer operator does not suffice; one has to deal with the product of different operators. This can be done using perturbation theory if the change in time is very slow [39]. However, if the change in time is more violent, perturbation theory fails, and a new approach is needed. This has been recently achieved in [17] using Hilbert metrics on invariant cones of densities.

Even though the above list of results is very partial, I hope it gives an idea of the breadth of the field and of the many directions along which the research is developing.

References

[1] Baladi, Viviane; Tsujii, Masato *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms.* Ann. Inst. Fourier (Grenoble) 57 (2007), no. 1, 127–154.

[2] Baladi, Viviane; Demers, Mark F.; Liverani, Carlangelo *Exponential decay of correlations for finite horizon Sinai billiard flows.* Invent. Math. 211 (2018), no. 1, 39–177.

[3] Bálint, Péter; Gouëzel, Sébastien *Limit theorems in the stadium billiard.* Comm. Math. Phys. 263 (2006), no. 2, 461–512.

[4] Blank, Michael; Keller, Gerhard; Liverani, Carlangelo *Ruelle-Perron-Frobenius spectrum for Anosov maps.* Nonlinearity 15 (2002), no. 6, 1905–1973.

[5] Bunimovich, L. A.; Sinaĭ, Ya. G. *Statistical properties of Lorentz gas with periodic configuration of scatterers.* Comm. Math. Phys. 78 (1980/81), no. 4, 479–497.

[6] Bunimovich, L. A.; Sinaĭ, Ya. G.; Chernov, N. I. *Statistical properties of two-dimensional hyperbolic billiards.* (Russian) Uspekhi Mat. Nauk 46 (1991), no. 4(280), 43–92, 192; translation in Russian Math. Surveys 46 (1991), no. 4, 47–106

[7] Bunimovich, Leonid A. *Mushrooms and other billiards with divided phase space.* Chaos 11 (2001), no. 4, 802–808.

[8] Chernov, N., *Decay of correlations and dispersing billiards.* J. Statist. Phys. 94 (1999), no. 3-4, 513–556.

[9] Chernov, Nikolai; Markarian, Roberto *Chaotic billiards.* Mathematical Surveys and Monographs, 127. American Mathematical Society, Providence, RI, 2006. xii+316 pp.

[10] Chernov, N.; Dolgopyat, D. *Brownian Brownian motion. I.* Mem. Amer. Math. Soc. 198 (2009), no. 927, viii+193 pp. ISBN: 978-0-8218-4282-9

[11] Baladi, Viviane; Demers, Mark F.; Liverani, Carlangelo *Exponential decay of correlations for finite horizon Sinai billiard flows.* Invent. Math. 211 (2018), no. 1, 39–177.

[12] Demers, Mark F.; Liverani, Carlangelo Stability of statistical properties in two-dimensional piecewise hyperbolic maps. Trans. Amer. Math. Soc. 360 (2008), no. 9, 4777–4814.

[13] Demers, Mark F.; Kiamari, Niloofar; Liverani, Carlangelo *Transfer operators in hyperbolic dynamics—an introduction.* 33 o Colóquio Brasileiro de Matemática. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2021. 238 pp. ISBN: 978-65-89124-26-9

[14] Demers, Mark F.; Zhang, Hong-Kun *Spectral analysis of the transfer operator for the Lorentz gas.* J. Mod. Dyn. 5 (2011), no. 4, 665–709.

[15] Demers, Mark F.; Zhang, Hong-Kun A functional analytic approach to perturbations of the Lorentz gas. Comm. Math. Phys. 324 (2013), no. 3, 767–830.

[16] Demers, Mark F.; Zhang, Hong-Kun *Spectral analysis of hyperbolic systems with singularities.* Nonlinearity 27 (2014), no. 3, 379–433.

[17] Demers, Mark F.; Liverani, Carlangelo Projective cones for sequential dispersing billiards. Comm. Math. Phys. 401 (2023), no. 1, 841–923. 37C83.

[18] do Carmo, Manfredo Perdigão *Riemannian geometry.* Translated from the second Portuguese edition by Francis Flaherty. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. xiv+300 pp. ISBN: 0-8176-3490-8

[19] Dolgopyat, Dmitry *Limit theorems for partially hyperbolic systems.* Trans. Amer. Math. Soc. 356 (2004), no. 4, 1637–1689.

[20] Dolgopyat, Dmitry; Szász, Domokos; Varjú, Tamás *Recurrence properties of planar Lorentz process.* Duke Math. J. 142 (2008), no. 2, 241–281.

[21] Donnay, Victor; Liverani, Carlangelo, *Potentials on the two-torus for which the Hamiltonian flow is ergodic.* Comm. Math. Phys. 135 (1991), no. 2, 267–302.

[22] Ferrero, P.; Schmitt, B. *Produits aléatoires d'opérateurs matrices de transfert.* (French) [Random products of transfer matrix operators] Probab. Theory Related Fields 79 (1988), no. 2, 227–248.

[23] González-Tokman, Cecilia; Quas, Anthony *A concise proof of the multiplicative ergodic theorem on Banach spaces.* J. Mod. Dyn. 9 (2015), 237–255.

[24] Gouëzel, Sébastien; Liverani, Carlangelo *Banach spaces adapted to Anosov systems.* Ergodic Theory Dynam. Systems 26 (2006), no. 1, 189–217.

[25] Katok, Anatole; Strelcyn, Jean-Marie; Ledrappier; Przytycki, F. *Invariant manifolds, entropy and billiards; smooth maps with singularities.* Lecture Notes in Mathematics, 1222. Springer-Verlag, Berlin, 1986.

[26] Knauf, Andreas, *Ergodic and topological properties of Coulombic periodic potentials.* Comm. Math. Phys. 110 (1987), no. 1, 89–112.

[27] Krámli, A.; Simányi, N.; Szász, D. *A "transversal" fundamental theorem for semi-dispersing billiards.* Comm. Math. Phys. 129 (1990), no. 3, 535–560.

[28] Liverani, Carlangelo; Wojtkowski, Maciej P. *Generalization of the Hilbert metric to the space of positive definite matrices.* Pacific J. Math. 166 (1994), no. 2, 339–355.

[29] Liverani, Carlangelo; Wojtkowski, Maciej P. *Ergodicity in Hamiltonian systems.* Dynamics reported, 130–202, Dynam. Report. Expositions Dynam. Systems (N.S.), 4, Springer, Berlin, 1995.

[30] Liverani, Carlangelo *Decay of correlations.* Ann. of Math. (2) 142 (1995), no. 2, 239–301.

[31] Liverani, Carlangelo; Saussol, Benoît; Vaienti, Sandro *A probabilistic approach to intermittency.* Ergodic Theory Dynam. Systems 19 (1999), no. 3, 671–685.

[32] Oseledec, V. I. *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems.* (Russian) Trudy Moskov. Mat. Obšč. 19 (1968), 179–210.

[33] Sarig, Omri *Subexponential decay of correlations.* Invent. Math. 150 (2002), no. 3, 629–653.

[34] Simányi, Nándor, *The K-property of N billiard balls. II. Computation of neutral linear spaces.* Invent. Math. 110 (1992), no. 1, 151–172.

[35] Simányi, Nándor, *The K-property of N billiard balls. I* Invent. Math. 108 (1992), no. 3, 521–548.

[36] Simányi, Nándor; Szász, Domokos *Hard ball systems are completely hyperbolic.* Ann. of Math. (2) 149 (1999), no. 1, 35–96.

[37] Simányi, Nándor *Proof of the Boltzmann-Sinai ergodic hypothesis for typical hard disk systems.* Invent. Math. 154 (2003), no. 1, 123–178.

[38] Sinaĭ, Ja. G. *Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards.* (Russian) Uspehi Mat. Nauk 25 (1970), no. 2(152), 141–192.

[39] Stenlund, Mikko; Young, Lai-Sang; Zhang, Hongkun *Dispersing billiards with moving scatterers.* Comm. Math. Phys. 322 (2013), no. 3, 909–955.

[40] Wojtkowski, Maciej *Invariant families of cones and Lyapunov exponents.* Ergodic Theory Dynam. Systems 5 (1985), no. 1, 145–161.

[41] Wojtkowski, Maciej *Principles for the design of billiards with nonvanishing Lyapunov exponents.* Comm. Math. Phys. 105 (1986), no. 3, 391–414.

[42] Viana, Marcelo *Lectures on Lyapunov exponents.* Cambridge Studies in Advanced Mathematics, 145. Cambridge University Press, Cambridge, 2014. xiv+202 pp.

[43] Young, Lai-Sang *Statistical properties of dynamical systems with some hyperbolicity.* Ann. of Math. (2) 147 (1998), no. 3, 585–650.

[44] Young, Lai-Sang *Recurrence times and rates of mixing.* Israel J. Math. 110 (1999), 153–188.

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