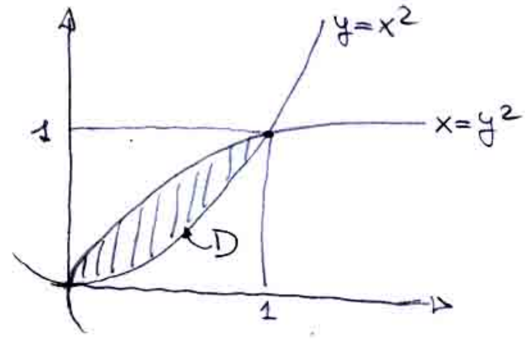
ESERCIZIO 1.

$$\iint_D y^2 \, dx\,dy \quad \text{con } D = \{(x,y) : x^2 \leq y \text{ e } y^2 \leq x\}$$



visto che la funzione da integrare dipende solo da y si preferisce impostare il calcolo così

$$= \int_{y=0}^{1} y^2 \left(\int_{x=y^2}^{x=\sqrt{y}} dx \right) dy = \int_0^1 y^2 \cdot (\sqrt{y} - y^2)\, dy$$

$$= \int_0^1 (y^{\frac{5}{2}} - y^4)\, dy = \left[\frac{2}{7} y^{\frac{7}{2}} - \frac{1}{5} y^5 \right]_0^1 = \frac{3}{35}$$

ESERCIZIO 2.

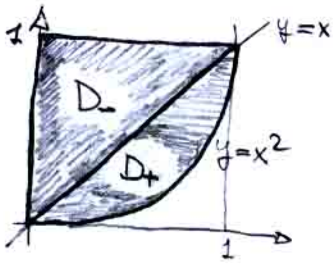
$$\iint_D |xy| \, dx\,dy \quad \text{con } D = \{(x,y) : x^2 + y^2 \leq 1\}$$

$$= 4 \int_0^1 x \left(\int_0^{\sqrt{1-x^2}} y \, dy \right) dx = 4 \int_0^1 x \left[\frac{y^2}{2} \right]_0^{\sqrt{1-x^2}} dx$$

$$= 2 \int_0^1 x (1 - x^2)\, dx = 2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2}$$

ESERCIZIO 3.

$$\iint_D |x-y|\,dx\,dy \quad \text{con } D=\{(x,y):\ 0\le x\le 1,\ x^2\le y\le 1\}$$



$$=\iint_{D_+}(x-y)\,dx\,dy+\iint_{D_-}(y-x)\,dx\,dy$$

$$=\int_{x=0}^{1}\left(\int_{y=x^2}^{x}(x-y)\,dy\right)dx+\int_{x=0}^{1}\left(\int_{y=x}^{1}(y-x)\,dy\right)dx$$

$$=\int_0^1\left[xy-\frac{y^2}{2}\right]_{x^2}^{x}dx+\int_0^1\left[\frac{y^2}{2}-xy\right]_x^1 dx$$

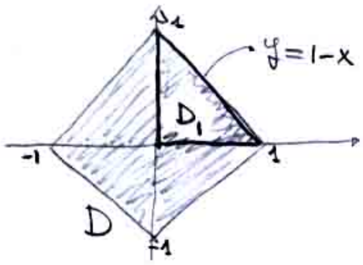
$$=\int_0^1\left(\cancel{x^2}-\cancel{\frac{x^2}{2}}-x^3+\frac{x^4}{2}+\frac{1}{2}-x-\cancel{\frac{x^2}{2}}+x^2\right)dx$$

$$=\left[-\frac{x^4}{4}+\frac{x^5}{10}+\frac{1}{2}x-\frac{x^2}{2}+\frac{x^3}{3}\right]_0^1=-\frac{1}{4}+\frac{1}{10}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}+\frac{1}{3}=\frac{11}{60}$$

ESERCIZIO 4.

$$\iint_D x^2y^2\,dx\,dy \quad \text{con } D=\{|x|+|y|\le 1\}$$

Per simmetrie di D e $f(x,y)=x^2y^2$

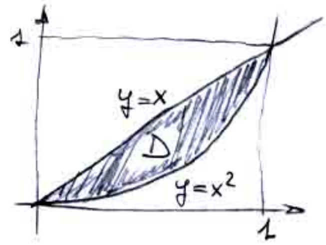


$$=4\iint_{D_1}x^2y^2\,dx\,dy=4\int_{x=0}^{1}x^2\left(\int_{y=0}^{1-x}y^2\,dy\right)dx$$

$$=4\int_0^1 x^2\left[\frac{y^3}{3}\right]_0^{1-x}=\frac{4}{3}\int_0^1 x^2(1-x)^3\,dx=\dots=\frac{1}{45}$$

ESERCIZIO 5.

$$\iint_D \frac{xe^y}{y}\,dx\,dy \quad \text{con } D=\{(x,y):\ 0\le x\le 1,\ x^2\le y\le x\}$$



$$=\int_{x=0}^{1} x\left(\int_{y=x^2}^{x}\frac{e^y}{y}\,dy\right)dx = ?$$

$$=\int_{y=0}^{1}\frac{e^y}{y}\int_{x=y}^{\sqrt{y}} x\,dx = \int_0^1 \frac{e^y}{y}\left[\frac{x^2}{2}\right]_y^{\sqrt{y}} dy$$

$$=\frac{1}{2}\int_0^1 \left(e^y - ye^y\right)dy = \frac{1}{2}\left[e^y\right]_0^1 - \frac{1}{2}\left[ye^y - e^y\right]_0^1 = \frac{e-2}{2}$$

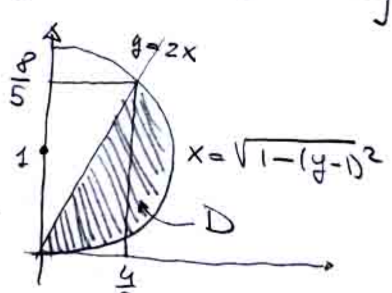
ESERCIZIO 6.

$$\iint \frac{1}{x+y}\,dx\,dy \quad \text{con } D=\{(x,y):\ 1\le x\le 2,\ 0\le y\le x\}$$

$$=\int_1^2\left(\int_0^x \frac{1}{x+y}\,dy\right)dx = \int_1^2 \left[\log(x+y)\right]_0^x dx = \int_1^2 \log\left(\frac{2\cancel{x}}{\cancel{x}}\right)dx = \log 2$$

ESERCIZIO 7.

$$\iint_D x\,dx\,dy \quad \text{con} \quad D=\{(x,y):\ 0\le y\le 2x,\ x^2+(y-1)^2\le 1\}$$



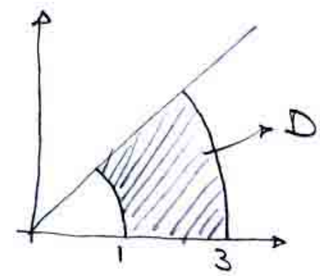
$$=\int_{y=0}^{8/5}\left(\int_{x=\frac{y}{2}}^{\sqrt{1-(y-1)^2}} x\,dx\right)dy = \frac{1}{2}\int_0^{8/5}\left(1-(y-1)^2-\frac{y^2}{4}\right)dy$$

$$=\frac{1}{2}\int_0^{8/5}\left(-\frac{5}{4}y^2+2y\right)dy = -\frac{5}{8}\left[\frac{y^3}{3}\right]_0^{8/5} + \left[\frac{y^2}{2}\right]_0^{8/5} = \frac{32}{75}$$

ESERCIZIO 8.

$$\iint_D \frac{y^2}{x^2}\,dx\,dy \quad \text{con } D=\left\{(x,y):\ \begin{array}{l}0\le y\le x,\\ 1\le x^2+y^2\le 9\end{array}\right\}$$

$$=\int_{\rho=1}^{3}\left(\int_{\theta=0}^{\pi/4}\frac{\rho^2\sin^2\theta}{\rho^2\cos^2\theta}\,d\theta\right)\rho\,d\rho$$



$$=\int_{\rho=1}^{3}\left[\operatorname{tg}\theta-\theta\right]_0^{\pi/4}\cdot\rho\,d\rho$$

$$D(\operatorname{tg}\theta)=\frac{1}{\cos^2\theta}=\operatorname{tg}^2\theta+1$$

$$\Rightarrow \int \operatorname{tg}^2\theta\,d\theta=\operatorname{tg}\theta-\theta+c$$

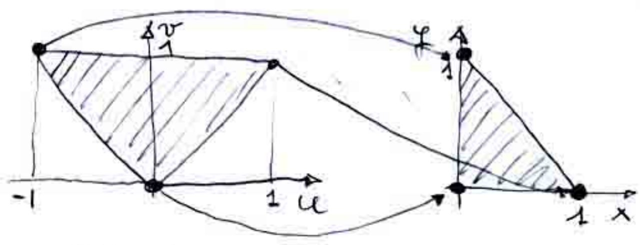
$$=\left(1-\frac{\pi}{4}\right)\cdot\left[\frac{\rho^2}{2}\right]_1^3=4-\pi$$

ESERCIZIO 9.

$$\iint_D e^{\frac{x-y}{x+y}}\,dx\,dy \quad \text{con } D=\{(x,y):\ x+y\le 1,\ x\ge 0,\ y\ge 0\}$$

Cambio di variabile

$$\begin{cases} u=x-y\\ v=x+y\end{cases}$$



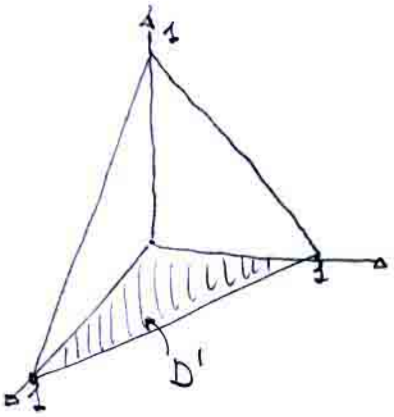
$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\xrightarrow{\det} 2$$

$$=\int_{v=0}^{1}\int_{u=-v}^{v} e^{\frac{u}{v}}\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

$$=\frac{1}{2}\int_{v=0}^{1}\left[e^{\frac{u}{v}}\cdot v\right]_{-v}^{v}du=\frac{e^2-1}{2e}\int_0^1 v\,dv=\frac{e^2-1}{4e}$$

ESERCIZIO 10.

$$\iiint_D \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$ con D il tetraedro delimitato dai piani coordinati e dal piano $x+y+z=1$.



$$= \iint_{D'} \left(\int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dz \right) dx\,dy$$

$$= \iint_{D'} \left[\frac{(x+y+z+1)^{-2}}{-2} \right]_0^{1-x-y} dx\,dy$$

$$= \int_{x=0}^{1} \left(\int_{y=0}^{1-x} \left(-\frac{1}{8} + \frac{1}{2(x+y+1)^2} \right) dy \right) dx$$

$$= -\frac{1}{8} \cdot |D'| + \frac{1}{2} \int_{x=0}^{1} \left[\frac{(x+y+1)^{-1}}{-1} \right]_0^{1-x} dx$$

$$= -\frac{1}{16} + \frac{1}{2} \int_{x=0}^{1} \left(-\frac{1}{2} + \frac{1}{x+1} \right) dx$$

$$= -\frac{1}{16} - \frac{1}{4} + \frac{1}{2} \left[\log(x+1) \right]_0^1 = -\frac{5}{16} + \frac{1}{2} \log 2$$

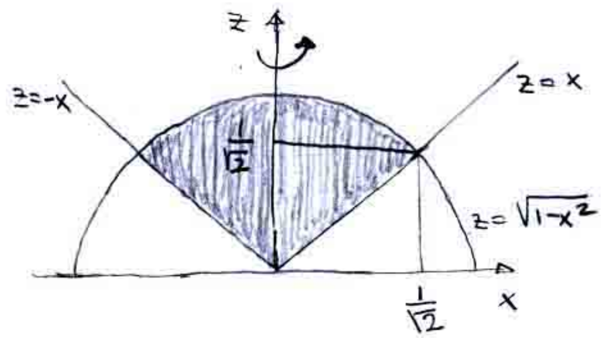
ESERCIZIO 11.

$\iiint_D \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$ con D dato dall'inter-

sezione della sfera $\{x^2+y^2+z^2\leq 1\}$, dal semipiano

$\{z\geq 0\}$ e dal cono $\{z^2\geq x^2+y^2\}$.

Il dominio è ottenuto ruotando la sezione



In coordinate sferiche:

$$= \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\frac{\pi}{4}} \rho\cdot(\rho^2\,\mathrm{sen}\,\varphi)\,d\varphi\,d\theta\,d\rho$$

$$= \left[\frac{\rho^4}{4}\right]_0^1\cdot\left[\theta\right]_0^{2\pi}\cdot\left[-\cos\varphi\right]_0^{\pi/4} = \pi\,\frac{(2-\sqrt2)}{4}$$

In coordinate polari

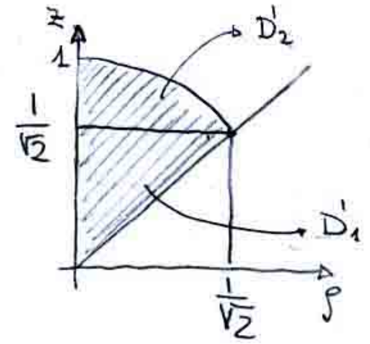
$$= \int_{\theta=0}^{2\pi}\left(\int_{\rho=0}^{1/\sqrt2}\left(\int_{z=\rho}^{\sqrt{1-\rho^2}}\sqrt{\rho^2+z^2}\,dz\right)\rho\,d\rho\right)d\theta$$

Dato che è più facile integrare $\rho\sqrt{\rho^2+z^2}$ in $d\rho$ conviene cambiare l'ordine di integrazione.

Prima integriamo in $d\theta$ ($\rho\sqrt{\rho^2+z^2}$ non dipende da θ)

$$= 2\pi \iint_{D'} \sqrt{\rho^2+z^2}\;\rho\, d\rho\, dz$$

con



$$D' = D'_1 \cup D'_2$$

Quindi integriamo rispetto a $d\rho$ e poi a dz

$$= 2\pi \underbrace{\int_{z=0}^{\frac{1}{\sqrt{2}}} \left(\int_{\rho=0}^{z}}_{D'_1} \sqrt{\rho^2+z^2}\;\rho\, d\rho \right) dz + 2\pi \underbrace{\int_{z=\frac{1}{\sqrt{2}}}^{1} \left(\int_{\rho=0}^{\sqrt{1-z^2}}}_{D'_2} \sqrt{\rho^2+z^2}\;\rho\, d\rho \right) dz$$

$$= 2\pi \int_{z=0}^{\frac{1}{\sqrt{2}}} \left[\frac{(\rho^2+z^2)^{3/2}}{3} \right]_{\rho=0}^{z} dz + 2\pi \int_{z=\frac{1}{\sqrt{2}}}^{1} \left[\frac{(\rho^2+z^2)^{3/2}}{3} \right]_{\rho=0}^{\sqrt{1-z^2}} dz$$

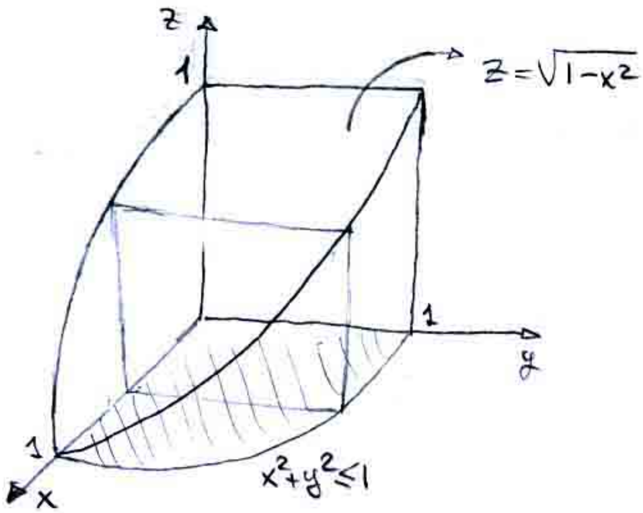
$$= 2\pi \int_{z=0}^{\frac{1}{\sqrt{2}}} \frac{2^{3/2}-1}{3} \cdot z^3\, dz + 2\pi \int_{z=\frac{1}{\sqrt{2}}}^{1} \frac{1-z^3}{3}\, dz$$

$$= 2\pi \cdot \left(\frac{2^{3/2}-1}{3} \right) \left[\frac{z^4}{4} \right]_0^{\frac{1}{\sqrt{2}}} + \frac{2\pi}{3} \left[z - \frac{z^4}{4} \right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= \pi \cdot \frac{2-\sqrt{2}}{4}$$

ESERCIZIO 12.

Calcolare il volume dell'intersezione S dei due cilindri $\{x^2+y^2\leq 1\}$ e $\{x^2+z^2\leq 1\}$.



L'ottante $\{x\geq 0, y\geq 0, z\geq 0\}$ contiene $\frac{1}{8}$ del volume totale.

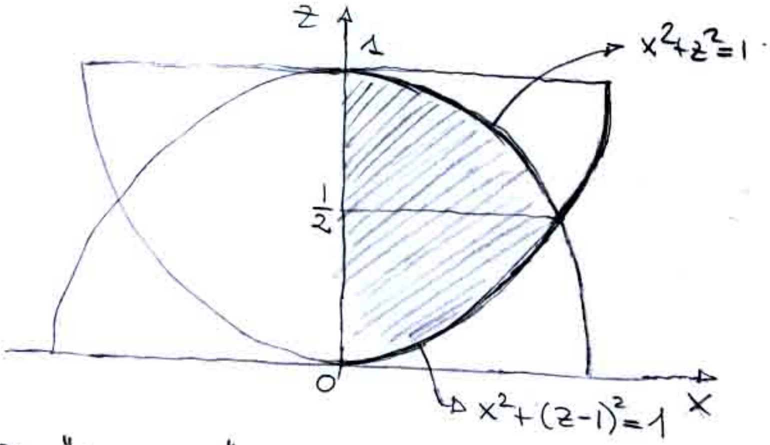
$$|S| = 8 \iint_{\{x^2+y^2\leq 1\}} \left(\int_{z=0}^{\sqrt{1-x^2}} dz \right) dx\,dy$$

$$= 8 \iint_{\{x^2+y^2\leq 1\}} \sqrt{1-x^2}\, dx\,dy = 8 \int_{x=0}^{1} \sqrt{1-x^2} \left(\int_{y=0}^{\sqrt{1-x^2}} dy \right) dx$$

$$= 8 \int_{x=0}^{1} \left(\sqrt{1-x^2}\right)^2 dx = 8 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{16}{3}$$

ESERCIZIO 13.

Calcolare $\iiint_D z^2 \, dx\,dy\,dz$ dove D è l'intersezione tra le sfere $\{x^2+y^2+z^2 \leq 1\}$ e $\{x^2+y^2+(z-1)^2 \leq 1\}$



sezione rispetto al piano xz.
Il dominio è simmetrico rispetto all'asse z

Per "sezioni"

$$= \int_{z=0}^{1/2} z^2 \iint_{\{x^2+y^2 \leq 1-(z-1)^2\}} dx\,dy + \int_{z=1/2}^{1} z^2 \iint_{\{x^2+y^2 \leq 1-z^2\}} dx\,dy$$

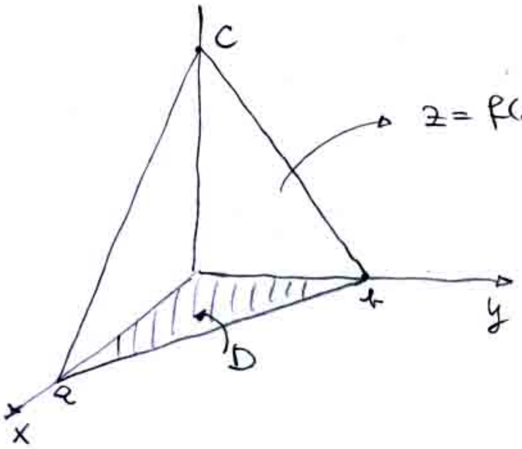
$$= \int_{z=0}^{1/2} z^2 \pi \cdot (1-(z-1)^2)\,dz + \int_{z=\frac{1}{2}}^{1} z^2 \cdot \pi\,(1-z^2)\,dz$$

$$= \pi \int_{z=0}^{1/2} (2z^3 - z^4)\,dz + \pi \int_{z=\frac{1}{2}}^{1} (z^2 - z^4)\,dz$$

$$= \pi \left[\frac{z^4}{2} - \frac{z^5}{5}\right]_0^{\frac{1}{2}} + \pi \left[\frac{z^3}{3} - \frac{z^5}{5}\right]_{1/2}^{1} = \pi \frac{59}{480}$$

ESERCIZIO 14.

Calcolare l'area della parte del piano $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ compresa tra i piani coordinati, con $a,b,c>0$.



$$z = f(x,y) = c\cdot\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

Quindi

$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1+\left(\frac{-c}{a}\right)^2+\left(\frac{-c}{b}\right)^2}$$

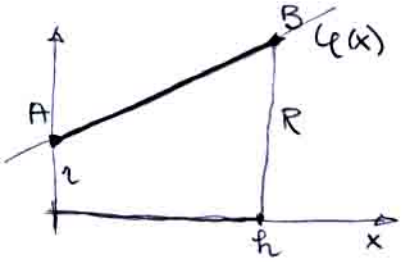
$$= \frac{1}{ab}\sqrt{(ab)^2+(cb)^2+(ac)^2}$$

$$|S| = \iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \frac{1}{ab}\sqrt{(ab)^2+(cb)^2+(ac)^2}\cdot|D|$$

$$= \frac{1}{2}\sqrt{(ab)^2+(cb)^2+(ac)^2} \qquad \text{perché } |D| = \frac{1}{2}a\cdot b.$$

ESERCIZIO 15.

Calcolare l'area laterale del tronco di cono retto di altezza h, e raggi di base R e r (con $R>r$)



La superficie può essere ottenuta ruotando il segmento AB di 2π attorno all'asse x.

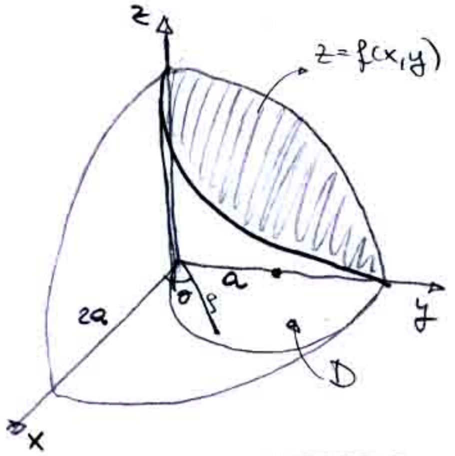
$$\varphi(x) = r + \frac{R-r}{h}\cdot x$$

$$|S| = 2\pi\int_{x=0}^{h}\sqrt{1+\varphi'(x)^2}\cdot\varphi(x)\,dx = 2\pi\cdot\int_{x=0}^{h}\sqrt{1+\left(\frac{R-r}{h}\right)^2}\cdot\left(r+\frac{R-r}{h}x\right)dx$$

$$= 2\pi\cdot\frac{\sqrt{h^2+(R-r)^2}}{h}\cdot\left[rx+\frac{R-r}{h}\cdot\frac{x^2}{2}\right]_0^h = \pi(R+r)\cdot\sqrt{h^2+(R-r)^2}$$

ESERCIZIO 16.

Calcolare l'area della parte di superficie della sfera $x^2+y^2+z^2=4a^2$ contenuta nel cilindro $x^2+(y-a)^2\leq a^2$ e nel semispazio $z\geq 0$.



Possiamo procedere come nell'esempio 9.

$$f(x,y)=\sqrt{4a^2-x^2-y^2}$$

e quindi

$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}=\frac{2a}{\sqrt{4a^2-x^2-y^2}}$$

Per simmetria la superficie indicata nella figura è uguale alla metà di quella da calcolare.

$$|S|=2\iint_D \frac{2a}{\sqrt{4a^2-x^2-y^2}}\,dx\,dy=4a\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{2a\sin\theta}\frac{1}{\sqrt{4a^2-\rho^2}}\,\rho\,d\rho\,d\theta$$

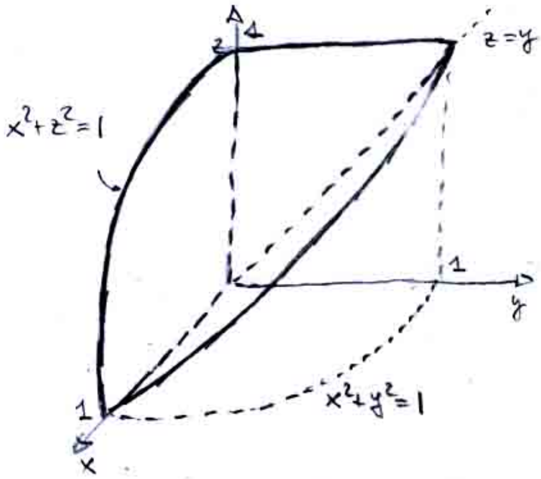
$$=4a\int_{\theta=0}^{\pi/2}\left[-(4a^2-\rho^2)^{1/2}\right]_0^{2a\sin\theta}d\theta=8a^2\int_0^{\pi/2}(1-\cos\theta)\,d\theta$$

$$=8a^2\left[\theta-\sin\theta\right]_0^{\pi/2}=4a^2(\pi-2).$$

ESERCIZIO 17.

Calcolare la superficie totale del solido

$$Q = \{ x^2+z^2 \leq 1,\ x \geq 0,\ z \geq y \geq 0 \}$$



La superficie totale è formata da

$S_1 = Q \cap \{x=0\}$,

$S_2 = Q \cap \{y=0\}$,

$S_3 = Q \cap \{z=y\}$ e

$S_4 = Q \cap \{x^2+z^2=1\}$.

S_1 è un triangolo di area $\frac{1}{2}$, S_2 è $\frac{1}{4}$ di un cerchio di area π.

Per S_3 considero $z = f_3(x,y) = y$ e $D_3 = \{ x^2+y^2 \leq 1, x \geq 0, y \geq 0 \}$

$$|S_3| = \iint_{D_3} \sqrt{1+0^2+1^2}\, dx dy = \sqrt{2} \cdot |D_3| = \sqrt{2} \cdot \frac{\pi}{4}$$

Per S_4 considero $z = f_4(x,y) = \sqrt{1-x^2}$ e $D_4 = D_3$

$$|S_4| = \iint_{D_4} \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2+0^2}\, dx dy = \iint_{D_4} \frac{1}{\sqrt{1-x^2}}\, dx dy$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}} \left(\int_{y=0}^{\sqrt{1-x^2}} dy \right) dx = \int_{x=0}^{1} 1\, dx = 1$$

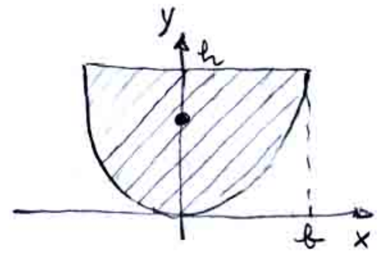
Quindi

$$|S| = |S_1| + |S_2| + |S_3| + |S_4| = \frac{\pi}{4}(1+\sqrt{2}) + \frac{3}{2}.$$

ESERCIZIO 18.

Calcolare la posizione del centro di massa del settore parabolico omogeneo

$$D = \left\{ (x,y) : \frac{hx^2}{b^2} \leq y \leq h \right\}$$



Calcoliamo prima $|D|$:

$$|D| = \iint_D dx\,dy = 2\int_{x=0}^{b} \int_{y=\frac{hx^2}{b^2}}^{h} dx\,dy = 2\int_{x=0}^{b} \left(h - \frac{hx^2}{b^2}\right) dx$$

$$= 2h\left[x - \frac{x^3}{3b^2}\right]_0^b = \frac{4bh}{3}$$

Per simmetria $\bar{x} = 0$. Infine

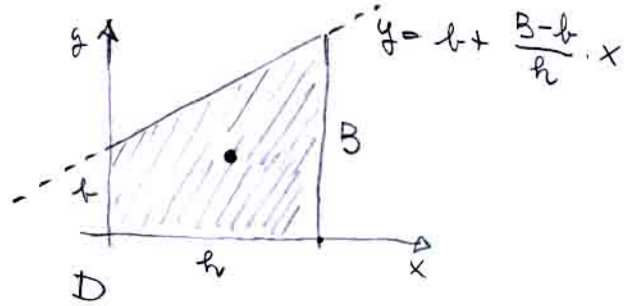
$$\bar{y} = \frac{1}{|D|} \iint_D y\,dx\,dy = \frac{3}{4bh} \cdot 2 \int_{x=0}^{b} \left(\int_{y=\frac{hx^2}{b^2}}^{h} y\,dy \right) dx$$

$$= \frac{3}{2bh} \int_{x=0}^{b} \left[\frac{y^2}{2}\right]_{\frac{hx^2}{b^2}}^{h} dx = \frac{3h}{4b} \int_{x=0}^{b} \left(1 - \frac{x^4}{b^4}\right) dx$$

$$= \frac{3h}{4b}\left[x - \frac{x^5}{5b^4}\right]_0^b = \frac{3h}{5}$$

ESERCIZIO 19.

Calcolare la posizione del centro di massa del trapezio



D

$$\bar{x} = \frac{1}{|D|}\iint_D x\,dx\,dy = \frac{2}{(B+b)h}\int_{x=0}^{h} x\left(\int_{y=0}^{b+\frac{B-b}{h}x} dy\right)dx$$

$$= \frac{2}{(B+b)h}\int_{x=0}^{h}\left(bx + \frac{B-b}{h}x^2\right)dx$$

$$= \frac{2}{(B+b)h}\cdot\left[\frac{bx^2}{2} + \frac{B-b}{h}\frac{x^3}{3}\right]_0^h = \frac{(2B+b)h}{3(B+b)}$$

e

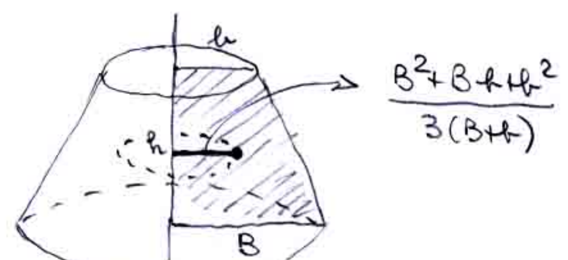
$$\bar{y} = \frac{1}{|D|}\iint_D y\,dx\,dy = \frac{2}{(B+b)h}\int_{x=0}^{h}\left(\int_{y=0}^{b+\frac{B-b}{h}x} y\,dy\right)dx$$

$$= \frac{\cancel{2}}{(B+b)h}\int_{x=0}^{h}\frac{1}{\cancel{2}}\left(b+\frac{B-b}{h}x\right)^2 dx = \frac{1}{(B+b)\cancel{h}}\cdot\frac{\cancel{h}}{B-b}\cdot\left[\left(b+\frac{B-b}{h}x\right)^3\frac{1}{3}\right]_0^h$$

$$= \frac{1}{3(B+b)}\cdot\frac{B^3-b^3}{B-b} = \frac{B^2+Bb+b^2}{3(B+b)}$$

Osservazioni:

1) Per la formula di Pappo-Guldino il volume del tronco di cono



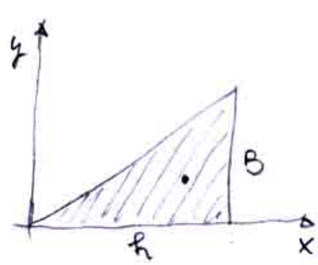
è dato da

$$V = \cancel{2}\pi\cdot\left(\frac{B^2+Bb+b^2}{3\cancel{(B+b)}}\right)\cdot\frac{\cancel{B+b}}{\cancel{2}}\cdot h = \frac{\pi h}{3}\left(B^2+Bb+b^2\right)$$

2) Nel caso particolare in cui $b=0$ si ottiene un triangolo rettangolo e $(\bar{x},\bar{y}) = \left(\frac{2}{3}h, \frac{B}{3}\right)$

Per un triangolo di vertici (x_i, y_i) $i=1,2,3$

$$(\bar{x},\bar{y}) = \left(\frac{1}{3}(x_1+x_2+x_3), \frac{1}{3}(y_1+y_2+y_3)\right)$$



ESERCIZIO 20.

Calcolare la posizione del centro di massa della semisfera omogenea

1) "piena" $D = \{ x^2+y^2+z^2 \leq R^2, \ z \geq 0 \}$

2) "vuota" $S = \{ x^2+y^2+z^2 = R^2, \ z \geq 0 \}$

In entrambi i casi, per simmetria, $\bar{x} = \bar{y} = 0$

Calcoliamo $\bar{z}$:

1)
$$\bar{z} = \frac{1}{|D|} \iiint_D z \, dx\,dy\,dz$$
$$= \frac{1}{\frac{2}{3}\pi R^3} \int_{\rho=0}^{R} \left(\int_{\theta=0}^{2\pi} \left(\int_{z=0}^{\sqrt{R^2-\rho^2}} z \, dz \right) d\theta \right) \rho \, d\rho$$
$$= \frac{1}{\frac{2}{3}\cancel{\pi} R^3} \int_{\rho=0}^{R} \cdot \cancel{2\pi} \cdot \left(\frac{R^2-\rho^2}{\cancel{2}} \right) \cdot \rho \, d\rho$$
$$= \frac{3}{2R^3} \left[\frac{R^2\rho^2}{2} - \frac{\rho^4}{4} \right]_0^R = \frac{3R}{8}$$

2)
$$\bar{z} = \frac{1}{|S|} \iint_{\{x^2+y^2 \leq R^2\}} z \, dS = \frac{1}{2\pi R^2} \iint_{\{x^2+y^2 \leq R^2\}} \sqrt{R^2-x^2-y^2} \cdot \frac{R}{\sqrt{R^2-x^2-y^2}} \, dx\,dy$$
$$= \frac{1}{2\pi R^2} R \cdot \left| \{x^2+y^2 \leq R^2\} \right| = \frac{\cancel{\pi} R^{\cancel{3}}}{2\cancel{\pi} \cancel{R^2}} = \frac{R}{2}$$

avendo considerato S come il grafico di

$$z = f(x,y) = \sqrt{R^2-x^2-y^2} \quad \text{sopra } \{x^2+y^2 \leq R^2\}$$

da cui $\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \frac{R}{\sqrt{R^2-x^2-y^2}}$

ESERCIZIO 21.

Calcolare il centro di massa del cubo $[0,a]^3$ nel caso la densità di massa sia $\delta(x,y,z)=x^2+y^2+z^2$.

La massa di $[0,a]^3$ vale

$$m=\iiint_{[0,a]^3}\delta(x,y,z)\,dx\,dy\,dz=\int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$=3\int_{x=0}^{a}x^2\cdot\left(\int_{y=0}^{a}\int_{z=0}^{a}dy\,dz\right)dx=3a^2\cdot\left[\frac{x^3}{3}\right]_0^a=a^5$$

Per simmetria rispetto a x,y,z, sia del cubo che della funzione densità $\delta(x,y,z)$ abbiamo che $\bar{x}=\bar{y}=\bar{z}$.

Calcoliamo $\bar{x}$:

$$\bar{x}=\frac{1}{m}\iiint_{[0,a^3]}x\,\delta\,dx\,dy\,dz=\frac{1}{a^5}\int_{x=0}^{a}x\left(\int_{y=0}^{a}\left(\int_{z=0}^{a}(x^2+y^2+z^2)\,dz\right)dy\right)dx$$

$$=\frac{1}{a^5}\int_{x=0}^{a}x\left(\int_{y=0}^{a}\left[x^2z+y^2z+\frac{z^3}{3}\right]_0^a dy\right)dz$$

$$=\frac{1}{a^5}\int_{x=0}^{a}x\left[x^2ay+a\frac{y^3}{3}+\frac{a^3}{3}y\right]_0^a dx$$

$$=\frac{1}{a^5}\left[\frac{x^4}{4}\cdot a^2+\frac{x^2}{\cancel{2}}\frac{\cancel{2}a^4}{3}\right]_0^a=\frac{7a}{12}$$

ESERCIZIO 22.

Calcolare I/m per il cerchio omogeneo

$$D = \{(x,y,z) : x^2+y^2 \leq R^2 \text{ e } z=0\}$$

rispetto a 1) l'asse x 2) l'asse z 3) la retta l $\{y=R, z=0\}$

1) $$\frac{I}{m} = \frac{1}{\pi R^2} \iint_D (y^2+z^2)\,dx\,dy \overset{z=0}{=}$$

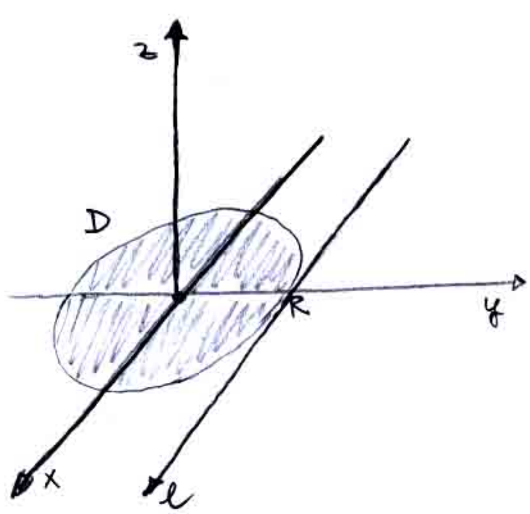
$$= \frac{1}{\pi R^2} \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} (\rho \operatorname{sen}\theta)^2 \cdot \rho \, d\rho \, d\theta$$

$$= \frac{1}{\pi R^2} \left[\frac{\rho^4}{4}\right]_0^R \cdot \int_0^{2\pi} (\operatorname{sen}\theta)^2 \, d\theta = \frac{R^4 \cdot \pi}{4\pi R^2} = \frac{R^2}{4}$$

2) $$\frac{I}{m} = \frac{1}{\pi R^2} \iint_D (x^2+y^2)\,dx\,dy$$

$$= \frac{1}{\pi R^2} \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \rho^2 \cdot \rho \, d\rho \, d\theta$$

$$= \frac{1}{\pi R^2} \cdot 2\pi \cdot \left[\frac{\rho^4}{4}\right]_0^R = \frac{R^2}{2}$$



3) La distanza di un punto $(x,y,0)$ di D dalla retta l è uguale a $|y-R|$. Quindi

$$\frac{I}{m} = \frac{1}{\pi R^2} \iint_D |y-R|^2 \, dx\,dy$$

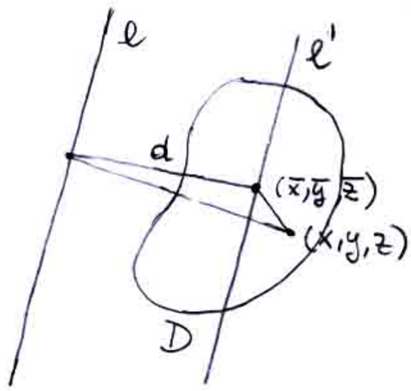
$$= \frac{1}{\pi R^2} \iint_D \left(y^2 - 2Ry + R^2\right) dx\,dy$$

$$= \frac{1}{\pi R^2} \iint_D y^2 \, dx\,dy - 2\frac{R}{\pi R^2} \iint_D y \, dx\,dy + \frac{R^2}{\pi R^2} \iint_D dx\,dy$$

$$= \underbrace{\frac{R^2}{4}}_{\text{(per il punto 1))}} - \underbrace{\frac{2}{\pi R} \cdot 0}_{\text{(perché } \bar{y}=0)} + \frac{R^2}{\cancel{\pi R^2}} \cdot \cancel{|D|}$$

$$= \frac{R^2}{4} + R^2 = \frac{5}{4} R^2$$

Osservazione: in generale il momento d'inerzia I di un solido D di massa m rispetto ad una retta l è uguale a



$$I = I_{CM} + m d^2$$

dove I_{CM} è il momento d'inerzia di D rispetto alla retta l' parallela a l e passante dal centro di massa $(\bar{x}, \bar{y}, \bar{z})$ di D e d è la distanza tra le due rette l e l'.

ESERCIZIO 23.

Calcolare $\frac{I}{m}$ per la sfera "vuota" omogenea

$$S = \{ x^2+y^2+z^2 = R^2 \}$$

rispetto all'asse z.

$$\frac{I}{m} = \frac{2}{4\pi R^2} \iint_{S^+} (x^2+y^2)\, dS \qquad (S^+ \text{ è la semisfera in } z\geq 0)$$

$$= \frac{2}{4\pi R^2} \iint_{\{x^2+y^2\leq e^2\}} (x^2+y^2) \cdot \frac{R}{\sqrt{R^2-x^2-y^2}}\, dx\, dy$$

$$= \frac{2R}{4\pi R^2} \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \rho^2 \cdot \frac{1}{\sqrt{R^2-\rho^2}}\, \rho\, d\rho\, d\theta$$

$$= \frac{\cancel{4\pi}}{\cancel{4\pi} R} \int_{\rho=0}^{R} \frac{\rho^3}{\sqrt{R^2-\rho^2}}\, d\rho \underset{\substack{t=R^2-\rho^2 \\ dt=-2\rho\, d\rho}}{=} \frac{1}{R} \int_{t=R^2}^{0} \frac{R^2-t}{\sqrt{t}} \cdot \frac{dt}{(-2)}$$

$$= \frac{1}{2R} \left[\frac{R^2 t^{1/2}}{1/2} - \frac{t^{3/2}}{3/2} \right]_0^{R^2} = \frac{R^2}{2} \left[2 - \frac{2}{3} \right] = \frac{2R^2}{3}$$

ESERCIZIO 24.

Calcolare I/m per il cubo di lato a rispetto alla retta passante per i centri di due facce opposte.

$$\frac{I}{m} = \frac{1}{a^3} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{a/2} \int_{-a/2}^{a/2} (x^2+y^2)\, dx\, dy\, dz = \frac{8}{a^3} \int_0^{a/2} \int_0^{a/2} \left[(x^2+y^2) \cdot z\right]_0^{a/2} dx\, dy$$

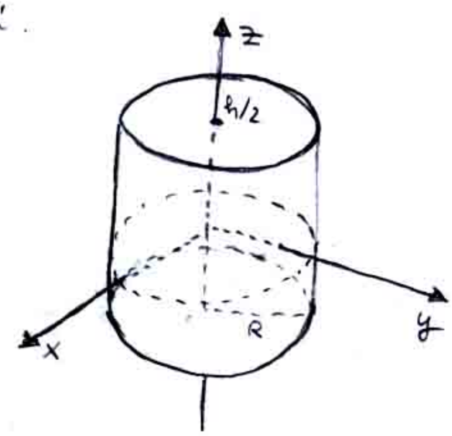
$$= \frac{4}{a^2} \int_0^{a/2} \left[x^2 y + \frac{y^3}{3} \right]_0^{a/2} dx = \frac{4}{a^2} \left[\frac{a}{2} \cdot \frac{x^3}{3} + \frac{a^3}{24} \cdot x \right]_0^{a/2} = \frac{a^2}{6}$$

ESERCIZIO 25.

Calcolare I/m per il cilindro omogeneo

$$D = \left\{ x^2+y^2 \le R^2,\ |z| \le \frac{h}{2} \right\}$$

rispetto ai tre assi coordinati.



1) asse z:

$$\frac{I}{m} = \frac{1}{\pi R^2 h} \iiint_D (x^2+y^2)\, dx\,dy\,dz$$

$$= \frac{1}{\pi R^2 h} \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \int_{z=-h/2}^{z=h/2} \rho^2 \cdot \rho\, d\rho\, d\theta\, dz$$

$$= \frac{1}{\cancel{\pi} R^2 \cancel{h}} \cdot 2\cancel{\pi} \cdot \cancel{h} \cdot \int_{\rho=0}^{R} \rho^3 d\rho = \frac{2}{R^2} \cdot \left[\frac{\rho^4}{4}\right]_0^R = \frac{R^2}{2}$$

2) asse x (stesso risultato per l'asse y)

$$\frac{I}{m} = \frac{1}{\pi R^2 h} \iiint_D (y^2+z^2)\, dx\,dy\,dz$$

$$= \frac{1}{\pi R^2 h} \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \int_{z=-h/2}^{h/2} (\rho^2 \operatorname{sen}^2\theta + z^2)\, \rho\, d\rho\, d\theta\, dz$$

$$= \frac{1}{\pi R^2 \cancel{h}} \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \rho^3 \operatorname{sen}^2\theta \cdot \cancel{h}\, d\rho\, d\theta +$$

$$+ \frac{1}{\pi R^2 h} \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \rho \cdot 2 \cdot \left(\frac{h}{2}\right)^3 \cdot \frac{1}{3}\, d\rho\, d\theta$$

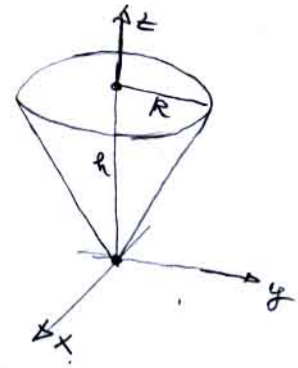
$$= \frac{1}{\cancel{\pi} R^2} \left[\frac{\rho^4}{4}\right]_0^R \cdot \cancel{\pi} + \frac{h^2}{\cancel{\pi} R^2 \cdot 12} \cdot \left[\frac{\rho^2}{2}\right]_0^R \cdot 2\cancel{\pi} = \frac{R^2}{4} + \frac{h^2}{12}.$$

ESERCIZIO 26.

Calcolare I/m per il cono omogeneo

$$D = \left\{ \frac{x^2+y^2}{R^2} \le \frac{z^2}{h^2} \,,\; 0 \le z \le h \right\}$$

rispetto ai tre assi coordinati.



1) asse z:

$$\frac{I}{m} = \frac{3}{\pi R^2 h} \iiint_D (x^2+y^2)\, dx\,dy\,dz$$

$$= \frac{3}{\pi R^2 h} \int_{z=0}^{h} \int_{\rho=0}^{R\frac{z}{h}} \int_{\theta=0}^{2\pi} \rho^2 \cdot \rho\, d\rho\, d\theta\, dz$$

$$= \frac{6\cancel{\pi}}{\cancel{\pi} R^2 h} \int_{z=0}^{h} \left[\frac{\rho^4}{4}\right]_0^{\frac{Rz}{h}} dz = \frac{6R^4}{4R^2 h^5}\left[\frac{z^5}{5}\right]_0^h = \frac{3R^2}{10}$$

2) asse x (stesso risultato per l'asse y):

$$\frac{I}{m} = \frac{3}{\pi R^2 h} \iiint_D (y^2+z^2)\, dx\,dy\,dz$$

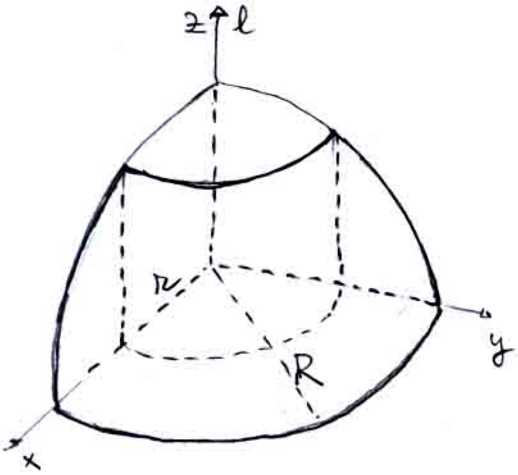
$$= \frac{3}{\pi R^2 h} \int_{z=0}^{h} \int_{\rho=0}^{R\frac{z}{h}} \int_{\theta=0}^{2\pi} (\rho^2 \sin^2\theta + z^2)\, \rho\, d\rho\, d\theta\, dz$$

$$= \frac{3\cancel{\pi}}{\cancel{\pi} R^2 h} \int_{z=0}^{h} \int_{\rho=0}^{\frac{Rz}{h}} \rho^3 d\rho\, dz + \frac{3 \cdot 2\cancel{\pi}}{\cancel{\pi} R^2 h} \int_{z=0}^{h} z^2 \int_{\rho=0}^{\frac{Rz}{h}} \rho\, d\rho\, dz$$

$$= \frac{3R^2}{20} + \frac{6}{\cancel{R^2} h} \cdot \frac{\cancel{R^2}}{2h^2} \cdot \left[\frac{z^5}{5}\right]_0^h = \frac{3R^2}{20} + \frac{3h^2}{5}$$

ESERCIZIO 27.

Calcolare I/m per una sfera omogenea di raggio R forata da un cilindro di raggio $r<R$ e asse l passante per il centro della sfera rispetto allo stesso asse l.



La figura rappresenta la parte di solido contenuto nell'ottante $\{x,y,z \geq 0\}$.

La massa vale (si assume $\delta = 1$)

$$m = 2\iint_{\{r^2 \leq x^2+y^2 \leq R^2\}} \sqrt{R^2-x^2-y^2}\,dx\,dy = 2\int_{\rho=r}^{R}\int_{\theta=0}^{2\pi} \sqrt{R^2-\rho^2}\,\rho\,d\rho\,d\theta$$

$$= 4\pi \int_{\rho=r}^{R} \sqrt{R^2-\rho^2}\,\rho\,d\rho = 2\pi\left[(R^2-\rho^2)^{3/2}\cdot\frac{2}{3}\right]_R^r = \frac{4\pi}{3}(R^2-r^2)^{3/2}$$

$$\frac{I}{m} = \frac{2}{m}\iint_{\{r^2 \leq \rho^2 \leq R^2\}} \rho^2\sqrt{R^2-\rho^2}\,\rho\,d\rho\,d\theta = \frac{4\pi}{m}\int_r^R \rho^3\sqrt{R^2-\rho^2}\,d\rho \overset{t=R^2-\rho^2}{=}$$

$$= \frac{4\pi}{m}\int_0^{R^2-r^2} (R^2-t)\cdot\sqrt{t}\,\frac{dt}{2} = \frac{2\pi}{m}\left[R^2\frac{t^{3/2}}{3/2} - \frac{t^{5/2}}{5/2}\right]_0^{R^2-r^2}$$

$$= \frac{2R^2+3r^2}{5}.$$