DYNAMICAL SYSTEMS FROM ODE'S TO ERGODIC THEORY

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Chapter 1

The origins: Differential equations

As this book is about Dynamical Systems, let's start by defining the object of study. The concept of Dynamical System is a very general one and it appears in many branches of mathematics from discrete mathematics, number theory, probability, geometry and analysis and has wide applications in physics, chemistry, biology, economy and social sciences.

Probably the most general formulation of such a concept is the action of a monoid over an algebra. Given a monoid $\mathbb{G}$ and an algebra $\mathcal{A}$, the (left)-action of $\mathbb{G}$ on $\mathcal{A}$ is simply a map $f : \mathbb{G} \times \mathcal{A} \to \mathcal{A}$ such that[1]

1. $f(gh, a) = f(g, f(h, a))$ for each $g, h \in \mathbb{G}$ and $a \in \mathcal{A}$;
2. $f(e, a) = a$ for every $a \in \mathcal{A}$, where e is the identity element of $\mathbb{G}$;
3. $f(g, a + b) = f(g, a) + f(g, b)$ for each $g \in \mathbb{G}$ and $a, b \in \mathcal{A}$;
4. $f(g, ab) = f(g, a)f(g, b)$ for each $g \in \mathbb{G}$ and $a, b \in \mathcal{A}$;

In our discussion we will be mainly motivated by physics. In fact, we will consider the cases in which $\mathbb{G} \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}_+, \mathbb{R}\}$[2] is interpreted as *time* and

[1] In an alternative, one can consider the action on a vector space, if one wants to include, e.g, stochastic processes.

[2] Although even in physics other possibilities are very relevant, e.g. in the case of Statistical Mechanics it is natural to consider the action of the space translations, i.e. the groups $\{\mathbb{Z}^d, \mathbb{R}^d\}$ for some $d \in \mathbb{N}$, $d > 1$.

$\mathcal{A}$, interpreted as the *observables* of the system,[3] is a commutative algebra consisting of functions over some set X. In addition, we will restrict ourselves to situations where the action over the algebra is induced by an action over the set X (this is a map $f : \mathbb{G} \times X \to X$ that satisfies condition 1, 2 above).[4] Indeed, given an action f of $\mathbb{G}$ on X and an algebra $\mathcal{A}$ of functions on X such that, for all $a \in \mathcal{A}$ and $g \in \mathbb{G}$, $b(\cdot) := a(f(g, \cdot)) \in \mathcal{A}$, it is natural to define $\tilde{f}(g, a)(x) := a(f(g, x))$ for all $g \in \mathbb{G}$, $a \in \mathcal{A}$ and $x \in X$. It is then easy to verify that $\tilde{f}$ satisfies conditions 1–4 above.

We will call *discrete time Dynamical System* the ones in which $\mathbb{G} \in \{\mathbb{N}, \mathbb{Z}\}$ and *continuous time Dynamical Systems* the ones in which $\mathbb{G} \in \{\mathbb{R}_+, \mathbb{R}\}$. Note that, in the first case, $f(n, x) = f(n-1+1, x) = f(1, f(n-1, x))$, hence defining $T : X \to X$ as $T(x) = f(1, x)$, holds $f(n, x) = T^n(x)$.[5] Thus in such a case we can (and will) specify the Dynamical System by writing only (X, T). In the case of continuos Dynamical Systems we will write $\phi_t(x) := f(t, x)$ and call ϕ_t a flow (if the group is $\mathbb{R}$) or a semi-flow (if the group is $\mathbb{R}_+$) and will specify the Dynamical System by writing (X, ϕ_t). In fact, in this notes we will be interested only in Dynamical Systems with more structure i.e. *topological*, *measurable* or *smooth* Dynamical Systems. By topological Dynamical Systems we mean a triplet $(X, \mathcal{T}, T)$, where $\mathcal{T}$ is a topology and T is continuos (if $B \in \mathcal{T}$, then $T^{-1}B \in \mathcal{T}$). By smooth we consider the case in which X has a differentiable structure and T is r-times differentiable for some $r \in \mathbb{N}$. Finally, a measurable Dynamical Systems is a quadruple (X, Σ, T, μ) where Σ is a σ-algebra, T is measurable (if $B \in \Sigma$, then $T^{-1}B \in \Sigma$) and μ is an invariant measure (for all $B \in \Sigma$, $\mu(T^{-1}B) = \mu(B)$).[6]

So far for general definitions that, to be honest, are not very inspiring. Indeed, what characterizes the modern Dynamical Systems is not so much the setting but rather the type of questions that are asked, first and foremost:

- **Which behaviors are visible in nature?** (stability and bifurcation theory).
- **What happens for very long times?** (statistics and asymptotic theory)

The rest of this book will deal in various ways with such questions.

The original motivation for the above setting and for these questions comes from the study of the motion which, after Newton, typically appears as so-

[3] Again other possibilities are relevant, e.g. the case of Quantum Mechanics (in the so called Heisenberg picture) where the algebra of the observable is non commutative and consists of the bounded operators over some Hilbert space.

[4] Again relevant cases are not included, for example all Markov Process where the evolution is given by the action of some semigroup.

[5] Obviously $T^2(x) = T \circ T(x) = T(T(x))$, $T^3(x) = T \circ T \circ T(x) = T(T(T(x)))$ and so on.

[6] The definitions for continuos Dynamical Systems are the same with $\{\phi_t\}$ taking the place of T.

lution of an *ordinary differential equation* (ODE). It is then natural to start with a brief reminder of basic ODE theory.[7]

In section 1.1 I will recall the theorem of existence and uniqueness of the solutions of an ODE. In addition, I will state the Gronwall inequality, a very useful inequality for estimating the growth rate of the solution of an ODE. Finally, a theorem yielding the smooth dependence of the solutions of an ODE from an external parameter or from the initial conditions is provided.

In section 1.2 is given a very brief account of linear equations with constant coefficients (by discussing the exponential of a matrix) and of Floquet theory. That is the study of the solutions of a linear equation with coefficients varying periodically in time. The basic result being that the asymptotic properties of the solutions can be understood by looking at the solutions after one period.

Finally, section 1.3 discusses the possibility of qualitative understanding the behavior of the solutions of ODE that cannot be solved explicitly (essentially all the ODEs). The arguments are very naive and are intended only to convince the reader that a) something can be done; b) a more sophisticated theory needs to be developed in order to have a satisfactory picture.

1.1 Few basic facts about ODE: a reminder

Our starting point is the initial Cauchy problem for ODE. That is, given a separable Banach space $\mathcal{B}$,[8] $V \in \mathcal{C}^0_{\text{loc}}(\mathcal{B} \times \mathbb{R}, \mathcal{B})$,[9] and $x_0 \in \mathcal{B}$, find an open interval $0 \ni I \subset \mathbb{R}$ and $x \in \mathcal{C}^1(I, \mathcal{B})$ such that

$$\begin{aligned} \dot{x}(t) &= V(x(t), t) \\ x(0) &= x_0. \end{aligned} \tag{1.1.1}$$

Remark 1.1.1 *I will be mainly interested in the case $\mathcal{B} = \mathbb{R}^d$, for some $d \in \mathbb{N}$. Thus, the reader uncomfortable with Banach spaces can safely substitute*

[7] In fact, also the solutions of a partial differential equation (PDE) may give rise to a Dynamical System, yet the corresponding theory is typically harder to investigate.

[8] A Banach spaces is a complete normed vector spaces. This means that a Banach space is a vector space V, over $\mathbb{R}$ or $\mathbb{C}$, equipped with a norm $\|\cdot\|$ such that every Cauchy sequence in V has a limit in V. By *separable* we mean that there exists a countable dense set. Check [RS80, Kat66] for more details or [DS88] for a lot more details.

[9] Given two Banach spaces $\mathcal{B}_1, \mathcal{B}_2$, an open set $U \subset \mathcal{B}_1$, and $q \in \mathbb{N}$ by $\mathcal{C}^q(U, \mathcal{B}_2)$ we mean the continuous functions from U to $\mathcal{B}_2$ that are q time (Fréchet) differentiable and the q-th differentials are continuous (see Problem 1.18 for a very quick discussion of differentiation in Banach spaces). Such a vector space can be equipped with the norm $\|\cdot\|_{\mathcal{C}^q}$ given by the sup of all its derivatives till the order q included. If we then consider the subset for which such a norm is finte, then we have again a vector space which is, in fact, a Banach space. We will call such a Banach space $\mathcal{C}^q(U, \mathcal{B}_2, \|\cdot\|_{\mathcal{C}^q})$, yet, when no confusion can arise, we will abuse of notation and call it simply $\mathcal{C}^q(U, \mathcal{B}_2)$. By $\mathcal{C}^q_{\text{loc}}(U, \mathcal{B}_2)$ we mean the vector space of the functions $f : U \to \mathcal{B}_2$ such that, for each $u \in U$ and $R > 0$ such that $\overline{B(u, R)} = \{v \in \mathcal{B}_1 \ : \|v - u\| \leq R\} \subset U$, $f \in \mathcal{C}^q(B(u, R), \mathcal{B}_2, \|\cdot\|_{\mathcal{C}^q})$. Note that, in general, $\mathcal{C}^q_{\text{loc}}$ is not a Banach space (in fact, it is a Fréchet space).

$\mathbb{R}^d$ to $\mathcal{B}$ in all the subsequent arguments. Yet, it is interesting that the theory can be developed for general Banach spaces at no extra cost. For simplicity, in the following we will always assume that all the Banach spaces are separable even if not explicitly mentioned. In essence, this is just a fancy way of saying that much of the following depends only on the Banach structure of $\mathbb{R}^d$, that is on the fact that $\mathbb{R}^d$ is a complete vector space with a norm (e.g. the euclidean norm) and, for example, nowhere is used the fact that $\mathbb{R}^d$ has a finite basis.

I will also briefly consider ODE on (finite dimensional) manifolds. Not much extra theory is needed in order to do this, since ODE on manifolds can always be reduced to the case $\mathbb{R}^d$ case, see section 1.1.5.

The first problem that comes to mind is

Question 1 *Does the Chauchy problem (1.1.1) always admit a solution? If there exists a solution is it unique?*

To address such an issue it is convenient to consider the equation[10]

$$x(t) = x_0 + \int_0^t V(x(s), s)ds \tag{1.1.2}$$

Problem 1.1 *Show that for each finite open interval $0 \in I \subset \mathbb{R}$, if $x \in \mathcal{C}^1(I, \mathcal{B})$ is a solution of (1.1.1), then it is a solution of (1.1.2). Show that if $x \in \mathcal{C}^0(I, \mathcal{B})$ is a solution of (1.1.2) then $x \in \mathcal{C}^1(I, \mathcal{B})$ and solves (1.1.1).*

1.1.1 Existence and uniqueness

The issue of existence and uniqueness of the solutions of (1.1.1) can be solved by applying the clasical Banach fixed point Theorem (see A.1.1), provided we make a stronger assumption on V.

Theorem 1.1.2 (Existence and Uniqueness theorem for ODE) *For each $V \in \mathcal{C}^1_{\text{loc}}(\mathcal{B} \times \mathbb{R}, \mathcal{B})$ and $x_0 \in \mathcal{B}$ there exists $\delta \in \mathbb{R}_+$ such that there exists a unique solution of (1.1.1) in $\mathcal{C}^1((-\delta, \delta), \mathcal{B})$.*[11]

PROOF. Let $\delta \in (0, 1)$. The reader can verify that the vector space $\mathcal{C}^0([-\delta, \delta], \mathcal{B})$, equipped with the norm $\|u\|_\infty := \sup_{t\in[-\delta,\delta]} \|u(t)\|_{\mathcal{B}}$ is a Banach space.[12] By definition there exist $\delta_0, R_0 \geq 0$ such that, for all $\delta \leq \delta_0$ and

[10] The most convenient meaning of the integral of a function with values in a Banach space is the *Bochner sense*, which reduces to the usual Lebesgue integral in the case $\mathcal{B} = \mathbb{R}^d$, see [Yos95] for definition and properties. Yet, for our purposes the equivalent of the Riemannian integral suffices and it is defined in the obvious manner. See Problem 1.20 for details.

[11] We equip $\mathcal{B} \times \mathbb{R}$ with the norm $\|(x, t)\| \leq \sup\{\|x\|_{\mathcal{B}}, |t|\}$, where $\|\cdot\|_{\mathcal{B}}$ is the norm of $\mathcal{B}$.

[12] The uniform limit of continuous functions is a continuos function.

$R \leq R_0$, $V \in \mathcal{C}^1(D_R, \mathcal{B})$, where $D_R = \{y \in \mathcal{C}^0([-\delta, \delta], \mathcal{B}) \,:\, \|y - x_0\|_\infty \leq R\}$. We can then define the operator $K : D_R \to \mathcal{C}^0([-\delta, \delta], \mathcal{B})$ by[13]

$$K(u)(t) := x_0 + \int_0^t V(u(s), s)ds.$$

Let $M_\delta = \sup_{|t|\leq\delta} \sup_{u\in D_R}\{\|V(u,t)\| + \|\partial_u V(u,t)\|\}$, note that M_δ is a decreasing function of δ. Then, for each $u \in D_R$ and $|t| \leq \delta$, (recall Problem 1.22)

$$\|K(u(t)) - x_0\| \leq \delta M_\delta \leq R$$

provided we chose $\delta M_\delta \leq R$. Thus K maps D_R into D_R. In addition, for each $u, v \in D_R$,

$$\|K(u) - K(v)\|_\infty \leq \delta M_\delta \|u - v\|_\infty \leq \frac{1}{2}\|u - v\|_\infty,$$

provided we chose $2\delta M_\delta \leq 1$. We can then apply Theorem A.1.1 and obtain a unique solution of the equation $Ku = u$ in D_R. This shows the existence and uniqueness of the solution of (1.1.2). The Theorem follows then by remembering Problem 1.1. □

Remark 1.1.3 *Note that in the proof of Theorem A.1.1 one can chose the same δ for an open set of initial condition.*

Remark 1.1.4 *The hypotheses of the above Theorem can be easily weakened to the case of V locally Lipschitz in x and continuous in t, yet only continuity does not suffice for uniqueness as shown by the example*

$$\dot{x} = \sqrt{x}$$
$$x(0) = 0.$$

which has the infinitely many solutions $x_a(t) = 0$ for $t \leq a$ and $x_a(t) = \frac{1}{4}(t-a)^2$ for $t \geq a$, $a \in \mathbb{R}$.[14]

Remark 1.1.5 *The restriction to an interval of size δ in Theorem A.1.1 cannot be avoided as shown by the example*

$$\dot{x} = x^2$$
$$x(0) = 1.$$

Its solution $x(t) = (1-t)^{-1}$ is not continuous, nor bounded, for $t = 1$.

[13] The meaning of $\mathcal{C}^0(K, \mathcal{B}_2)$ where K is a closed set of $\mathcal{B}_1$ is the usual one.

[14] If $\mathcal{B}$ is finite dimensional, then $V \in \mathcal{C}^0$ suffices for the existence of a solution. This follows by a direct application of Schauder fixed point Theorem to (1.1.2). For informations on such a fixed point theorem and fixed point theorems in general see [Zei86].

We have seen a mechanism whereby the solution cannot be defined for all times, the next Lemma shows that, for $\mathcal{C}^1$ vector fields, the above is the *only* mechanism.[15]

Lemma 1.1.6 *In the hypotheses of Theorem 1.1.2, if $x \in \mathcal{C}^1_{\text{loc}}((-\underline{\delta}, \delta), \mathcal{B})$ is a solution of (1.1.1) for some $\underline{\delta}, \delta > 0$, and if there exists $M > 0$ such that $\sup_{t\in[0,\delta)} \|x(t)\| \leq M$, then there exists $\bar{\delta} > \delta$ and $\bar{x} \in \mathcal{C}^1((-\underline{\delta}, \bar{\delta}), \mathcal{B})$ that solves (1.1.1) (i.e. the solution can be extended for longer times).*

Proof. Let $\{t_n\}$ be any sequence that converges to δ, then

$$\|x(t_n) - x(t_m)\| \leq \int_{t_n}^{t_m} \|V(x(s), s)\| ds \leq |t_n - t_m| \sup_{\|z\|\leq M} \sup_{s\in[0,\delta)} \|V(z, s)\|.$$

Thus $\{x(t_n)\}$ is a Cauchy sequence and admits a limit $x_* \in \mathcal{B}$ such that

$$x_* = \lim_{n\to\infty} x(t_n) = \lim_{t\to\delta} x(t) = x_0 + \int_0^\delta V(x(s), s) ds.$$

We can then consider the equation

$$y(t) = x_* + \int_0^t V(y(s), s + \delta) ds.$$

By Theorem 1.1.2 there exists δ_1 and $y \in \mathcal{C}^1((-\delta_1, \delta_1), \mathcal{B})$ which satisfy the above equation. Let then $\bar{\delta} = \delta + \delta_1$ and define

$$\bar{x}(t) := \begin{cases} x(t) & \text{fot all } t \in (-\underline{\delta}, \delta) \\ y(t - \delta) & \text{fot all } t \in [\delta, \bar{\delta}). \end{cases}$$

Clearly $\bar{x} \in \mathcal{C}^0((-\underline{\delta}, \bar{\delta}), \mathcal{B})$ and, for $t \in [\delta, \bar{\delta})$ holds true

$$\begin{aligned} \bar{x}(t) &= x_* + \int_\delta^t V(\bar{x}(s), s) ds = x_0 + \int_0^\delta V(\bar{x}(s), s) ds + \int_\delta^t V(\bar{x}(s), s) ds \\ &= x_0 + \int_0^t V(\bar{x}(s), s) ds. \end{aligned}$$

Thus, again remembering Problem 1.1, the Lemma follows. □

Remark 1.1.7 *Applying repeatedly Lemma 1.1.6 it follows that there exists a maximal open interval $J \subset \mathbb{R}$ such that the Cauchy problem (1.1.1) has a unique solution belonging to $\mathcal{C}^1_{\text{loc}}(J, \mathcal{B})$.*

[15] I state the result for positive times, for negative times it is the same.

1.1.2 Growald inequality

We have seen that the escape (growth) to infinity is the only obstruction to enlarging the domain of the solution.[16] The question remains: how large the maximal interval J in Remark 1.1.7 can be?

To understand better how the solution of an ODE can grow we need a technical but extremely useful Lemma.

Lemma 1.1.8 (Integral Gronwald inequality) *Let $L, T \in \mathbb{R}_+$ and $\xi, f \in \mathcal{C}^0([0,T],\mathbb{R})$. If, for all $t \in [0,T]$,*

$$\xi(t) \le L\int_0^t \xi(s)\,ds + f(t),$$

then

$$\xi(t) \le f(t) + L\int_0^t e^{L(t-s)} f(s)\,ds.$$

Proof. Let us first consider the case in which $f \equiv 0$. In this case the Lemma asserts $\xi(t) \le 0$. Indeed, since ξ is a continuos function there exists $t_* \in [0,(2L)^{-1}] \cap [0,T] =: I_1$ such that $\xi(t_*) = \sup_{t\in I_1} \xi(t)$. But then,

$$\xi(t_*) \le L\int_0^{t_*} \xi(s)\,ds \le \xi(t_*)Lt_* \le \frac{1}{2}\xi(t_*)$$

which implies $\xi(t_*) \le 0$ and hence $\xi(t) \le 0$ for each $t \in I_1$. If $I_1 = [0,T]$, then we are done, otherwise letting $t_1 := (2L)^{-1}$ we have

$$\xi(t) \le L\int_{t_1}^t \xi(s)\,ds$$

and we can make the same argument as before in the interval $[t_1, 2t_1]$. Iterating we have $\xi(t) \le 0$ for all $t \in [0,T]$.

To treat the general case we reduce it to the previous one. Let

$$\zeta(t) := \xi(t) - f(t) - L\int_0^t e^{L(t-s)} f(s)\,ds.$$

Then

$$\begin{aligned}
\zeta(t) &\le L\int_0^t \xi(s)\,ds - \int_0^t Le^{L(t-s)} f(s)\,ds \\
&= L\int_0^t \zeta(s)\,ds + L\int_0^t \left\{ f(s)ds + L\int_0^s e^{L(s-\tau)} f(\tau)d\tau \right\} \\
&\quad - \int_0^t Le^{L(t-s)} f(s)\,ds.
\end{aligned}$$

[16] Of course, this is the case only for regular vector fields. For other possibilities think of the case of collisions among planets.

Next, notice that

$$\begin{aligned}\int_0^t dsL\int_0^s e^{L(s-\tau)}f(\tau)\,d\tau &= L\int_0^t d\tau f(\tau)\int_\tau^t dse^{L(s-\tau)}\\ &= \int_0^t f(s)\{e^{L(t-s)}-1\}ds.\end{aligned}$$

Thus,

$$\zeta(t)\le L\int_0^t \zeta(s)\,ds.$$

We have then reduced the problem to the previous case which implies that it must be $\zeta(t)\le 0$ from which the Lemma follows. □

Let us see the usefulness of the above Lemma in a concrete example. Let $L(\mathcal{B},\mathcal{B})$ be the Banach space of the linear bounded operators from $\mathcal{B}$ to $\mathcal{B}$.[17]

Lemma 1.1.9 *For each $A\in\mathcal{C}^1_{\text{loc}}(\mathbb{R},L(\mathcal{B},\mathcal{B}))$, consider the Cauchy problem*

$$\begin{aligned}&\dot{x}(t)=A(t)x(t)\\ &x(0)=x_0.\end{aligned}$$

If $\|A(t)\|\le L$ for all $t\in\mathbb{R}$, then $\|x(t)\|\le e^{Lt}\|x_0\|$ for all $t\in\mathbb{R}$. In particular, the solution is defined on all $\mathbb{R}$.

Proof. If we write the equation in the equivalent integral form we have

$$\|x(t)\|\le\|x_0\|+\int_0^t\|A(s)x(s)\|\,ds\le\|x_0\|+L\int_0^t\|x(s)\|\,ds.$$

Let $\xi(t):=\|x(t)\|$, apply Lemma 1.1.8 for any $T\in\mathbb{R}_+$, the Lemma follows. □

Problem 1.2 *Explain why Lemma 1.1.9 does not apply to the following setting: $\mathcal{B}=\mathcal{C}^1(\mathbb{R}^n,\mathbb{R})$ and*

$$\dot{x}(t,z)=\alpha(z,t)\partial_z x(t,z),$$

for some $\alpha\in\mathcal{C}^1(\mathbb{R}^n,\mathbb{R})$, $\alpha(z,T+t)=\alpha(z,t)$, $T>0$. Compare with Problem 1.24.

[17]The norm of $L\in L(\mathcal{B},\mathcal{B})$ is given by $\|L\|:=\sup_{\substack{v\in\mathcal{B}\\ \|v\|=1}}\|Lv\|$. If $\mathcal{B}=\mathbb{R}^d$, then $L(\mathcal{B},\mathcal{B})$ is just the vector space of the $d\times d$ matrices.

1.1.3 Flows

In this section we analyze the case in which the vector field is time independent and grows at most linearly.

Lemma 1.1.10 *Given* $V \in \mathcal{C}^1_{\rm loc}(\mathcal{B}, \mathcal{B})$*, if there exists* $L, M \geq 0$ *such that* $\|V(x)\| \leq L\|x\| + M$*, then the solution of* (1.1.1) *exists for all times and for all initial conditions.*

PROOF. We argue by contradiction. Choose any initial condition $x_0 \in \mathcal{B}$ and let $I(x_0) = (-\delta_-(x_0), \delta_+(x_0))$ be the maximal interval on which the solution is defined. If $\delta_+(x_0) < \infty$, then for each $t \leq \delta_+(x_0)$

$$\|x(t)\| \leq \|x_0\| + L \int_0^t \|x(s)\| ds + Mt.$$

Thus Gronwald inequality implies

$$\|x(t)\| \leq e^{Lt} \left\{ \|x_0\| + ML^{-1} \right\}$$

for $t \in [0, \delta_+(x_0))$. Then, by Lemma 1.1.6, the solution can be extended, contrary to the assumption that $(-\delta_-(x_0), \delta_+(x_0))$ was the maximal interval. A similar argument holds for negative t. □

For each $x_0 \in \mathcal{B}$ and $t \in \mathbb{R}$ let $x(t, x_0)$ be the solution of (1.1.1) at time t.

Lemma 1.1.11 *For each* V *as in Lemma 1.1.10, setting* $\phi_t(x_0) := x(t, x_0)$*,* $\phi_{-t} = \phi_t^{-1}$ *for* $t \geq 0$*, we have that* $(\mathcal{B}, \phi_t)$*,* $t \in \mathbb{R}$*, is a Dynamical System.*

PROOF. All we need to prove is that ϕ_t is an action of $\mathbb{R}$ on $\mathcal{B}$. First of all note that ϕ_t is indeed invertible. If not then there would be $x, x' \in \mathcal{B}$ such that $\phi_t(x) = \phi_t(x')$. But then the uniqueness of the solutions of the ODE implies $x = x'$. Moreover it is easy to check that $\phi_{-t}(x_0) = x(-t, x_0)$. Finally, $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$. □

Remark 1.1.12 *We have thus proved that a large class of vector fields gives rise to* flows*.*

1.1.4 Dependence on a parameter

Having established the existence and uniqueness of the solution, the next natural questions present itself.

Question 2 *How do the solutions depend on the initial condition? How do the solutions depend on a change of the vector field?*

To discuss such issues it is convenient to analyze first the second question. More precisely, given $V \in \mathcal{C}^2_{\text{loc}}(\mathcal{B}\times\mathbb{R}\times\mathbb{R}^d, \mathcal{B})$ we consider the Chauchy problem

$$\begin{aligned} \dot{x}(t) &= V(x(t), t, \lambda) \\ x(0) &= x_0. \end{aligned} \tag{1.1.3}$$

Clearly the solution will depend on the parameter λ. The question is then: calling $x(t, \lambda)$ the solution of (1.1.3), for a given $t \in \mathbb{R}$ what can we say about the function $x(t, \cdot)$?

For simplicity let us consider the case $V \in \mathcal{C}^2(\mathcal{B} \times \mathbb{R} \times \mathcal{B}_1, \mathcal{B})$, the more general case $V \in \mathcal{C}^2_{\text{loc}}(\mathcal{B} \times \mathbb{R} \times \mathcal{B}_1, \mathcal{B})$ is similar and is left to the reader.

Theorem 1.1.13 (Smooth dependence on a parameter) *Given two Banach spaces $\mathcal{B}, \mathcal{B}_1$, let $V \in \mathcal{C}^2(\mathcal{B} \times \mathbb{R} \times \mathcal{B}_1, \mathcal{B})$. Let $X(t, x_0, \lambda)$ be the unique solution of (1.1.3), then $X(t, x_0, \cdot) \in \mathcal{C}^1_{\text{loc}}(\mathcal{B}_1, \mathcal{B})$.*

Proof. For each $x_0 \in \mathcal{B}$ consider the ODE for $\xi \in \mathcal{C}^1_{\text{loc}}(\mathbb{R} \times \mathcal{B}_1, L(\mathcal{B}_1, \mathcal{B}))$

$$\begin{aligned} \dot{\xi}(t, \lambda) &= \partial_x V(X(t, x_0, \lambda), t, \lambda) \cdot \xi(t, \lambda) + \partial_\lambda V(X(t, x_0, \lambda), t, \lambda) \\ \xi(0, \lambda) &= 0. \end{aligned} \tag{1.1.4}$$

We claim that $\xi(t) = \partial_\lambda X(t, x_0, \lambda)$.[18] To verify the claim it suffices to prove that there exists $C > 0$ such that, for $h \in \mathcal{B}_1$ small enough, if $\zeta(t, h, \lambda) := X(t, x_0, \lambda+h) - X(t, x_0, \lambda) - \xi(t)h$, then $\|\zeta(t, h)\| \leq C\|h\|^2$. By Taylor formula we have[19]

$$\begin{aligned} \dot{\zeta}(t, h) &= V(X(t, x_0, \lambda + h), t, \lambda + h) - V(X(t, x_0, \lambda), t, \lambda) \\ &\quad - \partial_x V(X(t, x_0, \lambda), t) \cdot \xi(t)h - \partial_\lambda V(X(t, x_0, \lambda), t, \lambda)h \\ &= \partial_x V(X(t, x_0, \lambda), t) \cdot \zeta(t, h) + R(t) \end{aligned} \tag{1.1.5}$$

where, in the last line, we have used

$$\begin{aligned} &V(X(t, x_0, \lambda + h), t, \lambda) - V(X(t, x_0, \lambda), t, \lambda) \\ &= \partial_x V(X(t, x_0, \lambda), t, \lambda) \cdot (X(t, x_0, \lambda + h), t, \lambda) - X(t, x_0, \lambda)) \\ &\quad + \mathcal{O}(\|X(t, x_0, \lambda + h), t, \lambda) - X(t, x_0, \lambda)\|^2), \end{aligned}$$

and

$$\begin{aligned} \|R(t)\| &\leq C\left(\|X(t, x_0, \lambda + h) - X(t, x_0, \lambda)\|^2 + \|h\|^2\right) \\ &\leq 2C(\|\zeta(t, h)\|^2 + (1 + \|\xi(t)\|^2)\|h\|^2). \end{aligned}$$

[18] If $\mathcal{B} = \mathbb{R}^d$ e $\mathcal{B}_1 = \mathbb{R}^m$ then ξ is just a $d \times m$ matrix.

[19] Note that we cannot Taylor expand $X(t, x_0, \lambda + h)$ with respect to h, since we do not know yet that X is differentiable with respect to λ.

with $C = \|V\|_{\mathcal{C}^2}$. Note that $\zeta(0) = 0$. We can then conclude by using Lemma 1.1.8. Indeed such a Lemma applied to (1.1.4) implies $\|\xi(t)\| \leq e^{C_1 t}$, for some $C_1 > 0$. Next, let $T > 0$ be the maximal time such that $\|\zeta(t,h)\| \leq 1/2$ and $e^{2C_1 T} \leq 2$. Then, for $t \leq T$, (1.1.5) yields

$$\|\zeta(t,h)\| \leq \int_0^t 2C\|\zeta(s)\|ds + 3\|h\|^2$$

and Lemma 1.1.8, again, implies the announced estimate. □

Problem 1.3 *Prove the analogous of Theorem 1.1.13 when* $V \in \mathcal{C}^1_{\text{loc}}$.

The above theorem allow to easily prove the following fundamental result on the smooth dependence on parameters of an ODE.

Theorem 1.1.14 (Smooth dependence on initial conditions) *Let* $V \in \mathcal{C}^r(\mathcal{B} \times \mathbb{R}, \mathcal{B})$, $r \geq 1$. *For* $x_0 \in \mathcal{B}$ *let* $X(t, x_0)$ *be the unique solution of* (1.1.1). *Then, for each* $t \in \mathbb{R}$, $X(t, \cdot) \in \mathcal{C}^r_{\text{loc}}(\mathcal{B}, \mathcal{B})$. *Moreover,* $\xi = \partial_{x_0} X$ *solves*

$$\begin{aligned} \dot{\xi}(t) &= \partial_x V(X(t, x_0), t) \cdot \xi(t) \\ \xi(0) &= \mathbb{1}. \end{aligned} \tag{1.1.6}$$

Proof. Set $z = x - x_0$ and consider the resulting equation

$$\begin{aligned} \dot{z} &= V(z + x_0, t) =: \bar{V}(z, t, x_0) \\ z(0) &= 0. \end{aligned}$$

One can then consider x_0 as an external parameter, applying Theorem 1.1.13 yields the result for $r = 1$. On the other hand, (1.1.6) is itself a differential equation depending on a parameter with a $\mathcal{C}^1$ vector field and a $\mathcal{C}^1$ dependence on the parameter x_0 , provided $r \geq 2$. So we can apply Theorem 1.1.13 again, and so on for r times, which proves the theorem. □

1.1.5 ODE on Manifolds–few words

Let us remind that a *topological manifold* is a second countable Hausdorff space which is locally homeomorphic to Euclidean space. A *chart* over a topological manifold M is a pair (U, ϕ) such that $U \subset M$ is an open set and $\phi : U \to \mathbb{R}^n$, for some $n \in \mathbb{N}$, is an homeomorphism between U and the open set $\Phi(U)$. An *atlas* on a topological manifold is a countable collection of charts $\{(U_\alpha, \phi_\alpha)\}$. We say that an atlas is $\mathcal{C}^k$ if $\phi_\alpha \circ \phi_\beta^{-1}$ is $\mathcal{C}^k$ when is defined. We say that two $\mathcal{C}^k$ atlas are equivalent if their union is a $\mathcal{C}^k$ atlas. A $\mathcal{C}^k$ manifold is a topological manifold equipped with an equivalence class of $\mathcal{C}^k$ atlas (often called a *differentiable structure*).

Although most often we will be concerned with manifolds embedded in some $\mathbb{R}^d$, also other possibilities will be relevant. Let us consider two examples.

Problem 1.4 *Show that $\mathbb{R}^d$ is a $\mathcal{C}^\infty$ manifold.*[20]

Problem 1.5 *Let $f \in \mathcal{C}^k(\mathbb{R}^d, \mathbb{R})$, and consider $M = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \,:\, y = f(x)\}$. Consider the atlas consisting of the chart (M, ϕ) where $\phi(x, y) = x$. This is a $\mathcal{C}^\infty$ manifold.*

Problem 1.6 *Check that $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is a $\mathcal{C}^\infty$ manifold.*

Given two differentiable manifolds ($\mathcal{C}^k$ manifolds with $k \geq 1$) M_1, M_2 and a map $f : M_1 \to M_2$ we say that $f \in \mathcal{C}^r(M_1, M_2)$, $r \leq k$, if for each atlas $\{(U_\alpha, \phi_\alpha)\}$ of M_1 and atlas $\{(V_\beta, \psi_\beta)\}$ of M_2, holds true $\psi_\beta \circ f \circ \phi_\alpha^{-1} \in \mathcal{C}^r$ on their domains of definition.

Given a differentiable manifold M and $x \in M$, we say that two curves $\gamma_1, \gamma_2 \in \mathcal{C}^1((-1, 1), M)$, such that $\gamma_1(0) = \gamma_1(0) = x$, are equivalent at x if for each chart (U, ϕ) such that $x \in U$ holds true $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$. A *tangent vector* at x is an equivalence class of curves.

Problem 1.7 *Show that if M is localy homeorphic to $\mathbb{R}^d$, then the set of tangent vectors at any $x \in M$ form canonically a d dimensional vector space.*[21]

We will use $\mathcal{T}_x M$ to designate the *tangent space* at x, that is the set of the tangent vectors at x. The tangent bundle is the disjoint union of the tangent spaces, i.e. $\mathcal{T}M = \cup_{x \in M}\{x\} \times \mathcal{T}_x M$. Finally, a *vector field* is a section of the tangent bundle, i.e. $\tilde{V} : M \to \mathcal{T}M$ such that $\tilde{V}(x) = (x, V(x))$, $V(x) \in \mathcal{T}_x M$. Form now on, with a slight abuse of notation, we will identify $\tilde{V}$ with V. Also, given $f \in \mathcal{C}^1(M_1, M_2)$, since the image of a $\mathcal{C}^1$ curve is a $\mathcal{C}^1$ curve, ve have naturally defined a map $f_* : \mathcal{T}M_1 \to \mathcal{T}M_2$.

Problem 1.8 *If $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^n)$ discuss the relation between f_* and the derivative Df.*

We have finally the language to define O.D.E. on manifolds, in fact the Cauchy problem is exactly given again by (1.1.1), only now V is a, possibly time dependent, $\mathcal{C}^1$ vector field.

Problem 1.9 *Suppose that x_0 belongs to some chart (U, ϕ), show that the solution of*

$$\begin{aligned} \dot{x} &= V(x, t) \\ x(0) &= x_0 \end{aligned}$$

for a sufficiently small time can be obtained by the solution of an appropriate O.D.E. in $\phi(U)$.

[20] Note that, contrary to $\mathcal{C}^k$, $\mathcal{C}^\infty$ is not a Banach space (there is no good norm). It is possible to give to it the structure of a Fréchet space [RS80], but we will refrain from such subtleties. We just consider $\mathcal{C}^\infty = \cap_{n \in \mathbb{N}} \mathcal{C}^n$ as a vector space.

[21] If (U, ϕ) is a chart containing x, and γ_1, γ_2 two curves, think of the curves $\gamma_\lambda(t) = \gamma_1(\lambda t)$ and $\phi^{-1}(\phi(\gamma_1(t)) + \phi(\gamma_2(t)) - \phi(x))$.

Problem 1.10 *Given a finite atlas $\{(U_\alpha, \phi_\alpha)\}$, show that there exists a smooth partition of unity subordinated to the atlas, that is a collections $\{\varphi_\alpha\} \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $\sum_\alpha \varphi_\alpha = 1$ and $\operatorname{supp} \varphi_\alpha \subset U_\alpha$.*

Problem 1.11 *Given a smooth vector field V consider*

$$\begin{aligned} \dot{x} &= V(x) \\ x(0) &= x_0 \end{aligned} \tag{1.1.7}$$

with $x_0 \in U_\alpha$ for some element of an atlas $\{(U_\alpha \phi_\alpha)\}$. Let $z_\alpha(t)$ be the solution of

$$\begin{aligned} \dot{z}_\alpha &= (\phi_\alpha)_* V(z_\alpha) \\ z_\alpha(0) &= \phi_\alpha(x_0) \end{aligned}$$

and suppose that $\phi_\alpha^{-1}(z(1)) \in U_\beta$. Consider then the solution of

$$\begin{aligned} \dot{z}_\beta &= (\phi_\beta)_* V(z_\beta) \\ z_\beta(1) &= \phi_\beta(\phi_\alpha^{-1}(z_\alpha(1))). \end{aligned}$$

Show that there exists $t_1 > 1$ such that

$$\begin{aligned} x(t) &= \phi_\alpha^{-1}(z_\alpha(t)) \quad \text{for } t \in [0, 1] \\ x(t) &= \phi_\beta^{-1}(z_\beta(t)) \quad \text{for } t \in (1, t_1) \end{aligned}$$

is a solution of (1.1.7) in the time interval $[0, t_1)$.

Remark 1.1.15 *We have seen that the theory of ODE on manifolds can be reduced locally to the case of $\mathbb{R}^d$. Yet, the reader should be aware that the global properties of the solutions can be very different. We will comment at length on this issue later on.*

1.2 Linear ODE and Floquet theory

Let us briefly discuss the simplest possible differential equation: the affine ones. For simplicity, we restrict ourselves to the case $\mathcal{B} = \mathbb{R}^d$ for some $d \in \mathbb{N}$.

1.2.1 Linear equations

Consider

$$\begin{aligned} \dot{x} &= Ax \\ x(0) &= x_0. \end{aligned} \tag{1.2.8}$$

Problem 1.12 *Show, by induction, that for each $n \in \mathbb{N}$ the solution of* (1.2.8) *satisfies*

$$x(t) = \sum_{k=0}^{n} \frac{1}{k!} A^k t^k x_0 + \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n A^{n+1} x(t_n).$$

Taking the limit for $n \to \infty$ in the above expression one readily obtains $x(t) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n x_0$. That this is a solution can be verified directly inserting this formula in (1.2.8) (and noticing that the series and the series obtained by deviating term by term are uniformly convergent). By the standard analytic functional calculus for matrices (and operators, see Appendix C) we can thus write $x(t) = e^{At} x_0$. The above discussion provides a general solution for all equations of the type (1.2.8).

In reality life it is not that simple: if one has a concrete matrix A and wants to compute e^{At}, this may be quite unpleasant. A general strategy, although not necessarily the simplest one, is to perform a linear change of variables $x = Uz$. Then $\dot{z} = U^{-1}AUz$, and U is chosen so that $\Lambda = U^{-1}AU$ is in Jordan normal form. Then

$$x(t) = Uz(t) = Ue^{\Lambda t} z_0 = Ue^{\Lambda t} U^{-1} x_0.$$

It suffices then to know how to take exponentials of Jordan blocks, and this can be computed by using the defining series.

Problem 1.13 *Compute $e^{\Lambda t}$ for*

$$\Lambda = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \; ; \quad \Lambda = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \; ; \quad \Lambda = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}.$$

Another, equivalent, point of view is to look for solutions of the type $x(t) = e^{at} v$, substituting in the first of (1.2.8) one obtains $av = Av$. Thus, as we know already, each eigenvalue of A provides a solution of (1.2.8) (ignoring the initial condition). If there exists real eigenvectors $\{v_i\}_{i=1}^d$ which span all $\mathbb{R}^d$ then one can write the general solution, depending on d parameters α_i, as $x(t) = \sum_{i=1}^d \alpha_i v_i e^{a_i t}$, where a_i is he eigenvalue associated to the eigenvector v_i. One can then satisfy the initial condition by solving $x_0 = \sum_{i=1}^d \alpha_i v_i$. The same can be done is the eigenvectors are complex, by working in $\mathbb{C}^d$ instead then $\mathbb{R}^d$. If Jordan blocks are present one can look for solutions of the form $x(t) = \sum_{k=0}^p \frac{1}{(p-k)!} t^k e^{at} v_k$, compare this formula with your solution of Problem 1.13.

Remark 1.2.1 *Note that if the matrix A does not have eigenvalues with zero real part, then (by spectral decomposition) one can write $\mathbb{R}^d = V_- \oplus V_+$, where*

$AV_\pm = V_\pm$ and A restricted to V_- has eigenvalues with negative real part while on V_+ has eigenvalues with positive real part. Hence if $x_0 \in V_-$ it will hold $\lim_{n\to\infty} x(t) = 0$, and if $x_0 \in V_+$ it will hold $\lim_{n\to\infty} \|x(t)\| = \infty$. If $x_0 \notin V_-$ we can write it as $x_0 = x_- + x_+$, where $x_\pm \in V_\pm$. Hence $\lim_{n\to\infty} \|x(t)\| = \infty$ and the trajectory will escape to infinity while getting exponentially close to the subspace V_+. This is our first long time *result.*

A slightly more complex situation is given by

$$\begin{aligned} \dot{x} &= Ax + b(t) \\ x(0) &= x_0, \end{aligned} \tag{1.2.9}$$

where $b \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^d)$. The solution of (1.2.9) is given by[22]

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}b(s)ds. \tag{1.2.10}$$

1.2.2 Floquet theory

Let us consider the simplest case of a linear time dependent equation: there exists a continuous function $A \in \mathcal{C}^0_{\rm loc}(\mathbb{R}, L(\mathbb{R}^d, \mathbb{R}^d))$ and $T \in \mathbb{R}_+$ such that, for all $t \in \mathbb{R}$, $A(t+T) = A(t)$. More precisely, let $\Phi(x_0, s, t)$ be the solution of the Cauchy problem[23]

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \\ x(s) &= x_0. \end{aligned} \tag{1.2.11}$$

Problem 1.14 *Verify the following facts for each $x_0, y_0 \in \mathcal{B}$ and for each $a, b, t, s, \tau \in \mathbb{R}$*

- $\Phi(ax_0 + by_0, s, t) = a\Phi(x_0, s, t) + b\Phi(y_0, s, t)$,
- $\Phi(x_0, s, t) = \Phi(\Phi(x_0, s, \tau), \tau, t)$,
- $\Phi(x_0, s + T, t + T) = \Phi(x_0, s, t)$.

By the first property of Problem 1.14 there exists $K \in \mathcal{C}^1_{\rm loc}(\mathbb{R}^2, L(\mathbb{R}^d, \mathbb{R}^d))$ such that $\Phi(x_0, s, t) = K(s, t)x_0$, the second property implies that $K(\tau, t)K(s, \tau) = K(s, t)$, the third that $K(s + T, t + T) = K(s, t)$. The next step is the first occurrence in this book of a very simply but very powerful idea to analyze dynamical systems: a Poincaré section. Essentially the idea consist in looking at the system only at specially selected moments in time. In this case it is convenient to look at $t \in \{nT\}_{n\in\mathbb{Z}}$. That is, we want to investigate $\Phi(x_0, 0, nT) =: F(x_0, n)$.

[22] Look for a solution of the form $x(t) = e^{At}z(t)$ and find the differential equation for z.

[23] The solution is well defined for all times by Lemma 1.1.10.

Lemma 1.2.2 *The couple* $(\mathbb{R}^d, F)$ *is a discrete Dynamical System.*

PROOF. We have to show that F is an action of $\mathbb{Z}$ on $\mathbb{R}^d$. Let $f(x_0) := F(x_0, 1)$.

$$\begin{aligned} F(x_0, n) &= \Phi(x_0, 0, nT) = \Phi(\Phi(x_0, 0, (n-1)T), (n-1)T, nT)) \\ &= \Phi(\Phi(x_0, 0, (n-1)T), 0, T)) = f(\Phi(x_0, 0, (n-1)T)) = f^n(x_0). \end{aligned}$$

In addition, note that the uniqueness of the solutions of the ODE implies that if $f(x_0) = 0$, then $x_0 = 0$. Now, by construction, $f(x_0) = K(0,T)x_0$, thus $K(0,T)$ is an invertible matrix. Hence $F(x_0, -n) = f^{-n}(x_0)$ for all $n \in \mathbb{N}$. □

By using the functional calculus (see Problem C.19) one can define $B := T^{-1} \ln K(0,T)$, so $e^{BT} = K(0,T)$. Let us now consider $P(t) := K(0,t)e^{-Bt}$.

$$\begin{aligned} P(t+T) &= K(0, t+T)e^{-B(t+T)} = K(T, t+T)K(0,T)K(0,T)^{-1}e^{-Bt} \\ &= K(0,t)e^{-Bt} = P(t). \end{aligned}$$

We have just proven the following result.

Theorem 1.2.3 (Floquet theorem) *The solutions of the equation* (1.2.11) *can be written as* $x(t) = P(t)e^{Bt}K(s,0)x_0$ *where* $P(t+T) = P(t)$ *is periodic.*

Note that the matrix B can be complex valued. This can be avoided at a little extra cost.

Problem 1.15 *Prove that the solutions of the equation* (1.2.11) *can be written as* $x(t) = P(t)e^{Bt}x_0$ *where* B *is real and* $P(t+2T) = P(t)$ *is periodic of period* $2T$.

Note that Theorem 1.2.3 implies that the long time behavior is completely contained in the eigenvalues of the matrix B often called *floquet exponents*.

Problem 1.16 *Find the solutions of*

$$\dot{x} = a(t)Ax$$

where $a \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ *is periodic of period* T *and* A *is a fixed matrix.*

Problem 1.17 *Given a fixed matrix* A *and a function at matrix values* $B(t)$ *of period* T, *consider the equation* $\dot{x} = (A + \varepsilon B(t))x$, $\varepsilon \in \mathbb{R}$. *Show that, for* ε *small enough, calling* ν_i *the Floquet exponents and setting* $\lambda_i = e^{\nu_i}$ *(often called* Floquet multiplier*), the* λ_i *are* ε*-close to the eigenvalues of* A.

1.3 Qualitative study of ODE

The previous discussion has shed some light on the behavior of linear ODE, unfortunately the interesting ODE are typically non linear. Although some nonlinear ODE can be solved explicitly (see any ODE book for examples) typically this is not possible, hence the need of a qualitative theory. As for the qualitative study of functions this can be done quite naively in one dimension, while higher dimensions requires some non trivial theory. Let us see such a naive qualitative theory for ODE via few examples.

1.3.1 The one dimensional case

This situation is very similar to the study of the graph of a function of one variable. Indeed to draw the graph one studies the first derivative and here the first derivative is specified by the equation. Let us consider a couple of simple examples. Consider

$$\begin{aligned}\dot{x} &= e^{-x^2} + x - 2 = V(x) \\ x_0 &= 0.\end{aligned}$$

One cannot integrate the function $V(x)^{-1}$ (which would yield an explicit solution of the ODE), yet from the equation follows that there exists a close to 2 such that $\dot{x}$ is negative if $x \le a$ and positive otherwise. This implies that the solution starts to be decreasing and keeps decreasing forever.

Next, consider

$$\begin{aligned}\dot{x} &= 1 - 2tx \\ x_0 &= a.\end{aligned}$$

Such an equation cannot be solved by separation of variables, yet the above arguments still apply. In particular, for $t \ge 0$, we have $\dot{x}(t) < 0$ iff $x(t) > \frac{1}{2t}$. On the other hand if $x(t) > \frac{1}{2t}$ it will be so forever. In fact, consider $g(t) = x(t) - \frac{1}{2t}$, then $g'(t) = \dot{x}(t) + \frac{1}{2t^2}$. So if $g(t_*) = 0$, then $g'(t_*) > 0$ hence for $t < t_*$ one has $g(t) < 0$. Thus the solution will increase until it will intersect the curve $\frac{1}{2t}$ and then it will start decreasing but always staying above such a curve. Accordingly, for $t \ge t_*$ we can write $x(t) = \frac{1+\alpha(t)}{2t}$ with $\alpha \ge 0$. Then $\dot{x}(t) = -\alpha(t)$, that is

$$\frac{1}{2t} \le x(t) = \frac{1}{2t_*} - \int_{t_*}^{t} \alpha(s) ds \tag{1.3.12}$$

moreover $-\frac{1+\alpha(t)}{2t^2} + \frac{\dot{\alpha}(t)}{2t} = -\alpha(t)$

$$\dot{\alpha}(t) = -(2t - \frac{1}{t})\alpha(t) + \frac{1}{t}$$

which means that either $\alpha(t) \leq \frac{1}{2t^2-1}$ or it is decreasing. But if it is decreasing it must decrease to zero otherwise (1.3.12) would be false for large t. Accordingly it must be $\lim_{t\to\infty} \alpha(t) = 0$.

1.3.2 Autonomous equations in two dimensions

In this case the basic idea is to consider one component as a function of the other and in this way reduce to the previous case. Let us see some examples.

Van Der Pol equation

Consider the equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (1-3x^2)y - x. \end{aligned} \tag{1.3.13}$$

Clearly $(0,0)$ is the unique zero of the vector field. If we linearise (1.3.13) around zero we have

$$\frac{d}{dt}(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix has eigenvalues $\lambda_{\pm} = \frac{1\pm\sqrt{3}i}{2}$ hence the fixed point is repelling and the solutions spiral away from it.

The next question is if a similar motion takes place also far away from the origin. To this end we want to forget the time dependence and concentrate only on the shape of the trajectories. Thus we can represent trajectories on the xy plane. Indeed, apart from the point $(0,0)$, either $\dot{x}$ or $\dot{y}$ are different from zero. In the first case one can locally invert $x(t)$ and write $y(x) = y(t(x))$. When this is possible one obtains

$$\frac{dy}{dx} = 1 - 3x^2 - \frac{x}{y},$$

which can be studied as in the previous examples. With a bit of work one can see that the trajectory spirals around zero, but exactly how?

To better understand the behaviour of the solution we introduce a "Lyapunov" like function.

$$L(x,y) = 2(x - x^3 - y)^2 + (x-y)^2 + 3x^2.$$

If $(x(t), y(t))$is a solution of (1.3.13), then a direct computation yields

$$\frac{d}{dt}L(x(t),y(t)) = x^2\left[6 - x^2 - 3(x-y)^2 - 3y^2\right].$$

Accordingly, L is decreasing outside an ellipse. Since $2ab \leq a^2 + b^2$,[24]

$$\begin{aligned} L(x, y) &= 3(x-y)^2 - 4(x-y)x^3 + 2x^6 + 3x^2 \geq (x-y)^2 + 3x^2 \\ &= 4x^2 - 2xy + y^2 \geq 2x^2 + \frac{1}{2}y^2. \end{aligned}$$

Hence, the level sets $K_\alpha = \{(x, y) \in \mathbb{R}^2 \ : \ L(x, y) \leq \alpha\}$ are contained in the ellipses $\{(x, y) \in \mathbb{R}^2 \ : \ 2x^2 + \frac{1}{2}y^2 \leq \alpha\}$ and hence are compact.

Thus, far away from the origin the trajectory spirals inwardly. It follows, by the continuity with respect to the initial data, that there exists an $a_* \geq 0$ such that the corresponding solution is a periodic orbit.

Lotka-Volterra equation

$$\begin{aligned} \dot{x} &= ax - Ax^2 - \lambda xy \\ \dot{y} &= -dy + \lambda xy. \end{aligned}$$

This equation is meant to describe the evolution of two populations one feeding on the other (predator-prey). It also has periodic solutions, try to prove it using qualitative methods.

Second order in one dimension

Consider the equation

$$\begin{aligned} &\ddot{x} = -\gamma \dot{x} + \frac{x^2}{1 + x^4} \\ &x(0) = 0; \quad \dot{x}(0) = v. \end{aligned}$$

Setting $(z, w) = (x, \dot{x})$, we can write it as

$$\begin{aligned} \dot{z} &= w \\ \dot{w} &= -\gamma w + \frac{z^2}{1 + z^4} \end{aligned}$$

which is the type discussed above.

Clearly if we consider still higher dimensional cases the above naive approach cannot help us very much, hence the need of a more sophisticated theory.

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[24] It follows from $(a - b)^2 \geq 0$.

Problems

1.18. Given two Banach spaces $\mathcal{B}_1, \mathcal{B}_2$ and a function$f : \mathcal{B}_1 \to \mathcal{B}_2$ we can define the partial derivative at $x \in \mathcal{B}_1$ in the direction $v \in \mathcal{B}_1$ (Gâteaux derivative) by

$$\partial_v f(x) = \lim_{h\to 0} h^{-1}\left[f(x+hv) - f(x)\right],$$

if the limit exists. On the other hand we say that f is Fréchet differentiable at x if there exists $A \in L(\mathcal{B}_1, \mathcal{B}_2)$ (the space of the continous linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$) such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0,$$

and A is called the Fréchet differential at of f at x (often written $Df(x)$). Show that if f is Fréchet differentiable at zero, then it is continuous and Gâteaux differentiable.

1.19. Let $f \in \mathcal{C}^0(\mathcal{B}_0, \mathcal{B}_1)$ and $g \in \mathcal{C}^0(\mathcal{B}_1, \mathcal{B}_2)$ such that f is Fréchet differentiable at $x \in \mathcal{B}_0$ and g is Fréchet differentaible at $f(x) \in \mathcal{B}_1$. Show that $g \circ f \in \mathcal{C}^0(\mathcal{B}_0, \mathcal{B}_2)$ is Fréchet differentaible at x and that $D(g\circ f)(x) = Dg(f(x)) \cdot Df(x) \in L(\mathcal{B}_0, \mathcal{B}_2)$. Of course, this is nothing else than a glorified version of the *chain rule*.

1.20. Given a compact interval $I \subset \mathbb{R}$, a Banach space $\mathcal{B}$, and a continuous function $f \in \mathcal{C}^0(I, \mathcal{B})$, shows that one can define the equivalent of the Riemannian integral.

1.21. Prove the fundamental theorem of calculus in this setting. That is, for $f \in \mathcal{C}^1(\mathcal{B}_1, \mathcal{B}_2)$ let $Df(x) \in L(\mathcal{B}_1, \mathcal{B}_2)$ be the Fréchet differential at $x \in \mathcal{B}_1$, then for each $x, y \in \mathcal{B}_1$

$$f(y) = f(x) + \int_0^1 Df(x + t(y-x)) \cdot (x-y)dt.$$

1.22. Show that, for all $f \in \mathcal{C}^0([a,b], \mathcal{B})$,

$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b \|f(t)\| dt.$$

1.23. Study the solutions of the following equations for all possible initial conditions and $p \in \mathbb{N}$

$$\dot{x} = |x|^p$$
$$\dot{x} = x(\ln|x|)^p$$

1.24. Let $K \in \mathcal{C}^1(\mathbb{R} \times [0, 1])$. Show that the equation

$$\partial_t u(t, s) = \int_0^1 K(t + s, \tau) u(t, \tau)^2 d\tau$$
$$u(0, s) = s^2.$$

has a unique continuos solution for t small enough.

1.25. Under the same hypotheses of Problem 1.17 show that if $\int_0^T B(s)ds = 0$ and the eigenvalues of A have all multiplicity one, then the Floquet multiplier differ from the eigenvalues of e^{AT} only of order ε^2.

1.26. Study the equation

$$(1 + x)y\dot{y} + (x + y^2) = 0.$$

1.27. Study the equation (Bernoulli)

$$\dot{y} + p(x)y = q(x)y^n.$$

1.28. Study the equation

$$\ddot{x} = -\gamma\dot{x} - x^3.$$

Hints to solving the Problems

In this section, and in the parallel sections in later chapters, I give hints for the solution of some of the Problems.

It is a very good idea to try very hard to solve the problems *before* looking at the hints: it is impossible to appreciate the solution if one has no feeling for the difficulties in the problem. The only way I know to get such a feeling is to *seriously* try to solve it.

Also, keep in mind that I suggest one way to proceed, often other ways are possible and maybe better.

1.1 The proof is the same as the standard proof for the case $\mathcal{B} = \mathbb{R}^d$. However to see this you have to do Problems 1.18 and 1.20 to understand exactly what the derivate and integral mean in this more general case.

1.12 For $n = 0$ it is just (1.1.2). To verify it for any n it suffices to show that

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n 1 = \frac{t^n}{(n+1)!}.$$

This follows since the domain of integration is $D = \{x \in [0, t]^{n+1} : t_{n+1} \leq t_n \leq \cdots \leq t\}$. On the other hand, for each permutation σ of the set $\{1, \ldots, n+1\}$ the sets $D_\sigma = \{x \in [0,t]^{n+1} \,:\, t_{\sigma_{n+1}} \leq t_{\sigma_n} \leq \cdots \leq t\}$ have the same measure, all the D_σ are disjoint and the union of all of them gives $[0,t]^{n+1}$.

1.15 First notice that if a matrix has no eigenvalues on the negative axis then the contour γ in C.3.2 can be taken symmetric around the real axis and, by using C.3.2 with the standard definition of ln with a cut on the negative real axis, this defines $\ln K(0,T)$ with real entries (since the formula for his complex conjugate is the same). In general use the spectral decomposition to write $K(0,T) = C + D$ where $\sigma(C) \cap \mathbb{R}_- = \emptyset$ and $\sigma(D) \subset \mathbb{R}_-$. Then $\sigma(D^2) \subset \mathbb{R}_+$, hence $B = \frac{1}{T}\ln C + \frac{1}{2T}\ln D^2$ is real and $e^{2BT} = C^2 + D^2 = K(0,T)^2$. The rest of the argument is as before.

1.17 Show that the solution satisfies

$$x(t) = e^{At}x_0 + \varepsilon \int_0^t e^{A(t-s)}B(s)x(s)ds.$$

and apply the perturbation theory in Appendix C.

1.20 Let $I = [a,b]$. Since the function is continuos, it is uniformly continuous, hence for $\varepsilon > 0$ there exists $\delta > 0$ such that, for each partition $\xi = \{[x_0,x_1], \ldots, [x_{n-1},x_n]\}$, $x_0 = a, x_n = b$, $x_{n+1} - x_n \leq \delta$, holds $\sup_{z,y\in[x_{n+1},x_n]} \|f(z) - f(y)\| \leq \varepsilon$. Accordingly, for each choice of $z_n, y_n \in [x_{n+1}, x_n]$ we have

$$\left\| \sum_{k=0}^{n-1} f(z_k)(x_{k+1} - x_k) - \sum_{k=0}^{n-1} f(y_k)(x_{k+1} - x_k) \right\| \leq \varepsilon.$$

By similar arguments one can compare the sum defined on one partition with the sum defined on a finer partition. Finally sum on different partitions can be compared with the sum on the coarser partition finer of both. This shows that all sufficiently fine partitions yield the same approximate value, hence one can consider the partitions $\xi_n = \{[a + i\frac{b-a}{n}, a + (i+1)\frac{b-a}{n}]\}_{i=0}^{n-1}$ and define

$$\int_I f(t)dt := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a + i\frac{b-a}{n})\frac{b-a}{n}.$$

By the above discussion this is equivalent to the same limit taken along any other partition the diameter of which elements tend uniformly to zero.

1.24 Consider the Banach space $\mathcal{B} = \mathcal{C}^0([0,1],\mathbb{R})$. Then $u(t,\cdot) \in \mathcal{B}$ and one can apply Theorem 1.1.2.

1.25 By Problem 1.17 we know that the solution at time T is given by the matrix $D(\varepsilon) := e^{AT}\left[\mathbb{1} + \varepsilon\int_0^T e^{-As}B(s)e^{As}ds\right]$. By the results in Appendix C it follows that, for ε small enough, the eigenvalues of $D(\varepsilon)$ are still simple and analytic on ε. Thus, let $\lambda(\varepsilon)$ one of such eigenvalues and $\Pi(\varepsilon)$ the associated eigenprojector. We have $D(\varepsilon)\Pi(\varepsilon) = \lambda(\varepsilon)\Pi(\varepsilon)$. Differentiating yields $\dot D(\varepsilon)\Pi(\varepsilon)+D(\varepsilon)\dot\Pi(\varepsilon) = \dot\lambda(\varepsilon)\Pi(\varepsilon)+\lambda(\varepsilon)\dot\Pi(\varepsilon)$. Multiplying on the right by $\Pi(\varepsilon)$, since $\Pi(\varepsilon)D(\varepsilon) = D(\varepsilon)\Pi(\varepsilon)$, we have

$$\Pi(\varepsilon)\dot D(\varepsilon)\Pi(\varepsilon) = \dot\lambda(\varepsilon)\Pi(\varepsilon).$$

Since $\Pi(\varepsilon)v = \langle a(\varepsilon), v\rangle b(\varepsilon)$ for some vectors a, b analytic in ε, $\dot\lambda(\varepsilon) = \langle a(\varepsilon), \dot D(\varepsilon)b(\varepsilon)\rangle$. We can now apply such a general formula to our specific case:

$$\begin{aligned}\langle a(0), \dot D(0)b(0)\rangle &= \langle a(0), e^{AT}\int_0^T e^{-As}B(s)e^{As}b(0)ds\rangle \\ &= \langle a(0), e^{AT}\int_0^T e^{-As}B(s)e^{As}b(0)ds\rangle \\ &= \lambda(0)\int_0^T \langle a(0), B(s)b(0)\rangle ds = 0.\end{aligned}$$

Notes

This chapter is super condensed and has no pretension to exhaust the theory of ODE. If one wants to have a better understanding of the field and some ideas of how an ODE can be solved in special cases better consult [HS74, Arn92, CL55].

Appendices

Appendix A

Fixed Points Theorems (an idiosincratic selection)

In this appendix, I provide some standard and less standard fixed-point theorems. These constitute a very partial introduction to the subject. The choice of the topics is motivated by the needs of the previous chapters.

A.1 Banach Fixed Point Theorem

Theorem A.1.1 (Fixed point contraction) *Given a Banach space $\mathcal{B}$, a bounded closed set $A \subset \mathcal{B}$ and a map $K : A \to \mathcal{B}$ if*

i) $K(A) \subset A$,

ii) there exists $\sigma \in (0,1)$ *such that* $\|K(v) - K(w)\| \le \sigma\|v - w\|$ *for each* $v, w \in A$,

then there exists a unique $v_* \in A$ *such that* $Kv_* = v_*$.

PROOF. Since A is bounded $\sup_{x,y\in A} \|x-y\| = L < \infty$, i.e. it has a finite diameter. Let $a_0 \in A$ and consider the sequence of points defined recursively by $a_{n+1} = K(a_n)$ and the sequence of sets $A_0 = A$ and $A_{n+1} = K(A_n) \subset A$. Let $d_n := \sup_{x,y\in A_n} \|x - y\|$ be the diameter of A_n. Then if $x, y \in A_n$, we have

$$\|K(y) - K(x)\| \le \sigma\|x - y\| \le \sigma d_n.$$

That is $d_{n+1} \le \sigma d_n \le \sigma^n L$. This means that, for each $n, m \in \mathbb{N}$, $a_n, a_0 \in A$ and $a_m, a_{n+m} \in A_m$, hence $\|a_{n+m} - a_m\| \le \sigma^m L$. That is, $\{a_n\} \subset A$ is a Cauchy sequence and, being $\mathcal{B}$ a Banach space, it must have an accumulation point $v_* \in \mathcal{B}$. Moreover, since A is closed, it must be $v_* \in A$. Clearly

$$\begin{aligned}\|Kv_* - v_*\| &= \lim_{n\to\infty} \|Kv_* - a_n\| = \lim_{n\to\infty} \|Kv_* - Ka_{n-1}\| \\ &\leq \lim_{n\to\infty} \sigma \|v_* - a_{n-1}\| = 0.\end{aligned}$$

Hence, v_* is a fixed point. Next, suppose there exists $u \in A$ such that $Ku = u$. Then

$$\|u - v_*\| = \|K(u - v_*)\| \leq \sigma \|u - v*\|$$

implies $u = v_*$. □

Corollary A.1.2 *Given a Banach space $\mathcal{B}$ and a map $K : \mathcal{B} \to \mathcal{B}$ with the property that there exists $\sigma \in (0, 1)$ such that $\|K(v) - K(w)\| \leq \sigma\|v - w\|$ for each $v, w \in \mathcal{B}$, then there exists a unique $v_* \in \mathcal{B}$ such that $Kv_* = v_*$.*

Proof. To prove the theorem, for each $L \in \mathbb{R}_+$ consider the sets $B_L := \{v \in \mathcal{B} : \|v\| \leq L\}$. Then $\|K(v)\| \leq \|K(v) - K(0)\| + \|K(0)\| \leq \sigma\|v\| + \|K(0)\| \leq \sigma L + \|K(0)\|$. Thus, for each $L \geq (1 - \sigma)^{-1}\|K(0)\|$ we have that $K(B_L) \subset B_L$. The existence follows by applying Theorem A.1.1. The uniqueness follows from the same argument used at the end of the proof of Theorem A.1.1. □

A.2 Brouwer's Fixed Point Theorems

The next result is interesting since it relates the geometrical properties of the domain to the existence of a fixed point. However, one should note that the fixed point may not be unique. In the following, I provide an elementary proof. Other proofs based on algebraic topology exist, but are outside the scope of this book.
Before stating the Theorem, we need a combinatoric lemma about simplices that will be fundamental in the proof. First, recall the definition of simplex.

Definition A.2.1 (Geometric n-simplex) *Let $v_0, v_1, \ldots, v_n$ be* affinely independent points *in $\mathbb{R}^m$, $m \geq n$.*[1] *The n*-simplex *spanned by these points is*

$$\Delta^n(v_1, \ldots, v_{n+1}) = \Big\{x \in \mathbb{R}^m : x = \sum_{i=1}^{n+1} \lambda_i v_i,\ \lambda_i \geq 0,\ \sum_{i=1}^{n+1} \lambda_i = 1\Big\}.$$

The standard n-simplex *in $\mathbb{R}^{n+1}$ is*

$$\Delta^n := \Delta^n(e_1, \ldots, e_{n+1}) = \Big\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \geq 0,\ \sum_{i=1}^{n+1} x_i = 1\Big\}.$$

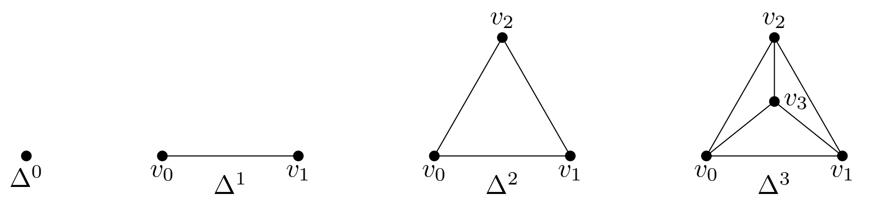

Figure A.1: Low-dimensional examples

Definition A.2.2 (Coloring) *Let Δ^n be the standard n–simplex, and let T be a simplicial subdivision (triangulation) of Δ^n. We call $\mathcal{V}(T)$ the set of vertices of the simplices in T. A* s-coloring *of T is a function $\ell : \mathcal{V}(T) \to \{1, \dots, n+1\}$ such that if v lies on the face of Δ^n opposite e_i (that is, $v_i = 0$), then $\ell(v) \neq i$. A simplex with vertices in $\mathcal{V}(T)$ is* fully colored *if, calling V the set of its vertices, $f|_V$ is invertible on its image.*

Lemma A.2.3 (Sperner's Lemma) *Let Δ^n be the standard n–simplex. Let T be a simplicial subdivision (triangulation) of Δ^n. Any s-colouring of T contains at least one fully colored simplex.*

Proof. The proof is by induction on n.
Let us start with $n = 1$. Here Δ^1 is the interval with endpoints e_0, e_1. The labeling rule forces e_0 to have label 0 and e_1 to have label 1. If all the subdivisions have vertices with the same color, then e_0 and e_1 would have the same color, contrary to the assumption.
Assume the lemma is true for dimension $n-1$. Consider Δ^n. By assumption, there is at least one fully colored $(n-1)$–simplex $\Delta(v_1, \dots v_n)$, $v_i \in \mathcal{V}(T)$, lying on the boundary $\partial\Delta^n$. Let $\Delta_1 := \Delta(v_1, \dots v_{n+1}) \in T$ be the n-simplex containing $\Delta(v_1, \dots, v_n)$. If $\ell(v_{n+1}) \neq \ell(v_i)$ for all $i \leq n$, then we have a fully colored simplex and we are done. Otherwise, there is a unique j such that $v_{n+1} = v_j$. We then consider the simplex $\Delta(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_{n+1})$, which is fully colored by construction. Note that each face of an element of T belongs to two elements of T, unless it belongs to $\partial\Delta^n$ in which case it belongs to a unique element of T. So there exists a unique $v_{n+2} \in \mathcal{V}(T)$ such that $v_{n+2} \neq v_j$ and $\Delta_2 := \Delta(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_{n+1}, v_{n+2}) \in T$. Again, either is fully colored, or we can erase the vertex with the same color as v_{n+2} and obtain another fully covered $n-1$-simplex. In this way, we can construct a sequence of simplices $\{\Delta_k\} \in T$.
Next, we show that $\Delta_k = \Delta_j \implies k = j$. Suppose the contrary, and let k be the smallest integer for which there exists $j < k$ such that $\Delta_k = \Delta_j$. Let $w_l^+ \in$

[1]A set of points $v_0, v_1, \dots, v_n \in \mathbb{R}^m$ is called *affinely independent* if the collection of vectors $v_1 - v_0,\ v_2 - v_0,\ \dots,\ v_n - v_0$ are linearly independent.

$\mathcal{V}(T)$ be the last vertex added to obtain Δ_l and $w_l^- \in \mathcal{V}(T)$ the unique vertex in Δ_l such that $\ell(w_l^+) = \ell(w_l^-)$. Consequently, if $V(\Delta)$ are the verteces of Δ, we have $V(\Delta_k) = [V(\Delta_{k-1})\setminus\{w_{k-1}^-\}]\cup\{w_k^+\}$ and $V(\Delta_{j+1}) = [V(\Delta_j)\setminus\{w_j^-\}]\cup\{w_{j+1}^+\}$. By contruction $\Delta(V(\Delta_k) \setminus \{w_k^+\})$, $\Delta(V(\Delta_k) \setminus \{w_k^-\})$, $\Delta(V(\Delta_j) \setminus \{w_j^-\})$, and $\Delta(V(\Delta_j) \setminus \{w_j^+\})$ are all fully coloured. Since, by hypothesis, $V(\Delta_k) = V(\Delta_j)$, it must be $w_k^\pm \in \{w_j^-, w_j^+\}$, otherwise $\Delta(V(\Delta_k) \setminus \{w_k^\pm\})$ could not be fully colored. So, either $w_k^\pm = w_j^\pm$, or $w_k^\pm = w_j^\mp$. If $w_k^+ = w_j^+$, and $j > 1$, then it must be $\Delta_{k-1} = \Delta_{j-1}$ contradicting the hypothesis that k is the smaller integer for which this happens. If $j = 1$, then note that $w_1^+ \notin \partial\Delta^n$ while $w_k^+ \in \partial\Delta^n$ since otherwise Δ_{k-1} would have a vertex outiside Δ^n. It remains the possibility $w_k^+ = w_j^-$, this implies $\Delta_{k-1} = \Delta_{j+1}$ again contradicting the hypothesis unless $k = j + 2$. But this would imply $\Delta_j = \Delta_{j+2}$ which is impossible, as one can check directly.
The above implies that all the Δ_k are different, but they are only finitely many, so the construction must eventually stop, and the only possibility to stop is when a fully colored simplex appears, whereby concluding the proof. □

Theorem A.2.4 (Fixed Point Theorem for simplices) *Every continuous map $f\colon \Delta^n \to \Delta^n$ has a fixed point.*

Proof. Let $x \in \Delta^n$ such that $f_i(x) \geq x_i$ for each $i \in \{1, \dots, n+1\}$, then

$$0 = 1 - 1 = \sum_{i=1}^{d+1}(f_i(x) - x_i), \tag{A.2.1}$$

which implies $f(x) = x$. It thus suffices to show that such a point exists. We argue by contradiction, assume that for every x there exists some i with $f_i(x) < x_i$.
For each $k \in \mathbb{N}$, consider a triangulation T_k of Δ^n with simplices of size smaller than 2^{-k}. For each vertex v of T_k, we set $\ell(v) = \arg\max_i\{v_i - f_i(v)\}$. By our assumption, we have $v_{\ell(v)} > f_{\ell(v)}(v)$. If v lies on the face $\{x_j = 0\}$, then clearly $f_j(v) \geq 0 = v_j$, so $\ell(v) \neq j$. Thus, we have defined an s-coloring of T_k. It follows that there exists a simplex $\Delta_k \in T_k$ which is fully colored. Let $x_k \in \Delta_k =: \Delta(v_{k,1}, \dots v_{k,n+1})$, by compactness the sequence $\{x_k\}$ admits a convergent subsequence $\{x_{k_j}\}$. Let $\bar{x} = \lim_{j\to\infty} x_{k_j}$. It follows that $\bar{x} = \lim_{j\to\infty} v_{k_j,l}$, for each $l \in \{1, \dots, n+1\}$. Since the Δ_k are fully colored, for each i and j there exists $l_{j,i}$ such that $f(v_{k_j,l_{i,j}})_i < (v_{k_j,l_{i,j}})_i$. By the continuity of f, it follows

$$\bar{x}_i \leq f(\bar{x})_i$$

for each $i \in \{1, \dots, n+1\}$, hence the contradiciton. The lemma follows. □

To obtain a more general result, we need to recall a useful characterization of convex sets.

Lemma A.2.5 *Let $K \subset \mathbb{R}^n$ be a non-empty compact convex set with nonempty interior. Then K is homeomorphic to the standard n–simplex Δ^n.*

Proof. Choose $x_0 \in \text{int}(K)$ and $z_0 = (\frac{1}{n+1}, \dots, \frac{1}{n+1}) \in \mathbb{R}^{n+1}$. Let R be a rotation that sends e_{d+1} into the vector $[n+1]^{-\frac{1}{2}}(1, \dots, 1)$. Consider the map $\Phi_0(x) = z_0 + R(x - x_0, 0)$ and let $\tilde{K} = \Phi_0(K)$. By construction, $\tilde{K}$ belongs to the same hyperplane containing Δ^n. For each $z \in \tilde{K}$, the half line $\{z_0 + t(z - z_0) \ : \ t \geq 0\}$ intersects the boundary ∂K at a unique point $a(z)$ and the boundary $\partial \Delta^n$ at a unique point $b(z)$. Define a continuous map $\phi_1 : \tilde{K} \to \Delta^n$ by

$$\phi_1(x) = z_0 + \frac{\|b(z)\|}{\|a(z)\|}(z - z_0).$$

Clearly, $\phi = \phi_1 \circ \phi_0$ is the wanted homeomorphism. □

Theorem A.2.6 (Brouwer Fixed Point Theorem) *For every non-empty compact convex set $K \subset \mathbb{R}^n$ and continuous map $f\colon K \to K$, f has a fixed point.*

Proof. By Lemma A.2.5, there exists a homeomorphism $\phi : K \to \Delta^n$. Define $F = \phi \circ f \circ \phi^{-1} : \Delta^n \to \Delta^n$. Theorem A.2.4 implies that there exist $\bar{x} \in \Delta^n$ such that $F(\bar{n}) = \bar{x}$. Hence, setting $x_* = \phi^{-1}(\bar{x})$ we have $f(x_*) = x_*$. □

To conclude this section, we show how Brouwer's result can be extended to the infinite-dimensional setting by an approximation procedure.

Theorem A.2.7 (Schauder Fixed-Point Theorem) *Let $\mathcal{B}$ be a Banach space and $K \subset \mathcal{B}$ a nonempty, compact, convex subset. Let $f : K \to K$ be continuous. Then f has a fixed point.*

Proof. Since K is compact, for each $\varepsilon > 0$ there exists a finite set $\{x_1, \dots, x_N\} \subset \mathring{K}$ such that

$$K \subset \bigcup_{i=1}^{N} B_\varepsilon(x_i),$$

where $B_\varepsilon(x_i)$ denotes the open ball of radius ε around x_i. Let

$$K_\varepsilon := \text{conv}\{x_1, \dots, x_N\} \subset K$$

be the convex hull of the points $\{x_i\}$. Then K_ε is a compact, convex, and finite-dimensional set since it is contained in span$\{x_1, \dots, x_N\}$. Next, define

$$\phi_i(x) = \begin{cases} \varepsilon - \|x - x_i\| & \text{for} \|x - x_i\| \le \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

and

$$P_\varepsilon(x) = \left[\sum_{i=1}^N \phi_i(x)\right]^{-1} \sum_{i=1}^N \phi_i(x) x_i.$$

Note that $P_\varepsilon(\mathcal{B}) = K_\varepsilon$, P_ε is continuos and, for all $x \in K$

$$\|P_\varepsilon(x) - x\| = \left\| \left[\sum_{i=1}^N \phi_i(x)\right]^{-1} \sum_{i=1}^N \phi_i(x)(x_i - x) \right\| \le \varepsilon. \tag{A.2.2}$$

We can then define the continuous function

$$f_\varepsilon := P_\varepsilon \circ f : K_\varepsilon \to K_\varepsilon.$$

By Brouwer's fixed-point theorem, there exists

$$x_\varepsilon \in K_\varepsilon \quad \text{such that} \quad f_\varepsilon(x_\varepsilon) = x_\varepsilon.$$

Since K is compact, there exists a convergent subsequence $\{x_{\varepsilon_j}\}$, let x_* be the limit. Consequently, reacalling (A.2.2), we have

$$\|f(x_{\varepsilon_j}) - x_{\varepsilon_j}\| = \|f(x_{\varepsilon_j}) - f_{\varepsilon_j}(x_{\varepsilon_j})\| = \|f(x_{\varepsilon_j}) - P_{\varepsilon_j}(f(x_{\varepsilon_j}))\| \le \varepsilon_j.$$

Taking the limit $j \to \infty$, by the continuity of f, we have the wanted fixed point $f(x_*) = x_*$. □

A.3 Hilbert metric and Birkhoff theorem

One may wonder if there are cases in which the fixed point provided by the Brower and Shauder theory is unique. In general, the answer is negative, but much more can be said for linear maps. In particular, we will see that the Banach fixed-point theorem can produce unexpected results if used with respect to an appropriate metric. We thus start with a short digression on projective metrics.

Projective metrics are widely used in geometry, not to mention the importance of their generalizations (e.g. Kobayashi metrics) for the study of complex manifolds [IK00]. It is quite surprising that they play a major rôle also in our situation, [Liv95].

Here we limit ourselves to a few words on the Hilbert metric, a quite important tool in hyperbolic geometry.

A.3.1 Projective metrics

Let $C \subset \mathbb{R}^n$ be a strictly convex compact set. For each two point $x, y \in C$ consider the line $\ell = \{\lambda x + (1 - \lambda y) \mid \lambda \in \mathbb{R}\}$ passing through x and y. Let $\{u, v\} = \partial C \cap \ell$ and define[2]

$$\Theta(x, y) = \left| \ln \frac{\|x - u\|\|y - v\|}{\|x - v\|\|y - v\|} \right|$$

(the logarithm of the cross ratio). By remembering that the cross ratio is a projective invariant and looking at Figure A.2, it is easy to check that Θ is indeed a metric. Moreover, the distance of an inner point from the boundary is always infinite. One can also check that if the convex set is a disc, then the disc with the Hilbert metric is nothing but the Poincaré disc.

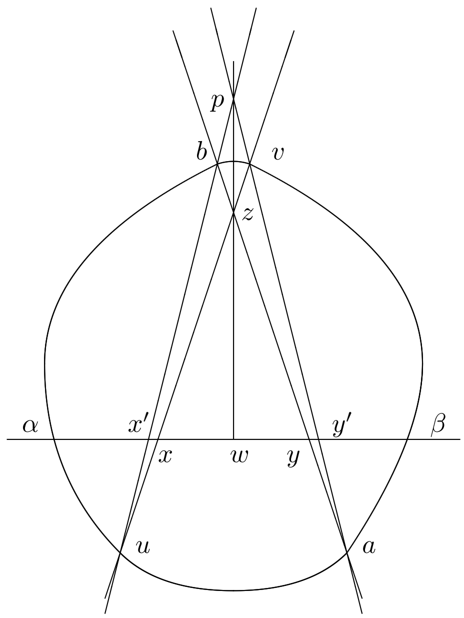

Figure A.2: Hilbert metric

The objects that we will use in our subsequent discussion are not convex sets but rather convex cones, yet their projectivization is a convex set, and one can define the Hilbert metric on it (whereby obtaining a semi-metric for the original cone). It turns out that there exists a more algebraic way of defining

[2] Remark that u, v can also be ∞.

such a metric, which is easier to use in our context. Moreover, there exists a simple connection between vector spaces with a convex cone and vector lattices (in a vector lattice one can always consider the positive cone). This justifies the next digression in lattice theory.[3]

Consider a topological vector space $\mathbb{V}$ with a partial ordering "$\preceq$," that is a vector lattice.[4] We require the partial order to be "continuous," i.e. given $\{f_n\} \in \mathbb{V}$ $\lim_{n\to\infty} f_n = f$, if $f_n \succeq g$ for each n, then $f \succeq g$. We call such vector lattices "integrally closed." [5]

We define the closed convex cone [6] $\mathcal{C} = \{f \in \mathbb{V} \mid f \neq 0,\ f \succeq 0\}$ (hereafter, the term "closed cone" $\mathcal{C}$ will mean that $\mathcal{C} \cup \{0\}$ is closed), and the equivalence relation "$\sim$": $f \sim g$ iff there exists $\lambda \in \mathbb{R}^+ \backslash \{0\}$ such that $f = \lambda g$. If we call $\widetilde{\mathcal{C}}$ the quotient of $\mathcal{C}$ with respect to $\sim$, then $\widetilde{\mathcal{C}}$ is a closed convex set. Conversely, given a closed convex cone $\mathcal{C} \subset \mathbb{V}$, enjoying the property $\mathcal{C} \cap -\mathcal{C} = \emptyset$, we can define an order relation by

$$f \preceq g \iff g - f \in \mathcal{C} \cup \{0\}.$$

Henceforth, each time that we specify a convex cone, we will assume the corresponding order relation and vice versa. The reader must therefore be advised that "$\preceq$" will mean different things in different contexts.

It is then possible to define a projective metric Θ (Hilbert metric),[7] in $\mathcal{C}$, by the construction:

$$\begin{aligned}
\alpha(f,\, g) &= \sup\{\lambda \in \mathbb{R}^+ \mid \lambda f \preceq g\} \\
\beta(f,\, g) &= \inf\{\mu \in \mathbb{R}^+ \mid g \preceq \mu f\} \\
\Theta(f,\, g) &= \log\left[\frac{\beta(f,\, g)}{\alpha(f,\, g)}\right]
\end{aligned}$$

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

The relevance of the above metric in our context is due to the following Theorem by Garrett Birkhoff.

[3] For more details, see [Bir57], and [Nus88] for an overview of the field.

[4] We are assuming the partial order to be well-behaved with respect to the algebraic structure: for each $f,\, g \in \mathbb{V}$ $f \succeq g \iff f - g \succeq 0$; for each $f \in \mathbb{V}$, $\lambda \in \mathbb{R}^+ \backslash \{0\}$ $f \succeq 0 \Longrightarrow \lambda f \succeq 0$; for each $f \in \mathbb{V}$ $f \succeq 0$ and $f \preceq 0$ imply $f = 0$ (antisymmetry of the order relation).

[5] To be precise, in the literature "integrally closed" is used in a weaker sense. First, $\mathbb{V}$ does not need a topology. Second, it suffices that for $\{\alpha_n\} \in \mathbb{R}$, $\alpha_n \to \alpha$; $f,\, g \in \mathbb{V}$, if $\alpha_n f \succeq g$, then $\alpha f \succeq g$. Here we will ignore these and other subtleties: our task is limited to a brief account of the results relevant to the present context.

[6] Here, by "cone," we mean any set such that, if f belongs to the set, then λf belongs to it as well, for each $\lambda > 0$.

[7] In fact, we define a semi–metric, since $f \sim g \Rightarrow \Theta(f,\, g) = 0$. The metric that we describe corresponds to the conventional Hilbert metric on $\widetilde{\mathcal{C}}$.

Theorem A.3.1 *Let $\mathbb{V}_1$, and $\mathbb{V}_2$ be two integrally closed vector lattices; $\mathcal{L} : \mathbb{V}_1 \to \mathbb{V}_2$ a linear map such that $\mathcal{L}(\mathcal{C}_1) \subset \mathcal{C}_2$, for two closed convex cones $\mathcal{C}_1 \subset \mathbb{V}_1$ and $\mathcal{C}_2 \subset \mathbb{V}_2$ with $\mathcal{C}_i \cap -\mathcal{C}_i = \emptyset$. Let Θ_i be the Hilbert metric corresponding to the cone $\mathcal{C}_i$. Setting $\Delta = \sup_{f,\, g \in \mathcal{L}(\mathcal{C}_1)} \Theta_2(f,\, g)$ we have*

$$\Theta_2(\mathcal{L}f,\, \mathcal{L}g) \leq \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f,\, g) \qquad \forall f,\, g \in \mathcal{C}_1$$

($\tanh(\infty) \equiv 1$*).*

Proof. The proof is provided for the reader's convenience.

Let $f,\, g \in \mathcal{C}_1$, on the one hand if $\alpha = 0$ or $\beta = \infty$, then the inequality is obviously satisfied. On the other hand, if $\alpha \neq 0$ and $\beta \neq \infty$, then

$$\Theta_1(f,\, g) = \ln \frac{\beta}{\alpha}$$

where $\alpha f \preceq g$ and $\beta f \succeq g$, since $\mathbb{V}_1$ is integrally closed. Notice that $\alpha \geq 0$, and $\beta \geq 0$ since $f \succeq 0$, $g \succeq 0$. If $\Delta = \infty$, then the result follows from $\alpha \mathcal{L} f \preceq \mathcal{L} g$ and $\beta \mathcal{L} f \succeq \mathcal{L} g$. If $\Delta < \infty$, then, by hypothesis,

$$\Theta_2\left(\mathcal{L}(g - \alpha f),\, \mathcal{L}(\beta f - g)\right) \leq \Delta$$

which means that there exist $\lambda,\, \mu \geq 0$ such that

$$\begin{aligned} \lambda \mathcal{L}(g - \alpha f) &\preceq \mathcal{L}(\beta f - g) \\ \mu \mathcal{L}(g - \alpha f) &\succeq \mathcal{L}(\beta f - g) \end{aligned}$$

with $\ln \frac{\mu}{\lambda} \leq \Delta$. The previous inequalities imply

$$\begin{aligned} \frac{\beta + \lambda\alpha}{1 + \lambda} \mathcal{L} f &\succeq \mathcal{L} g \\ \frac{\mu\alpha + \beta}{1 + \mu} \mathcal{L} f &\preceq \mathcal{L} g. \end{aligned}$$

Accordingly,

$$\begin{aligned} \Theta_2(\mathcal{L}f,\, \mathcal{L}g) &\leq \ln \frac{(\beta + \lambda\alpha)(1 + \mu)}{(1 + \lambda)(\mu\alpha + \beta)} = \ln \frac{e^{\Theta_1(f,\, g)} + \lambda}{e^{\Theta_1(f,\, g)} + \mu} - \ln \frac{1 + \lambda}{1 + \mu} \\ &= \int_0^{\Theta_1(f,\, g)} \frac{(\mu - \lambda) e^{\xi}}{(e^{\xi} + \lambda)(e^{\xi} + \mu)} d\xi \leq \Theta_1(f,\, g) \frac{1 - \frac{\lambda}{\mu}}{\left(1 + \sqrt{\frac{\lambda}{\mu}}\right)^2} \\ &\leq \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f,\, g). \end{aligned}$$

□

Remark A.3.2 *If $\mathcal{L}(\mathcal{C}_1) \subset \mathcal{C}_2$, then it follows that $\Theta_2(\mathcal{L}f, \mathcal{L}g) \leq \Theta_1(f, g)$. However, a uniform rate of contraction depends on the diameter of the image being finite.*

In particular, if an operator maps a convex cone strictly inside itself (in the sense that the diameter of the image is finite), then it is a contraction in the Hilbert metric. This implies the existence of a "positive" eigenfunction (provided the cone is complete with respect to the Hilbert metric), and, with some additional work, the existence of a gap in the spectrum of $\mathcal{L}$ (see [Bir79] for details). The relevance of this theorem for the study of invariant measures and their ergodic properties is obvious.

It is natural to wonder about the strength of the Hilbert metric compared to other, more usual, metrics. While, in general, the answer depends on the cone, it is nevertheless possible to state an interesting result.

Lemma A.3.3 *Let $\|\cdot\|$ be a norm on the vector lattice $\mathbb{V}$, and suppose that, for each $f, g \in \mathbb{V}$,*

$$-f \preceq g \preceq f \Longrightarrow \|f\| \geq \|g\|.$$

Then, given $f, g \in \mathcal{C} \subset \mathbb{V}$ for which $\|f\| = \|g\|$,

$$\|f - g\| \leq \left(e^{\Theta(f, g)} - 1\right) \|f\|.$$

Proof. We know that $\Theta(f, g) = \ln \frac{\beta}{\alpha}$, where $\alpha f \preceq g$, $\beta f \succeq g$. This implies that $-g \preceq 0 \preceq \alpha f \preceq g$, i.e. $\|g\| \geq \alpha \|f\|$, or $\alpha \leq 1$. In the same manner, it follows that $\beta \geq 1$. Hence,

$$\begin{aligned} g - f &\preceq (\beta - 1) f \preceq (\beta - \alpha) f \\ g - f &\succeq (\alpha - 1) f \succeq -(\beta - \alpha) f \end{aligned}$$

which implies

$$\|g - f\| \leq (\beta - \alpha)\|f\| \leq \frac{\beta - \alpha}{\alpha}\|f\| = \left(e^{\Theta(f, g)} - 1\right) \|f\|.$$

□

Many normed vector lattices satisfy the hypothesis of Lemma 1.3 (e.g. Banach lattices[8]); nevertheless, we will see that some important examples treated in this paper do not.

[8] A Banach lattice $\mathbb{V}$ is a vector lattice equipped with a norm satisfying the property $\| |f| \| = \|f\|$ for each $f \in \mathbb{V}$, where $|f|$ is the least upper bound of f and $-f$. For this definition to make sense it is necessary to require that $\mathbb{V}$ is "directed," i.e. any two elements have an upper bound.

A.3.2 An application: Perron-Frobenius

Consider a matrix $L : \mathbb{R}^n \to \mathbb{R}^n$ of all strictly positive elements: $L_{ij} \geq \gamma > 0$. The Perron-Frobenius theorem states that there exists a unique eigenvector v^+ such that $v_i^+ > 0$, in addition, the corresponding eigenvalue λ is simple, maximal and positive. There are quite a few proofs of this theorem; one is based on Birkhoff's theorem. Consider the cone $\mathcal{C}^+ = \{v \in \mathbb{R}^2 \mid v_i \geq 0\}$, then obviously $L\mathcal{C}^+ \subset \mathcal{C}^+$. Moreover an explicit computation (see

Problem A.1 *shows that*

$$\Theta(v, w) = \ln \sup_{ij} \frac{v_i w_j}{v_j w_i}. \tag{A.3.3}$$

Then, setting $M = \max_{ij} L_{ij}$, it follows that

$$\Theta(Lv, Lw) \leq 2 \ln \frac{M}{\gamma} := \Delta < \infty.$$

We then have a contraction in the Hilbert metric, and the result follows from the usual fixed points theorems. Note that, since $\Theta(v, \lambda v) = 0$, for all $\lambda \in \mathbb{R}^+$, the fixed point $v_+ \in \mathbb{R}^n$ is only projective, that is $Lv_+ = \lambda v_+$ for some $\lambda \in \mathbb{R}$; in other words, we have an eigenvalue.

Remark that L^* satisfies the same conditions as L, thus there exists $w^+ \in \mathcal{C}^+$, $\mu \in \mathbb{R}^+$, such that $L^* w^+ = \mu w^+$. Next, define $\rho_1(v) = |\langle w^+, v\rangle|$ and $\rho_2(v) = \|v\|$. It is easy to check that there are two homogeneous forms of degree one adapted to the cone.

In addition, if $\rho_1(v) = \rho_2(v)$, then $\rho_1(L^n v) = \rho_1(L^n w)$. Hence, by Lemma A.3.3

$$\begin{aligned} \|L^n v - L^n w\| &\leq \left(e^{\Theta(L^n v, L^n w)} - 1\right) \min\{\|L^n v\|, \|L^n w\|\} \\ &\leq K\Lambda^n \min\{\|L^n v\|, \|L^n w\|\}, \end{aligned} \tag{A.3.4}$$

for some constant K depending only on v, w. The estimate A.3.4 means that all the vectors in the cone grow at the same rate. In fact, for all $v \in \text{int}\mathcal{C}$,

$$\|\lambda^{-n} L^n v - \lambda^{-n} L^n w\| \leq K\Lambda^n.$$

Hence, $\lim_{n\to\infty} \lambda^{-n} L^n v = v_+$.

Finally, consider $\mathbb{V}_1 = \{v \in \mathbb{V} \mid \langle w^+, v\rangle = 0\}$. Clearly $L\mathbb{V}_1 \subset \mathbb{V}_1$ and $\mathbb{V}_1 \oplus \text{span}\{v_+\} = \mathbb{V}$. Let $w \in \mathbb{V}_1$, clearly there exists $\alpha \in \mathbb{R}^+$ such that $\alpha v_+ + w \in \mathcal{C}$,[9] thus

$$\|L^n w\| \leq \|L^n(\alpha v_+ + w) - \alpha L^n v_+\| \leq L\Lambda^n \lambda^n.$$

[9] this is a special case of the general fact that any vector can be written as the linear combination of two vectors belonging to the cone.

This immediately implies that L restricted to the subspace $\mathbb{V}_1$ has spectral radius less than $\lambda\Lambda$. In other words, λ is the maximal eigenvalue; it is simple, and any other eigenvalue must be smaller than $\lambda\Lambda$. We have thus obtained an estimate of the spectral gap between the first and the second eigenvalue.

Notes

For more details on Hilbert metrics see [Bir79], and [Nus88] for an overview of the field.

Appendix B

Implicit function theorem (a quantitative version)

In this appendix we recall the implicit function Theorem. We provide an explicit proof because we use in the text a quantitative version of the theorem so it is important to keep track of the various constants.

B.1 The theorem

Let $n, m \in \mathbb{N}$ and $F \in \mathcal{C}^1(\mathbb{R}^{m+n}, \mathbb{R}^m)$ and let $(x_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $F(x_0, \lambda_0) = 0$. For each $\delta > 0$ let $V_\delta = \{(x, \lambda) \in \mathbb{R}^{n+m} \ : \ \|x - x_0\| \leq \delta, \|\lambda - \lambda_0\| \leq \delta\}$.

Theorem B.1.1 *Assume that $\partial_x F(x_0, \lambda_0)$ is invertible and choose $\delta > 0$ such that $\sup_{(x,\lambda)\in V_\delta} \|\mathbb{1} - [\partial_x F(x_0, \lambda_0)]^{-1} \partial_x F(x, \lambda)\| \leq \frac{1}{2}\}$. Let $B_\delta = \sup_{(x,\lambda)\in V_\delta} \|\partial_\lambda F(x, \lambda)\|$ and $M = \|\partial_x F(x_0, \lambda_0)^{-1}\|$. Set $\delta_1 = (2MB_\delta)^{-1}\delta$ and $\Lambda_{\delta_1} := \{\lambda \in \mathbb{R}^m \ : \ \|\lambda - \lambda\| < \delta_1\}$. Then there exists $g \in \mathcal{C}^1(\Lambda_{\delta_1}, \mathbb{R}^m)$ such that all the solutions of the equation $F(x, \lambda) = 0$ in the set $\{(x, \lambda) \in \mathcal{B}_1 \times \mathcal{B}_2 \ : \ \|\lambda - \lambda_0\| < \delta_1, \|x - x_0\| < \delta\}$ are given by $(g(\lambda), \lambda)$. In addition,*

$$\partial_\lambda g(\lambda) = -(\partial_x F(g(\lambda), \lambda))^{-1} \partial_\lambda F(g(\lambda), \lambda).$$

We will do the proof in several steps.

B.1.1 Existence of the solution

Let $A(x, \lambda) = \partial_x F(x, \lambda)$, $M = \|A(x_0, \lambda_0)^{-1}\|$.

We want to solve the equation $F(x, \lambda) = 0$, various approaches are possible. Here we will use a simplification of Newton method, made possible by the fact that we already know a good approximation of the zero we are looking for. Let λ be such that $\|\lambda-\lambda_0\| < \delta_1 \leq \delta$. Consider $U_\delta = \{x \in \mathbb{R}^n \;:\; \|x-x_0\| \leq \delta\}$ and the function $\Theta_\lambda : U_\delta \to \mathbb{R}^n$ defined by[1]

$$\Theta_\lambda(x) = x - A(x_0,\lambda_0)^{-1}F(x,\lambda). \tag{B.1.1}$$

Problem B.1 *Prove that, for $x \in U(\lambda)$, $F(x,\lambda) = 0$ is equivalent to $x = \Theta_\lambda(x)$.*

Next,

$$\|\Theta_\lambda(x_0) - \Theta_{\lambda_0}(x_0)\| \leq M\|F(x_0,\lambda)\| \leq MB_\delta\delta_1.$$

In addition, $\|\partial_x\Theta_\lambda\| = \|\mathbb{1} - A(x_0,\lambda_0)^{-1}A(x,\lambda)\| \leq \frac{1}{2}$. Thus,

$$\|\Theta_\lambda(x) - x_0\| \leq \frac{1}{2}\|x - x_0\| + \|\Theta_\lambda(x_0) - x_0\| \leq \frac{1}{2}\|x - x_0\| + MB_\delta\delta_1 \leq \delta.$$

The existence of $x \in U_\delta$ such that $\Theta_\lambda(x) = x$ follows then by the standard fixed point Theorem A.1.1. We have so obtained a function $g : \{\lambda \;:\; \|\lambda - \lambda_0\| \leq \delta_1\} = \Lambda_{\delta_1} \to \mathbb{R}^n$ such that $F(g(\lambda),\lambda) = 0$. it remains the question of the regularity.

B.1.2 Lipschitz continuity and Differentiability

Let $\lambda, \lambda' \in \Lambda_{\delta_1}$. By (B.1.1)

$$\|g(\lambda) - g(\lambda')\| \leq \frac{1}{2}\|g(\lambda) - g(\lambda')\| + MB_\delta|\lambda - \lambda'|$$

This yields the Lipschitz continuity of the function g. To obtain the differentiability we note that, by the differentiability of F and the above Lipschitz continuity of g, for $h \in \mathbb{R}^m$ small enough,

$$\|F(g(\lambda+h),\lambda+h) - F(g(\lambda),\lambda) + \partial_x F[g(\lambda+h) - g(\lambda)] + \partial_\lambda F h\| = o(\|h\|).$$

Since $F(g(\lambda+h),\lambda+h) = F(g(\lambda),\lambda) = 0$, we have that

$$\lim_{h\to 0}\|h\|^{-1}\|g(\lambda+h) - g(\lambda) + [\partial_x F]^{-1}\partial_\lambda F h\| = 0$$

which concludes the proof of the Theorem, the continuity of the derivative being obvious be the obtained explicit formula.

[1] The Newton method would consist in finding a fixed point for the function $x - A(x,\lambda)^{-1}F(x,\lambda)$. This gives a much faster convergence and hence is preferable in applications, yet here it would make the estimates a bit more complicated.

B.2 Generalization

First of all note that the above theorem implies the inverse function theorem. Indeed if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function such that $\partial_x f$ is invertible at some point x_0, then one can consider the function $F(x, y) = f(x) - y$. Applying the implicit function theorem to the equation $F(x, y) = 0$ it follows that $y = f(x)$ are the only solution, hence the function is locally invertible.

The above theorem can be generalized in several ways.

Problem B.2 *Show that if F in Theorem B.1.1 is $\mathcal{C}^r$, then also g is $\mathcal{C}^r$.*

Problem B.3 *Verify that if $\mathcal{B}_1, \mathcal{B}_2$ are two Banach spaces and in Theorem B.1.1 we have $\mathcal{B}_1$ instead of $\mathbb{R}^n$ and $\mathcal{B}_2$ instead of $\mathbb{R}^m$ the Theorem remains true and the proof remains exactly the same.*

As I mentioned the statement of Theorem B.1.1 is suitable for quantitative applications.

Problem B.4 *Suppose that in Theorem B.1.1 we have $F \in \mathcal{C}^2$, then show that we can chose*

$$\delta = [2\|D\partial_x F\|_\infty]^{-1}.$$

Appendix C

Perturbation Theory (a super-fast introduction)

The following is really super condensate (although self-consistent). If you want more details see [RS80, Kat66] in which you probably can find more than you are looking for.

C.1 Bounded operators

In the following we will consider only *separable* Banach spaces, i.e. Banach spaces that have a countable dense set.[1]

Given a Banch space $\mathcal{B}$ we can consider the set $L(\mathcal{B}, \mathcal{B})$ of the linear bounded operators from $\mathcal{B}$ to itself. We can then introduce the norm $\|B\| = \sup_{\|v\|\leq 1} \|Bv\|$.

Problem C.1 *Show that $(L(\mathcal{B}, \mathcal{B}), \|\cdot\|)$ is a Banach space. That is that $\|\cdot\|$ is really a norm and that the space is complete with respect to such a norm.*

Problem C.2 *Show that the $n \times n$ matrices form a Banach Algebra.*[2]

Problem C.3 *Show that $L(\mathcal{B}, \mathcal{B})$ form a Banach algebra.*[3]

[1] Recall that a Banach space is a complete normed vector space (in the following we will consider vector spaces on the field of complex numbers), that is a normed vector space in which all the Cauchy sequences have a limit in the space. Again, if you are uncomfortable with Banach spaces, in the following read $\mathbb{R}^d$ instead of $\mathcal{B}$ and matrices instead of operators, but be aware that we have to develop the theory without the use of the determinant that, in general, is not defined for operators on Banach spaces.

[2] A Banach Algebra $\mathcal{A}$ is a Banach space where it is defined the multiplications between element with the usual properties of an algebra and, in addition, for each $a, b \in \mathcal{A}$ holds $\|ab\| \leq \|a\| \cdot \|b\|$.

[3] The multiplication is given by the composition.

To each $A \in L(\mathcal{B}, \mathcal{B})$ are associated two important subspaces: the range $R(A) = \{v \in \mathcal{B} \;:\; \exists\, w \in \mathcal{B} \text{ such that } v = Aw\}$ and the kernel $N(A) = \{v \in \mathcal{B} \;:\; Av = 0\}$.

Problem C.4 *Prove, for each $A \in L(\mathcal{B}, \mathcal{B})$, that $N(A)$ is a closed linear subspaces of $\mathcal{B}$. Show that this is not necessarily the case for $R(A)$ if $\mathcal{B}$ is not finite dimensional.*

An very special, but very important, class of operators are the projectors.

Definition C.1.1 *An operator $\Pi \in L(\mathcal{B}, \mathcal{B})$ is called a* projector *iff $\Pi^2 = \Pi$.*

Note that if Π is a projector, so is $\mathbb{1} - \Pi$. We have the following interesting fact.

Lemma C.1.2 *If $\Pi \in L(\mathcal{B}, \mathcal{B})$ is a projector, then $N(\Pi) \oplus R(\Pi) = \mathcal{B}$.*

PROOF. If $v \in \mathcal{B}$, then $v = \Pi v + (\mathbb{1} - \Pi)v$. Notice that $R(\mathbb{1} - \Pi) = N(\Pi)$ and $R(\Pi) = N(\mathbb{1} - \Pi)$. Finally, if $v \in N(\Pi) \cap R(\Pi)$, then $v = 0$, which concludes the proof. □

Another, more general, very important class of operators are the compact ones.

Definition C.1.3 *An operator $K \in L(\mathcal{B}, \mathcal{B})$ is called* compact *iff for any bounded set B the closure of $K(B)$ is compact.*

Remark C.1.4 *Note that not all the linear operator on a Banach space are bounded. For example consider the derivative acting on $\mathcal{C}^1((0,1), \mathbb{R})$.*

C.2 Functional calculus

First of all recall that all the Riemannian theory of integration works verbatim for function $f \in \mathcal{C}^0(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is a Banach space. We can thus talk of integrals of the type $\int_a^b f(t)dt$.[4] Next, we can talk of *analytic functions* for functions in $\mathcal{C}^0(\mathbb{C}, \mathcal{B})$: a function is analytic in an open region $U \subset \mathbb{C}$ iff at each point $z_0 \in U$ there exists a neighborhood $B \ni z_0$ and elements $\{a_n\} \subset \mathcal{B}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in B. \tag{C.2.1}$$

[4] This is special case of the so called Bochner integral [Yos95].

Problem C.5 *Show that if* $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ *is analytic in* $U \subset \mathbb{C}$*, then given any smooth closed curve* γ*, contained in a sufficiently small disk in* U*, holds*[5]

$$\int_\gamma f(z)dz = 0 \tag{C.2.2}$$

Then show that the same hold for any piecewise smooth closed curve with interior contained in U*, provided* U *is simply connected.*

Problem C.6 *Show that if* $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ *is analytic in a simply connected* $U \subset \mathbb{C}$*, then given any smooth closed curve* γ*, with interior contained contained in* U *and having in its interior a point* z*, hods the formula*

$$f(z) = \frac{1}{2\pi i} \int_\gamma (\xi - z)^{-1} f(\xi) d\xi. \tag{C.2.3}$$

Problem C.7 *Show that if* $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ *satisfies* (C.2.3) *for each smooth closed curve in a simply connected open set* U*, then* f *is analytic in* U*.*

C.3 Spectrum and resolvent

Given $A \in L(\mathcal{B}, \mathcal{B})$ we define the *resolvent*, called $\rho(A)$, as the set of the $z \in \mathbb{C}$ such that $(z\mathbb{1} - A)$ is invertible and the inverse belongs to $L(\mathcal{B}, \mathcal{B})$. The *spectrum* of A, called $\sigma(A)$ is the complement of $\rho(A)$ in $\mathbb{C}$.

Problem C.8 *Prove that, for each Banach space* $\mathcal{B}$ *and operator* $A \in L(\mathcal{B}, \mathcal{B})$*, if* $z \in \rho(A)$*, then there exists a neighborhood* U *of* z *such that* $(z\mathbb{1} - A)^{-1}$ *is analytic in* U*.*

From the above exercise follows that $\rho(A)$ is open, hence $\sigma(A)$ is closed.

Problem C.9 *Show that, for each* $A \in L(\mathcal{B}, \mathcal{B})$*,* $\sigma(A) \neq \emptyset$*.*

Problem C.10 *Show that if* $\Pi \in L(\mathcal{B}, \mathcal{B})$ *is a projector, then* $\sigma(\Pi) = \{0, 1\}$*.*

Up to now the theory for operators seems very similar to the one for matrices. Yet, the spectrum for matrices is always given by a finite number of points while the situation for operators can be very dfferenct.

[5] Of course, by $\int_\gamma f(z)dz$ we mean that we have to consider any smooth parametrization $g : [a, b] \to \mathbb{C}$ of γ, $g(a) = g(b)$, and then $\int_\gamma f(z)dz := \int_a^b f \circ g(t) g'(t) dt$. Show that the definition does not depend on the parametrization and that one can use piecewise smooth parametrizations as well.

Problem C.11 *Consider the operator* $\mathcal{L} : \mathcal{C}^0([0,1],\mathbb{C}) \to \mathcal{C}^0([0,1],\mathbb{C})$ *defined by*

$$(\mathcal{L}f)(x) = \frac{1}{2}f(x/2) + \frac{1}{2}f(x/2+1/2).$$

Show that $\sigma(\mathcal{L}) = \{z \in \mathbb{C} \ : \ |z| \leq 1\}$.

Problem C.12 *Show that, if* $A \in L(\mathcal{B},\mathcal{B})$ *and* p *is any polynomial, then for each* $n \in \mathbb{N}$ *and smooth curve* $\gamma \subset \mathbb{C}$*, with* $\sigma(A)$ *in its interior,*

$$p(A) = \frac{1}{2\pi i}\int_\gamma p(z)(z\mathbb{1} - A)^{-1}dz.$$

Problem C.13 *Show that, for each* $A \in L(\mathcal{B},\mathcal{B})$ *the limit*

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

exists.

The above limit is called the *spectral radius* of A.

Lemma C.3.1 *For each* $A \in L(\mathcal{B},\mathcal{B})$ *holds true* $\sup_{z\in\sigma(A)} |z| = r(A)$.

PROOF. Since we can write

$$(z\mathbb{1} - A)^{-1} = z^{-1}(\mathbb{1} - z^{-1}A)^{-1} = z^{-1}\sum_{n=0}^{\infty} z^{-n}A^n,$$

and since the series converges if it converges in norm, from the usual criteria for the convergence of a series follows $\sup_{z\in\sigma(A)}|z| \leq r(A)$. Suppose now that the inequality is strict, then there exists $0 < \eta < r(A)$ and a curve $\gamma \subset \{z \in \mathcal{C} \ : \ |z| \leq \eta\}$ which contains $\sigma(A)$ in its interior. Then applying Problem C.12 yields $\|A^n\| \leq C\eta^n$, which contradicts $\eta < r(A)$. □

Note that if $f(z) = \sum_{n=0}^{\infty} f_n z^n$ is an analytic function in all $\mathbb{C}$ (entire), then we can define

$$f(A) = \sum_{n=0}^{\infty} f_n A^n.$$

Problem C.14 *Show that, if* $A \in L(\mathcal{B},\mathcal{B})$ *and* f *is an entire function, then for each smooth curve* $\gamma \subset \mathbb{C}$*, with* $\sigma(A)$ *in its interior,*

$$f(A) = \frac{1}{2\pi i}\int_\gamma f(z)(z\mathbb{1} - A)^{-1}dz.$$

In view of the above fact, the following definition is natural:

Definition C.3.2 *For each $A \in L(\mathcal{B}, \mathcal{B})$, f analytic in a region U containing $\sigma(A)$, then for each smooth curve $\gamma \subset U$, with $\sigma(A)$ in its interior, define*

$$f(A) = \frac{1}{2\pi i} \int_\gamma f(z)(z\mathbb{1} - A)^{-1} dz. \tag{C.3.4}$$

Problem C.15 *Show that the above definition does not depend on the curve γ.*

Problem C.16 *For each $A \in L(\mathcal{B}, \mathcal{B})$ and functions f, g analytic on a domain $D \supset \sigma(A)$, show that $f(A) + g(A) = (f + g)(A)$ and $f(A)g(A) = (f \cdot g)(A)$.*

Problem C.17 *In the hypotheses of the Definition C.3.2 show that $f(\sigma(A)) = \sigma(f(A))$ and $[f(A), A] = 0$.*

Problem C.18 *Consider $f : \mathbb{C} \to \mathbb{C}$ entire and $A \in L(\mathcal{B}, \mathcal{B})$. Suppose that $\{z \in \mathbb{C} \ : \ f(z) = 0\} \cap \sigma(A) = \emptyset$. Show that $f(A)$ is invertible and $f(A)^{-1} = f^{-1}(A)$.*

Problem C.19 *Let $A \in L(\mathcal{B}, \mathcal{B})$. Suppose there exists a semi-line ℓ, starting from the origin, such that $\ell \cap \sigma(A) = \emptyset$. Prove that it is possible to define an operator $\ln A$ such that $e^{\ln A} = A$.*

Remark C.3.3 *Note that not all the interesting functions can be constructed in such a way. In fact, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is such that $A^2 = -\mathbb{1}$, thus it can be interpreted as a square rooth of $-\mathbb{1}$ but it cannot be obtained directly by a formula of the type (C.3.4).*

Problem C.20 *Suppose that $A \in L(\mathcal{B}, \mathcal{B})$ and $\sigma(A) = B \cup C$, $B \cap C = \emptyset$, suppose that the smooth closed curve $\gamma \subset \rho(A)$ contains B, but not C, in its interior, prove that*

$$P_B := \frac{1}{2\pi i} \int_\gamma (z\mathbb{1} - A)^{-1} dz$$

is a projector that does not depend on γ.

Note that by Problem C.17 easily follows that $P_B A = A P_B$. Hence, $AR(P_B) \subset R(P_B)$ and $AN(P_B) \subset N(P_B)$. Thus $\mathcal{B} = R(P_B) \oplus N(P_B)$ provides an invariant decomposition for A.

Problem C.21 *In the hypotheses of Problem C.20, prove that $A = P_B A P_B + (\mathbb{1} - P_B)A(\mathbb{1} - P_B)$.*

Problem C.22 *In the hypotheses of Problem C.20, prove that $\sigma(P_B A P_B) = B \cup \{0\}$. Moreover, if $\dim(R(P_B)) = D < \infty$, then the cardinality of B is $\leq D$.*

C.4 Perturbations

Let us consider $A, B \in L(\mathcal{B}, \mathcal{B})$ and the family of operators $A_\nu := A + \nu B$.

Lemma C.4.1 *For each $\delta > 0$ there exists $\nu_\delta \in \mathbb{R}$ such that, for all $|\nu| \leq \nu_\delta$, $\rho(A_\nu) \supset \{z \in \mathbb{C} \,:\, d(z, \sigma(A)) > \delta\}$.*

PROOF. Let $d(z, \sigma(A)) > \delta$, then

$$(z\mathbb{1} - A_\nu) = (z\mathbb{1} - A)\left[\mathbb{1} - \nu(z\mathbb{1} - A)^{-1}B\right] \tag{C.4.5}$$

Now $\|(z\mathbb{1} - A)^{-1}B\|$ is a continuous function in z outside $\sigma(A)$, moreover it is bounded outside a ball of large enough radius, hence there exists $M_\delta > 0$ such that $\sum_{d(z,\sigma(A))>\delta} \|(z\mathbb{1} - A)^{-1}B\| \leq M_\delta$. Choosing $\nu_\delta = (2M_\delta)^{-1}$ yields the result. □

Suppose that $\bar{z} \in \mathbb{C}$ is an isolated point of $\sigma(A)$, that is there exists $\delta > 0$ such that $\{z \in \mathbb{C} \,:\, |z - \bar{z}| \leq \delta\} \cap (\sigma(A) \setminus \{\bar{z}\}) = \emptyset$, then the above Lemma shows that, for ν small enough, $\{z \in \mathbb{C} \,:\, |z - \bar{z}| \leq \delta\}$ still contains an isolated part of the spectrum of $\sigma(A_\nu)$, let us call it B_ν, clearly $B_0 = \{\bar{z}\}$.

Problem C.23 *Let P_{B_ν} be defined as in Problem C.20. Prove that, for ν small enough, it is an analytic function of ν.*

Problem C.24 *If P, Q are two projectors and $\|P - Q\| < 1$, then $\dim(R(P)) = \dim(R(Q))$.*

The above two exercises imply that the dimension of the eigenspace $R(P_{B_\nu})$ is constant.

Next, we consider the case in which B_0 consist of one point and $\dim(R(P_{B_0})) = 1$, it follows that also B_ν must consist of only one point, let us set $P_\nu := P_{B_\nu}$.

Lemma C.4.2 *If $\dim(R(P_0)) = 1$, then A_ν has a unique eigenvalue z_ν in a neighborhood of $\bar{z}$, $z_0 = \bar{z}$. In addition z_ν is an analytic function of ν.*

PROOF. From the previous exercises it follows that P_ν is a rank one operator which depend analytically on ν. In addition, since P_ν is a rank one projector it must have the form $P_\nu w = v_\nu \ell_\nu(w)$, where $\ell_\nu \in \mathcal{B}'$.[6] Then $z_\nu P_\nu = P_\nu A_\nu P_\nu$. Next, setting $a(\nu) := \ell_0(P_\nu v_0) = \ell_\nu(v_0)\ell_0(v_\nu)$, we have that a is analytic and $a(0) = 1$. Thus $a \neq 0$ in a neighborhood of zero and $z_\nu = a(\nu)^{-1}\ell_0(P_\nu A_\nu P_\nu v_0)$ is analytic in such a neighborhood. □

Problem C.25 *If $\dim(R(P_0)) = 1$, then there exists $h_\nu \in \mathcal{B}$ and $\ell_\nu \in \mathcal{B}'$ such that $P_\nu f = h_\nu \ell_\nu(f)$ for each $f \in \mathcal{B}$. Prove that h_ν, ℓ_ν can be chosen to be analytic functions of ν.*

[6] By $\mathcal{B}'$, the dual space, we mean the set of bounded linear functionals on $\mathcal{B}$. Verify that is a Banach space with the norm $\|\ell\| = \sum_{w \in \mathcal{B}} \frac{|\ell(w)|}{\|w\|}$.

Hence in the case of $A \in L(\mathcal{B},\mathcal{B})$ with an isolated simple[7] eigenvalue $\bar{z}$ we have that the corresponding eigenvalue z_ν of $A_\nu = A + \nu B$, $B \in L(\mathcal{B},\mathcal{B})$, for ν small enough, depend smoothly from ν. In addition, using the notation of the previous Lemma, we can easily compute the derivative: differentiating $A_\nu v_\nu = z_\nu v_\nu$ with respect to ν and then setting $\nu = 0$, yields

$$Bv + Av_0' = z_0' v + \bar{z} v_0'.$$

But, for all $w \in \mathcal{B}$, $Pw = v\ell(w)$, with $\ell(Aw) = \bar{z}\ell(w)$ and $\ell(v) = 1$, thus applying ℓ to both sides of the above equation yields

$$z_0' = \ell(Bv).$$

Problem C.26 *Compute* v_0'.

Problem C.27 *What does it happen if the eigenspace associated to* $\bar{z}$ *is finite dimensional, but with dimension strictly larger than one?*

Hints to solving the Problems

C.1. The triangle inequality follows trivially from the triangle inequality of the norm of $\mathcal{B}$. To verify the completeness suppose that $\{B_n\}$ is a Cauchy sequence in $L(\mathcal{B},\mathcal{B})$. Then, for each $v \in \mathcal{B}$, $\{B_n v\}$ is a Cauchy sequence in $\mathcal{B}$, hence it has a limit, call it $B(v)$. We have so defined a function from $\mathcal{B}$ to teself. Show that such a function is linear and bounded, hence it defines an element of $L(\mathcal{B},\mathcal{B})$, which can easily be verified to be the limit of $\{B_n\}$.

C.2. Use the norm $\|A\| = \sup_{v\in\mathbb{R}^n} \frac{\|Av\|}{\|v\|}$.

C.3. Use the same norm as in Problem C.2.

C.4. The first part is trivial. For the second one can consider the vector space $\ell^2 = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{i=0}^\infty x_i^2 < \infty\}$. Equipped with the norm $\|x\| = \sqrt{\sum_{i=0}^\infty x_i^2}$ it is a Banach (actually Hilbert) space. Consider now the vectors $e_i \in \ell^2$ defined by $(e_i) = \delta_{ik}$ and the operator $(Ax)_k = \frac{1}{k}x_k$. Then $R(A) = \{x \in \ell^2 : \sum_{k=0}^\infty k^2 x_k^2 < \infty\}$, which is dense in ℓ^2 but strictly smaller.

C.5. Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

[7] That is with the associated eigenprojector of rank one.

C.6. Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

C.7. Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

C.8. Note that

$$(\zeta\mathbb{1} - A) = (z\mathbb{1} - A - (z-\zeta)\mathbb{1}) = (z\mathbb{1} - A)\left[\mathbb{1} - (z-\zeta)(z\mathbb{1} - A)^{-1}\right]$$

and that if $\|(z-\zeta)(z\mathbb{1}-A)^{-1}\| < 1$ then the inverse of $\mathbb{1} - (z-\zeta)(z\mathbb{1} - A)^{-1}$ is given by $\sum_{n=0}^{\infty}(z-\zeta)^n[(z\mathbb{1}-A)^{-1}]^n$ (the Von Neumann series–which really is just the geometric series).

C.9. If $\sigma(A) = \emptyset$, then $(z\mathbb{1} - A)^{-1}$ is an entire function, then the Von Neumann series shows that $(z\mathbb{1} - A)^{-1} = z^{-1}(\mathbb{1} - z^{-1}A)^{-1}$ goes to zero for large z, and then (C.2.3) shows that $(z\mathbb{1} - A)^{-1} = 0$ which is impossible.

C.10. Verify that $(z\mathbb{1} - \Pi)^{-1} = z^{-1}\left[\mathbb{1} - (z-1)^{-1}\Pi\right]$.

C.11. The idea is to look for eigenvalues by using Fourier series. Let $f = \sum_{k\in\mathbb{Z}} f_k e^{2\pi ikx}$ and consider the equation $\mathcal{L}f = zf$,

$$\sum_{k\in\mathbb{Z}} f_k \frac{1}{2}\left\{e^{\pi ikx} + e^{\pi ikx + \pi ik}\right\} = z\sum_{k\in\mathbb{Z}} f_k e^{2\pi ikx}.$$

Let us then restrict to the case in which $f_{2k+1} = 0$, then

$$\sum_{k\in\mathbb{Z}} f_{2k} e^{2\pi ikx} = z\sum_{k\in\mathbb{Z}} f_k e^{2\pi ikx}.$$

Thus we have a solution provided $f_{2k} = zf_k$, such conditions are satisfied by any sequence of the type

$$f_k = \begin{cases} z^j & \text{if } k = 2^j m, j \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

for $m \in \mathbb{N}$. It remains to verify that $\sum_{j=0}^{\infty} z^j e^{2\pi i 2^j x}$ belong to $\mathcal{C}^0$. This is the case if the series is uniformly convergent, which happens for $|z| < 1$. Thus all the points in $\{z \in \mathbb{C} \,:\, |z| < 1\}$ are point spectrum of infinite multiplicity. Since the spectrum is closed the statement of the Problem follows.

C.12. Let $p(z) = z^n$, then

$$\frac{1}{2\pi i}\int_\gamma z^n(z\mathbb{1} - A)^{-1}dz = A^n + \frac{1}{2\pi i}\int_\gamma (z^n - A^n)(z\mathbb{1} - A)^{-1}dz$$
$$= A^n + \sum_{k=0}^{n-1}\frac{1}{2\pi i}\int_\gamma z^k A^{n-k}dz = A^n.$$

The statement for general polynomial follows trivially.

C.14. Approximate by polynomials.

C.17. For $z \notin f(\sigma(A))$ it is well defined

$$K(z) := \frac{1}{2\pi i}\int_\gamma (z - f(\zeta))^{-1}(\zeta\mathbb{1} - A)^{-1}\,d\zeta,$$

with γ containing $\sigma(A)$ in its interior. By direct computation, using definition C.3.2, one can verify that $(z\mathbb{1} - f(A))K(z) = \mathbb{1}$, thus $\sigma(f(A)) \subset f(\sigma(A))$. On the other hand if, if f is not constant, then for each $z \in \mathcal{C}$ $f(z) - f(\xi) = (z-\xi)g(\xi)$. Hence, applying Definition C.3.2 and Problem C.16 it follows $f(z)\mathbb{1} - f(A) = (z-A)g(A)$ which shows that if $z \in \sigma(A)$, then $f(z) \in \sigma(A)$ (otherwise $(z - A)\left[g(A)(f(z)\mathbb{1} - f(A))^{-1}\right] = \mathbb{1}$).

C.19. Since one can define the logarithm on $\mathbb{C}\setminus\ell$, one can use Definition C.3.2 to define $\ln A$. It suffices to prove that if $f : U \to \mathcal{C}$ and $g : V \to \mathcal{C}$, with $\sigma(A) \subset U$, $f(U) \subset V$, then $g(f(A)) = g \circ f(A)$. Whereby showing that the definition C.3.2 is a reasonable one. Indeed, rememebring Problems C.17, C.18,

$$g(f(A)) = \frac{1}{2\pi i}\int_\gamma g(z)(z\mathbb{1} - f(A))^{-1}dz$$
$$= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_\gamma \frac{g(z)}{z - f(\xi)}(\xi\mathbb{1} - A)^{-1}dzd\xi$$
$$= \frac{1}{2\pi i}\int_{\gamma_1} g(f(\xi))(\xi\mathbb{1} - A)^{-1}d\xi = f \circ g(A).$$

From this imediately follows $e^{\ln A} = A$.

C.20. The non dependence on γ is obvious. A projector is characterized by the property $P^2 = P$. Thus

$$P_B^2 := \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2}(z\mathbb{1} - A)^{-1}(\zeta\mathbb{1} - A)^{-1}dzd\zeta$$
$$= \frac{1}{(2\pi i)^2}\int_{\gamma_1} dz\int_{\gamma_2} d\zeta(z - \zeta)^{-1}\left[(z\mathbb{1} - A)^{-1} - (\zeta\mathbb{1} - A)^{-1}\right].$$

If we have chosen γ_1 in the interior of γ_2, then $(z-\zeta)^{-1}(\zeta\mathbb{1}-A)^{-1}$ is analytic in the interior of γ_1, hence the corresponding integral gives zero. The other integral gives P_B, as announced.

C.21. Use the above decomposition and the fact that $(\mathbb{1}-P_B)$ is a projector.

C.22. The first part follows from the previous decomposition. Indeed, for z large (by Neumann series)

$$(z\mathbb{1}-A)^{-1} = (z\mathbb{1}-P_BAP_B)^{-1} + (z\mathbb{1}-(\mathbb{1}-P_B)A(\mathbb{1}-P_B))^{-1}.$$

Since the above functions are analytic in the respective resolvent sets it follows that $\sigma(A) \subset \sigma(P_BAP_B) \cup \sigma((\mathbb{1}-P_B)A(\mathbb{1}-P_B))$. Next, for $z \notin B$, define the operator

$$K(z) := \frac{1}{2\pi i}\int_\gamma (z-\xi)^{-1}(\xi\mathbb{1}-A)^{-1}\,d\xi,$$

where γ contains B, but no other part of the spectrum, in its interior. By direct computation (using Fubini and the standard facts about residues and integration of analytic functions) verify that

$$(z\mathbb{1}-P_BAP_B)K(z) = P_B.$$

This implies that, for $z \neq 0$, $(z\mathbb{1}-P_BAP_B)(K(z)+z^{-1}(\mathbb{1}-P_B)) = \mathbb{1}$, that is $(z\mathbb{1}-P_BAP_B)^{-1} = K(z)+z^{-1}(\mathbb{1}-P_B)$. Hence $\sigma(P_BAP_B) \subset B \cup \{0\}$. Since P_B has a kernel, zero must be in the spectrum. On the other hand the same argument applied to $\mathbb{1}-P_B$ yields $\sigma((\mathbb{1}-P_B)A)\mathbb{1}-P_B)) \subset C \cup \{0\}$, hence $\sigma(P_BAP_B) = B \cup \{0\}$.

The second property follows from the fact that P_BAP_B, when restricted to the space $R(P_B)$ is described by a $D \times D$ matrix A_B and the equation $\det(z\mathbb{1}-A_B) = 0$ is a polynomial of degree D in z and hence has exactly D solutions (counted with multiplicity).[8]

C.23. Use the representation in Problem C.20 and formula (C.4.5).

[8] This is the real reason why spectral theory is done over the complex rather than the real. You should be well aquatinted with the fact that a polynomial p of degree D has D root over $\mathbb{C}$ but, in case you have forgotten, consider the following: first a polynomial of degree larger than zero must have at least a root, otherwise $\frac{1}{p(z)}$ would be an entire function and hence

$$\frac{1}{p(z)} = \lim_{r\to\infty}\frac{1}{2\pi}\int_0^{2\pi} d\theta\,\frac{1}{p(z+re^{i\theta})} = 0.$$

Let z_1 be a root. By the Taylor expansion in z_1 follows the decomposition $p(z) = (z-z_1)p_1(z)$ where p_1 has degree $D-1$. The result follows by induction.

C.24. Note that $Q(\mathbb{1} + P - Q) = QP$, then $Q = QP(\mathbb{1} - (Q - P))^{-1}$, hence $\dim(R(P)) \geq \dim(R(Q))$, exchanging the role of P and Q the result follows.

C.25. Note that $\ell_\nu(h_\nu) = 1$ since P_ν is a projector, hence they are unique apart from a noralization factor. Then we can chose the normalization $\ell_\nu(h_0) = 1$ for all ν small enough. Thus $P_\nu f = h_\nu$, that is h_ν is analytic. Hence, for each $g \in \mathcal{B}$ and ν small, $\ell_\nu(g)\ell_0(h_\nu) = \ell_0(P_\nu g)$, which implies ℓ_ν analytic for ν small.

C.27 Think hard.[9]

[9] A good idea is to start by considering concrete examples, for instance

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Appendix D

Analytic Fredholm Theorem (fine rank)

Here we give a proof of the Analytic Fredholm alternative in a special case.

Theorem D.0.1 (Analytic Fredholm theorem–finite rank)[1] *Let D be an open connected subset of $\mathbb{C}$. Let $F : \mathbb{C} \to L(\mathbb{B}, \mathbb{B})$ be an analytic operator-valued function such that $F(z)$ is finite rank for each $z \in D$. Then, one of the following two alternatives holds true*

- $(\mathbb{1} - F(z))^{-1}$ *exists for no $z \in D$*
- $(\mathbb{1} - F(z))^{-1}$ *exists for all $z \in D \backslash S$ where S is a discrete subset of D (i.e. S has no limit points in D). In addition, if $z \in S$, then 1 is an eigenvalue for $F(z)$ and the associated eigenspace has finite multiplicity.*

Proof. First of all notice that, for each $z_0 \in D$ there exists $r > 0$ such that $D_{r(z_0)}(z_0) := \{z \in \mathbb{C} \,:\, |z - z_0| < r(z_0)\} \subset D$, and

$$\sup_{z \in D_{r(z_0)}(z_0)} \|F(z) - F(z_0)\| \leq \frac{1}{2}.$$

Clearly if we can prove the theorem in each such disk we are done.[2] Note that

$$\mathbb{1} - F(z) = \left(\mathbb{1} - F(z_0)(\mathbb{1} - [F(z) - F(z_0)])^{-1}\right)(\mathbb{1} - [F(z) - F(z_0)]).$$

[1] The present proof is patterned after the proof of the Analytic Fredholm alternative for compact operators (in Hilbert spaces) given in [RS80, Theorem VI.14]. There it is used the fact that compact operators in Hilbert spaces can always be approximated by finite rank ones. In fact the theorem holds also for compact operators in Banach spaces but the proof is a bit more involved.

[2] In fact, consider any connected compact set K contained in D. Let us suppose that for each $z_0 \in K$ we have a disk $D_{r(z_0)}(z_0)$ in the theorem holds. Since the disks $D_{r(z_0)/2}(z_0)$ form a covering for K we can extract a finite cover. If the first alternative holds in one such

Thus the invertibility of $\mathbb{1} - F(z)$ in $D_r(z_0)$ depends on the invertibility of $\mathbb{1} - F(z_0)(\mathbb{1} - [F(z) - F(z_0)])^{-1}$. Let us set $F_0(z) := F(z_0)(\mathbb{1} - [F(z) - F(z_0)])^{-1}$.

Let us start by looking at the equation

$$(\mathbb{1} - F_0(z))h = 0. \tag{D.0.1}$$

Clearly if a solution exists, then $h \in \text{Range}(F_0(z)) = \text{Range}(F(z_0)) := \mathbb{V}_0$. Since $\mathbb{V}_0$ is finite dimensional there exists a basis $\{h_i\}_{i=1}^N$ such that $h = \sum_i \alpha_i h_i$. On the other hand there exists an analytic matrix $G(z)$ such that[3]

$$F_0(z)h = \sum_{ij} G(z)_{ij} \alpha_j h_i.$$

Thus (D.0.1) is equivalent to

$$(\mathbb{1} - G(z))\alpha = 0,$$

where $\alpha := (\alpha_i)$.

The above equation can be satisfied only if $\det(\mathbb{1} - G(z)) = 0$ but the determinant is analytic hence it is either always zero or zero only at isolated points.[4]

Suppose the determinant different from zero, and consider the equation

$$(\mathbb{1} - F_0(z))h = g.$$

Let us look for a solution of the type $h = \sum_i \alpha_i h_i + g$. Substituting yields

$$\alpha - G(z)\alpha = \beta$$

where $\beta := (\beta_i)$ with $F_0(z)g =: \sum_i \beta_i h_i$. Since the above equation admits a solution, we have $\text{Range}(\mathbb{1} - F_0(z)) = \mathbb{B}$, Thus we have an everywhere defined inverse, hence bounded by the open mapping theorem.

We are thus left with the analysis of the situation $z \in S$ in the second alternative. In such a case, there exists h such that $(\mathbb{1} - F(z))h = 0$, thus

disk then, by connectness, it must hold on all K. Otherwise each $S \cap D_{r(z_0)/2}(z_0)$, and hence $K \cap S$, contains only finitely many points. The Theorem follows by the arbitrariness of K.

[3] To see the analyticity notice that we can construct linear functionals $\{\ell_i\}$ on $\mathbb{V}_0$ such that $\ell_i(h_j) = \delta_{ij}$ and then extend them to all $\mathbb{B}$ by the Hahn-Banach theorem. Accordingly, $G(z)_{ij} := \ell_j(F_0(z)h_i)$, which is obviously analytic.

[4] The attentive reader has certainly noticed that this is the turning point of the theorem: the discreteness of S is reduced to the discreteness of the zeroes of an appropriate analytic function: a determinant. A moment thought will immediately explain the effort made by many mathematicians to extend the notion of determinant (that is to define an analytic function whose zeroes coincide with the spectrum of the operator) beyond the realm of matrices (the so called Fredholm determinants).

one is an eigenvalue. On the other hand, if we apply the above facts to the function $\Phi(\zeta) := \zeta^{-1}F(z)$ analytic in the domain $\{\zeta \neq 0\}$ we note that the first alternative cannot take place since for $|\zeta|$ large enough $\mathbb{1} - \Phi(\zeta)$ is obviously invertible. Hence, the spectrum of $F(z)$ is discrete and can accumulate only at zero. This means that there is a small neighborhood around one in which $F(z)$ has no other eigenvalues, we can thus surround one with a small circle γ and consider the projector

$$\begin{aligned} P :=& \frac{1}{2\pi i}\int_\gamma (\zeta - F(z))^{-1} d\zeta = \frac{1}{2\pi i}\int_\gamma \left[(\zeta - F(z))^{-1} - \zeta^{-1}\right] d\zeta \\ =& \frac{1}{2\pi i}\int_\gamma F(z)\zeta^{-1}(\zeta - F(z))^{-1} d\zeta. \end{aligned}$$

By standard functional calculus it follows that P is a projector and it clearly projects on the eigenspace of the eigenvector one. But the last formula shows that P must project on a subspace of the range of $F(z)$, hence it must be finite dimensional. $\square$

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List of symbols

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