ESERCIZIO 1.

Calcolare le trasformate delle seguenti funzioni:

1) $f_1(t) = 4e^{5t} + 3\,\mathrm{sen}(t-1)\,u(t-1)$,

2) $f_2(t) = \cos^2(at)$,

3) $f_3(t) = t^2 \cos t$.

$$\mathcal{L}[f_1(t)](s) = 4\mathcal{L}[e^{5t}](s) + 3\,\mathcal{L}[\mathrm{sen}(t-1)\,u(t-1)](s)$$

$$= \frac{4}{s-5} + 3e^{-s}\frac{1}{s^2+1}.$$

$$\mathcal{L}[f_2(t)](s) = \mathcal{L}\left[\frac{1+\cos(2at)}{2}\right](s)$$

$$= \frac{1}{2}\mathcal{L}[1](s) + \frac{1}{2}\mathcal{L}[\cos(2at)](s)$$

$$= \frac{1}{2s} + \frac{1}{2}\cdot\frac{s}{(s^2+4a^2)}.$$

$$\mathcal{L}[f_3(t)](s) = \frac{d^2}{ds^2}\mathcal{L}[\cos(t)](s)$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{d}{ds}\left(\frac{1}{s^2+1} - \frac{s\cdot 2s}{(s^2+1)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right)$$

$$= \frac{-2s}{(s^2+1)^2} + (1-s^2)\,\frac{\cdot(-2)}{(s^2+1)^3}\cdot 2s$$

$$= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}$$

$$= \frac{2s^3-6s}{(s^2+1)^3}.$$

ESERCIZIO 2.

Calcolare $\mathcal{L}\left[e^{at}\cdot t\,\mathrm{sen}(bt)\right](s)$.

$$\mathcal{L}\left[\mathrm{sen}(bt)\right](s)=\frac{b}{s^2+b^2},$$

$$\mathcal{L}\left[t\,\mathrm{sen}(bt)\right](s)=-\frac{d}{ds}\left(\frac{b}{s^2+b^2}\right)=+\frac{2sb}{(s^2+b^2)^2},$$

$$\mathcal{L}\left[e^{at}t\,\mathrm{sen}(bt)\right](s)=\frac{2(s-a)b}{((s-a)^2+b^2)^2}.$$

ESERCIZIO 3.

Calcolare $\mathcal{L}\left[t^2\cos(2t)\right](s)$.

$$\mathcal{L}\left[t^2\cos(2t)\right](s)=\frac{d^2}{ds^2}\left(\mathcal{L}\left[\cos(2t)\right](s)\right)=\frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$$

$$=\frac{d}{ds}\left(\frac{s^2+4-s\cdot 2s}{(s^2+4)^2}\right)=\frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right)$$

$$=\frac{-2s(s^2+4)^2-(4-s^2)\cdot 2(s^2+4)\cdot 2s}{(s^2+4)^4}=\frac{2s(s^2-12)}{(s^2+4)^3}.$$

ESERCIZIO 4.

Calcolare $\mathcal{L}\left[\cos^3(t)\right](s)$.

Dato che

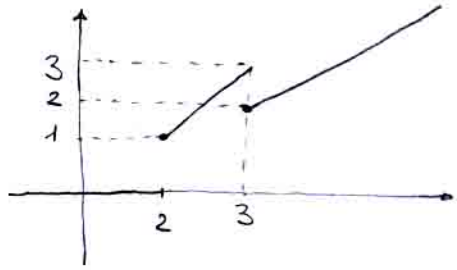
$$\cos^3 t=\left(\frac{e^{it}+e^{-it}}{2}\right)^3=\frac{1}{8}\left(e^{3it}+3e^{it}+3e^{-it}+e^{-3it}\right)$$

allora

$$\mathcal{L}\left[\cos^3(t)\right](s)=\frac{1}{8}\left(\frac{1}{s-3i}+\frac{3}{s-i}+\frac{3}{s+i}+\frac{1}{s+3i}\right)$$

$$=\frac{1}{4}\frac{s}{s^2+9}+\frac{3}{4}\frac{s}{s^2+1}=\frac{s(s^2+7)}{(s^2+1)(s^2+9)}.$$

ESERCIZIO 5.




Sia $f(t) = \begin{cases} 2t-3 & \text{per } t \in [2,3) \\ t-1 & \text{per } t \in [3,+\infty) \\ 0 & \text{altrove} \end{cases}$

Calcolare $\mathcal{L}[f](s)$.

Dato che $f(t) = (2t-3)\,u(t-2) + (t-1-2t+3)\,u(t-3)$

$$= (2t-3)\,u(t-2) - (t-2)\,u(t-3).$$

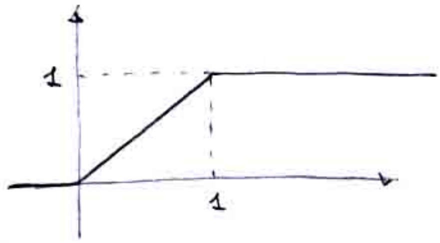
$$= 2(t-2)\,u(t-2) + u(t-2)$$

$$- (t-3)\,u(t-3) - u(t-3)$$

allora

$$\mathcal{L}[f](s) = 2\frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s} = \frac{e^{-2s}-e^{-3s}}{s} + \frac{2e^{-2s}-e^{-3s}}{s^2}.$$

ESERCIZIO 6.




Sia $f(t) = \min(t,1)$ per $t \geq 0$

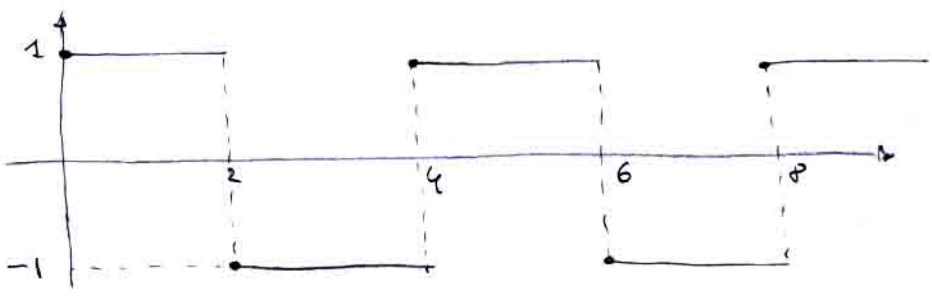
Calcolare $\mathcal{L}[f](s)$.

Dato che $f(t) = t\,u(t) + (1-t)\,u(t-1)$ allora

$$\mathcal{L}[f](s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} = \frac{1-e^{-s}}{s^2}.$$

ESERCIZIO 7.

Sia $f(t)$ la funzione periodica per $t \geq 0$




Calcolare $\mathcal{L}[f](s)$.

Il periodo è $T=4$. Sia $f_0(t)=\begin{cases}1 & \text{per } t\in[0,2),\\ -1 & \text{per } t\in[2,4),\\ 0 & \text{altrove.}\end{cases}$

allora $f(t)=\sum_{n=0}^{+\infty} f_0(t-4n)$.

Inoltre

$$f_0(t)=u(t)-2u(t-2)+u(t-4)$$

e così

$$\mathcal{L}[f_0](s)=\frac{1}{s}\left(1-2e^{-2s}+e^{4s}\right)=\frac{1}{s}\left(1-e^{-2s}\right)^2.$$

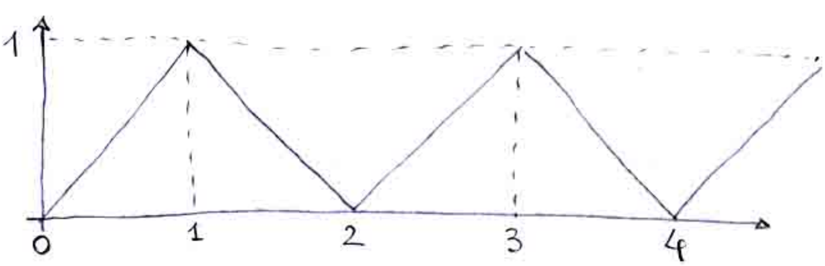
Infine

$$\mathcal{L}[f](s)=\frac{\mathcal{L}[f_0](s)}{1-e^{-Ts}}=\frac{(1-e^{-2s})^2}{s(1-e^{-4s})}=\frac{1-e^{-2s}}{s(1+e^{-2s})}$$

$$=\frac{1}{s}\cdot\frac{e^{s}-e^{-s}}{e^{s}+e^{-s}}=\frac{tgh(s)}{s}.$$

ESERCIZIO 8.

Sia $f(t)$ la funzione periodica per $t\geq 0$




Calcolare $\mathcal{L}[f](s)$.

Il periodo è $T=2$. Sia $f_0(t)=\begin{cases}t & \text{per } t\in[0,1)\\ 2-t & \text{per } t\in[1,2)\\ 0 & \text{altrove}\end{cases}$.

allora $f(t)=\sum_{n=0}^{+\infty} f_0(t-2n)$. Inoltre

$$f_0(t)=t\,u(t)+(2-2t)\,u(t-1)+(t-2)\,u(t-2)$$
$$=t\,u(t)-2(t-1)\,u(t-1)+(t-2)\,u(t-2)$$

e così

$$\mathcal{L}[f_0](s)=\frac{1}{s^2}-\frac{2e^{-s}}{s^2}+\frac{e^{-2s}}{s^2}=\frac{(1-e^{-s})^2}{s^2}.$$

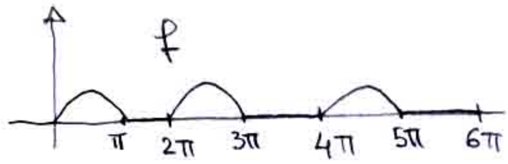
Infine

$$\mathcal{L}[f](s) = \frac{\mathcal{L}[f_0](s)}{1-e^{-Ts}} = \frac{(1-e^{-s})^2}{s^2(1-e^{-2s})} = \frac{1}{s^2}\,\frac{1-e^{-s}}{1+e^{-s}} = \frac{\mathrm{tgh}(s)}{s^2}.$$

## ESERCIZIO 9.

Sia $f_0(t) = \begin{cases} \mathrm{sen}(t) & \text{per } t \in [0,\pi) \\ 0 & \text{altrove} \end{cases}$ e

sia $f(t) = \sum_{n=0}^{+\infty} f_0(t - 2\pi n)$




Calcolare $\mathcal{L}[f](s)$.

Allora $f_0(t) = \mathrm{sen}(t)\,u(t) - \mathrm{sen}(t)\,u(t-\pi)$

$= \mathrm{sen}(t)\,u(t) + \mathrm{sen}(t-\pi)\,u(t-\pi)$

così

$$\mathcal{L}[f_0](s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

e

$$\mathcal{L}[f](s) = \frac{\mathcal{L}[f_0](s)}{1-e^{-2\pi s}} = \frac{1}{s^2+1}\cdot\frac{1+e^{-\pi s}}{1-e^{-2\pi s}} =$$

$$= \frac{1}{s^2+1}\cdot\frac{1}{1-e^{-\pi s}}.$$

## ESERCIZIO 10

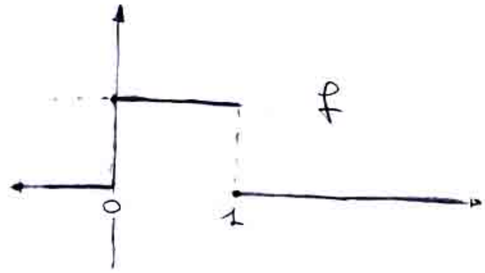
Calcolare $f(t) = t^2 * t^3$ e $\mathcal{L}[f](s)$.

$$f(t) = \int_0^t (t-\tau)^2(\tau)^3\,d\tau = \int_0^t (t^2\tau^3 - 2t\tau^4 + \tau^5)\,d\tau$$

$$= \left[t^2\frac{\tau^4}{4} - 2t\frac{\tau^5}{5} + \frac{\tau^6}{6}\right]_0^t = \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right)t^6 = \frac{t^6}{60},$$

$$\mathcal{L}[f](s) = \mathcal{L}[t^2](s)\,\mathcal{L}[t^3](s) = \frac{2!}{s^3}\cdot\frac{3!}{s^4} = \frac{12}{s^7} = \mathcal{L}\left[\frac{t^6}{60}\right](s).$$

ESERCIZIO 11.




Sia $f(t) = u(t) - u(t-1)$

Calcolare $f*f$ e $\mathcal{L}[f*f]$.

In questo caso, il calcolo di $f*f$ attraverso l'uso della definizione è un po' complicato.

Tali difficoltà possono essere superate calcolando prima $\mathcal{L}[f*f]$ e poi facendo l'anti-trasformata.

$$\mathcal{L}[f*f](s) = \left(\mathcal{L}[f](s)\right)^2 = \left(\frac{1-e^{-s}}{s}\right)^2$$

$$= \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}.$$

Ora

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) = t \cdot u(t),$$

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right](t) = (t-1)\,u(t-1),$$

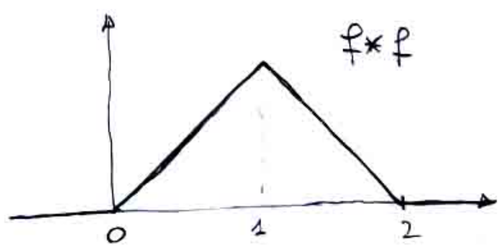
$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right](t) = (t-2)\,u(t-2),$$

e per linearità

$$(f*f)(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\right](t)$$

$$= t\,u(t) - 2(t-1)\,u(t-1) + (t-2)\,u(t-2)$$

$$= \begin{cases} t & \text{per } t \in [0,1) \\ 2-t & \text{per } t \in [1,2) \\ 0 & \text{altrove}. \end{cases}$$



ESERCIZIO 12.

Sia $f(t) = \operatorname{sen} t \cdot u(t)$ e $g(t) = \cos t \cdot u(t)$.
Calcolare $(f * g)(t)$.
Seguiamo il metodo del precedente esercizio.

$$\mathcal{L}[f*g](s) = \frac{1}{1+s^2} \cdot \frac{s}{1+s^2} = \frac{s}{(1+s^2)^2}.$$

allora

$$(f*g)(t) = \mathcal{L}^{-1}\left[\frac{s}{(1+s^2)^2}\right](t) =$$

$$= \operatorname{Res}\left(\frac{s e^{st}}{(1+s^2)^2}, i\right) + \operatorname{Res}\left(\frac{s e^{st}}{(1+s^2)^2}, -i\right)$$

$$= \left(\frac{d}{ds}\left(\frac{s e^{st}}{(s+i)^2}\right)\right)_{s=i} + \left(\frac{d}{ds}\left(\frac{s e^{st}}{(s-i)^2}\right)\right)_{s=-i}$$

$$= \left(\frac{s e^{st}}{(s+i)^2} \cdot \left(\frac{1}{s} + t - \frac{2}{s+i}\right)\right)_{s=i}$$

$$+ \left(\frac{s e^{st}}{(s-i)^2} \cdot \left(\frac{1}{s} + t - \frac{2}{s-i}\right)\right)_{s=-i}$$

$$= \frac{e^{it}}{4i} t - \frac{e^{-it} t}{4i} = \frac{t}{2}\left(\frac{e^{it} - e^{-it}}{2i}\right) = \frac{1}{2} t \cdot \operatorname{sen}(t).$$

ESERCIZIO 13.

Calcolare $\mathcal{L}^{-1}\left[\dfrac{8s^2 - 4s + 12}{s(s^2+4)}\right]$.

Proviamo a fare il calcolo usando la decomposizione in funzioni razionali semplici:

$$\frac{8s^2 - 4s + 12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C}{s^2+4} = \frac{(A+B)s^2 + Cs + 4A}{s(s^2+4)}$$

Risolvendo il sistema

$$\begin{cases} A+B=8 \\ C=-4 \\ 4A=12 \end{cases}$$

si ottiene $A=3$, $B=5$ e $C=-4$. Quindi per $t \geq 0$

$$\mathcal{L}^{-1}\left[\frac{8s^2-4s+12}{s(s^2+4)}\right] = 3\cdot\mathcal{L}^{-1}\left[\frac{1}{s}\right] + 5\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= 3 + 5\cos(2t) - 2\,\mathrm{sen}\,2t.$$

Ora proviamo a fare il calcolo usando i residui:

$$\mathcal{L}^{-1}\left[\frac{8s^2-4s+12}{s(s^2+4)}\right] = \mathrm{Res}\left(\frac{8s^2-4s+12}{s(s^2+4)}e^{st}, 0\right) + \mathrm{Res}(\dots, 2i) + \mathrm{Res}(\dots, -2i)$$

$$= \left(\frac{8s^2-4s+12}{s^2+4}\cdot e^{st}\right)_{s=0} + \underbrace{\left(\frac{8s^2-4s+12}{s(s+2i)}e^{st}\right)_{s=2i} + \left(\frac{8s^2-4s+12}{s(s-2i)}e^{st}\right)_{s=-2i}}_{\text{complessi coniugati}}$$

$$= 3 + 2\mathrm{Re}\left(\frac{8\cdot(-4) - 4\cdot 2i + 12}{2i\cdot 4i}e^{2it}\right)$$

$$= 3 + 2\mathrm{Re}\left(\frac{-8i-20}{-8}e^{2it}\right)$$

$$= 3 + \mathrm{Re}\left((2i+5)(\cos 2t + i\,\mathrm{sen}\,2t)\right)$$

$$= 3 + 5\cos 2t - 2\,\mathrm{sen}\,2t.$$

ESERCIZIO 14.

Calcolare $\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right]$.

$$\mathcal{L}^{-1}\left[\frac{2(s+1)}{(s+1)^2+4}\right] = 2e^{-t}\cdot\cos 2t.$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t$$

ESERCIZIO 15.

Calcolare $\mathcal{L}^{-1}\left[\dfrac{2e^{-s}}{s^3+3s^2+2s}\right]$.

Determiniamo prima

$$\mathcal{L}^{-1}\left[\frac{2}{s(s+1)(s+2)}\right] = \operatorname{Res}\left(\frac{2e^{st}}{s(s+1)(s+2)}, 0\right) + \operatorname{Res}(\dots,-1) + \operatorname{Res}(\dots,-2)$$

$$= \left(\frac{2e^{st}}{(s+1)(s+2)}\right)_{s=0} + \left(\frac{2e^{st}}{s(s+2)}\right)_{s=-1} + \left(\frac{2e^{st}}{s(s+1)}\right)_{s=-2}$$

$$= \left(1 - 2e^{-t} + e^{-2t}\right) = \left(1 - e^{-t}\right)^2 \quad \text{per } t \geq 0.$$

Così

$$\mathcal{L}^{-1}\left[\frac{2e^{-s}}{s^3+3s^2+2s}\right](t) = \left(1 - e^{-(t-1)}\right)^2 \cdot u(t-1).$$

ESERCIZIO 16.

Calcolare $\mathcal{L}^{-1}\left[\dfrac{1}{((s+1)^2+4)^2}\right]$.

Determiniamo prima

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)^2}\right] = \operatorname{Res}\left(\frac{e^{st}}{(s^2+4)^2}, 2i\right) + \operatorname{Res}\left(\frac{e^{st}}{(s^2+4)^2}, -2i\right).$$

Ora

$$\operatorname{Res}\left(\frac{e^{st}}{(s^2+4)^2}, 2i\right) = \left(\frac{d}{ds}\left(\frac{e^{st}}{(s+2i)^2}\right)\right)_{s=2i} = \left(\frac{t e^{st}(s+2i)^{\cancel{2}} - e^{st}\cdot 2\cancel{(s+2i)}}{(s+2i)^{\cancel{4}\,3}}\right)_{s=2i}$$

$$= e^{2it}\left(\frac{t\cdot 4i - 2}{(4i)^3}\right) = e^{2it}\left(-\frac{t}{16} - \frac{i}{32}\right).$$

Analogamente

$$\operatorname{Res}\left(\frac{e^{st}}{(s^2+4)^2}, -2i\right) = e^{-2it}\left(-\frac{t}{16} + \frac{i}{32}\right).$$

complessi coniugati

Così

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)^2}\right] = 2\,\mathrm{Re}\left(e^{2it}\left(-\frac{t}{16}-\frac{i}{32}\right)\right)$$

$$= \mathrm{Re}\left((\cos 2t + i \operatorname{sen} 2t)\left(-\frac{t}{8}-\frac{i}{16}\right)\right)$$

$$= -\frac{t}{8}\cos 2t + \frac{\operatorname{sen} 2t}{16} = \frac{1}{16}\left(-2t\cos 2t + \operatorname{sen} 2t\right)$$

e

$$\mathcal{L}^{-1}\left[\frac{1}{((s+1)^2+4)^2}\right] = e^{-t}\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)^2}\right] = \frac{e^{-t}}{16}\left(-2t\cos 2t + \operatorname{sen} 2t\right).$$

ESERCIZIO 17.

Risolvere il problema di Cauchy

$$\begin{cases} x''(t) - 2x'(t) + 2x(t) = 0 \\ x(0) = 5, \quad x'(0) = 0 \end{cases}$$

Allora

$$s^2 X - 5s - 0 - 2(sX - 5) + 2X = 0$$

da cui

$$X(s) = \frac{5s-10}{s^2-2s+2} = \frac{5(s-2)}{(s-1)^2+1} = \frac{5(s-1)}{(s-1)^2+1} - \frac{5}{(s-1)^2+1}$$

Quindi

$$x(t) = \mathcal{L}^{-1}[X] = 5\,\mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] - 5\,\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= 5e^t \cdot \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - 5e^t\,\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 5e^t(\cos t - \operatorname{sen} t) \qquad \text{per } t \geq 0.$$

## ESERCIZIO 18.

Risolvere il problema di Cauchy

$$\begin{cases} x''(t) + 2x'(t) + x(t) = \delta(t-1) \\ x(0)=0, \; x'(0)=0 \end{cases}$$

Allora

$$s^2 X + 2sX + X = e^{-s}$$

da cui

$$X(s) = \frac{e^{-s}}{(s+1)^2}.$$

Quindi

$$x(t) = \mathcal{L}^{-1}[X](t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right](t-1)\cdot u(t-1)$$

$$= e^{-(t-1)}\cdot(t-1)\cdot u(t-1) \qquad \text{per } t \geq 0$$

perché $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right](t) = e^{-t}\cdot t$.

## ESERCIZIO 19.

Risolvere il problema di Cauchy

$$\begin{cases} x''(t) + 2x'(t) + x(t) = 4e^{-t} \\ x(0)=2, \; x'(0)=-1. \end{cases}$$

Allora

$$s^2 X - 2s + 1 + 2(sX - 2) + X = \frac{4}{s+1}$$

e

$$X(s) = \frac{2s+3+\frac{4}{s+1}}{(s+1)^2} = \frac{2(s+1)+1+\frac{4}{s+1}}{(s+1)^2}$$

$$= \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}.$$

Quindi

$$x(t)=\mathcal{L}^{-1}[X]=2\,\mathcal{L}\left[\frac{1}{s+1}\right]+\mathcal{L}\left[\frac{1}{(s+1)^2}\right]+4\,\mathcal{L}\left[\frac{1}{(s+1)^3}\right]$$

$$=2e^{-t}+t\,e^{-t}+2e^{-t}\cdot t^2$$

$$=e^{-t}(2t^2+t+2) \qquad \text{per } t\geq 0.$$

ESERCIZIO 20.

Risolvere il problema di Cauchy

$$\begin{cases} x''(t)+3x'(t)+2x(t)=2u(t-2) \\ x(0)=0,\ x'(0)=2. \end{cases}$$

Allora

$$s^2X-0\cdot s-2+3(sX-0)+2X=2\frac{e^{-2s}}{s}$$

e dunque

$$X(s)=\frac{1}{s^2+3s+2}\cdot\left(2+\frac{2e^{-2s}}{s}\right)$$

$$=\frac{2}{(s+1)(s+2)}+\frac{2e^{-2s}}{s(s+1)(s+2)}.$$

Ma per $t\geq 0$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=\operatorname{Res}\left(\frac{e^{st}}{(s+1)(s+2)},-1\right)+\operatorname{Res}\left(\frac{e^{st}}{(s+1)(s+2)},-2\right)$$

$$=\left(\frac{e^{st}}{s+2}\right)_{s=-1}+\left(\frac{e^{st}}{s+1}\right)_{s=-2}=e^{-t}-e^{-2t}$$

e

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]=\operatorname{Res}(\dots,0)+\operatorname{Res}(\dots,-1)+\operatorname{Res}(\dots,-2)$$

$$=\left(\frac{e^{st}}{(s+1)(s+2)}\right)_{s=0}+\left(\frac{e^{st}}{s(s+2)}\right)_{s=-1}+\left(\frac{e^{st}}{s(s+1)}\right)_{s=-2}$$

$$= \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} = \frac{1}{2}(1-e^{-t})^2.$$

Così

$$x(t) = \mathcal{L}^{-1}[X] = 2(e^{-t} - e^{-2t}) + (1 - e^{-(t-2)})^2 \cdot u(t-2).$$

ESERCIZIO 21.

Risolvere il sistema

$$\begin{cases} x'(t) = x(t) - y(t) \\ y'(t) = x(t) + y(t) \end{cases} \quad \text{con } x(0) = 1 \text{ e } y(0) = 0.$$

Allora

$$\begin{cases} sX - 1 = X - Y \\ sY - 0 = X + Y \end{cases} \Rightarrow \begin{bmatrix} s-1 & 1 \\ -1 & s-1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

da cui, per $t \geq 0$

$$X = \frac{\begin{vmatrix} 1 & 1 \\ 0 & s-1 \end{vmatrix}}{\begin{vmatrix} s-1 & 1 \\ -1 & s-1 \end{vmatrix}} = \frac{s-1}{(s-1)^2+1} \xrightarrow{\mathcal{L}^{-1}} x(t) = e^t \cos t,$$

$$Y = \frac{\begin{vmatrix} s-1 & 1 \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} s-1 & 1 \\ -1 & s-1 \end{vmatrix}} = \frac{1}{(s-1)^2+1} \xrightarrow{\mathcal{L}^{-1}} y(t) = e^t \operatorname{sen} t.$$

ESERCIZIO 22.

Risolvere il sistema

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -x(t) + 2\,\text{sen}\,t \end{cases} \qquad \text{con } x(0)=0,\ y(0)=0.$$

Allora

$$\begin{cases} sX - 0 = Y \\ sY - 0 = -X + \dfrac{2}{s^2+1} \end{cases} \Rightarrow \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{s^2+1} \end{bmatrix}$$

da cui

$$X = \frac{\begin{vmatrix} 0 & -1 \\ \frac{2}{s^2+1} & s \end{vmatrix}}{\begin{vmatrix} s & -1 \\ 1 & s \end{vmatrix}} = \frac{2}{(s^2+1)^2}$$

allora per $t \geq 0$

$$x(t) = \mathcal{L}^{-1}[X] = \text{Res}\left(\frac{2e^{st}}{(s^2+1)^2}, i\right) + \text{Res}\left(\frac{2e^{st}}{(s^2+1)^2}, -i\right)$$

$$= 2\,\text{Re}\left(\frac{d}{ds}\left(\frac{2e^{st}}{(s+i)^2}\right)\right)_{s=i}$$

$$= 4\,\text{Re}\left(\frac{te^{st}(s+i)^2 - e^{st}\cdot 2(s+i)}{(s+i)^4}\right)_{s=i}$$

$$= 4\,\text{Re}\left(\frac{e^{it}\cdot(t\cdot(-4) - 4i)}{16}\right)$$

$$= \text{Re}\left((\cos t + i\,\text{sen}\,t)(-t-i)\right) = -t\cos t + \text{sen}\,t.$$

e

$$y(t) = x'(t) = -\cancel{\cos t} + t\,\text{sen}\,t + \cancel{\cos t} = t\,\text{sen}\,t.$$