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# József Wildt International Mathematical Competition

The Edition XXXIII$^{th}$, 2023 [1]

The solution of problems W1. - W55. must be mailed before 26. October 2023, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania,
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**W1.** Find the limit

$$\lim_{n\to\infty}\left(\sqrt[n+1]{((n+1)!)^a\left((2n+1)!!\right)^b}-\sqrt[n]{(n!)^a\left((2n-1)!!\right)^b}\right)$$

where $a, b \in R$.

D.M. Bătineţu-Giurgiu and Neculai Stanciu

**W2.** a). If $a, b, c \in R_+^*$ such that $abc = 1$, then prove the following inequality:

$$\frac{1}{a^3\left(F_nb+F_{n+1}c\right)}+\frac{1}{b^3\left(F_nc+F_{n+1}a\right)}+\frac{1}{c^3\left(F_na+F_{n+1}b\right)}\geq\frac{3}{F_{n+2}}$$

b). If $a, b, c \in R_+^*$ such that $ab + bc + ca = 3$, then prove the following inequality:

$$\frac{1}{a^3\left(F_nb+F_{n+1}c\right)}+\frac{1}{b^3\left(F_nc+F_{n+1}a\right)}+\frac{1}{c^3\left(F_na+F_{n+1}b\right)}\geq\frac{3}{F_{n+2}}$$

a). If $a, b, c \in R_+^*$ such that $a + b + c = 3$, then prove the following inequality:

$$\frac{1}{a^3\left(F_nb+F_{n+1}c\right)}+\frac{1}{b^3\left(F_nc+F_{n+1}a\right)}+\frac{1}{c^3\left(F_na+F_{n+1}b\right)}\geq\frac{3}{F_{n+2}}$$

D.M. Bătineţu-Giurgiu and Neculai Stanciu

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**W3.** For $a, b, c > 0$. Prove that

$$\left(a^b b^a\right)\left(b^c c^b\right)\left(c^a a^c\right) \le \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{2(a+b+c)}$$

Toyesh Prakash

**W4.** Let $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = af'(-x) + b, \forall x \in \mathbb{R}.$$

Ovidiu Furdui and Alina Sîntămărian

**W5.** Calculate

$$\sum_{n=1}^{\infty}\left[(2n-1)\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots\right)^2-\frac{2}{n}\right].$$

Ovidiu Furdui and Alina Sîntămărian

**W6.** Let $n > 1$ and $A, B \in M_n(C)$ suvh that $f(z) = e^{zA}Be^{-zA}$ is bounded for all $z \in C$. Compute

$$C = AB - BA \text{ and } D = A^2B + BA^2$$

Moubinool Omarjee

**W7.** Let $m \ge 0$ and $M$ be a point inside the triangle ABC. If we denote by $d_a, d_b, d_c$ the distances of the point $M$ to the sides $BC, CA$ and $AB$ respectively, let it be shown that:

$$\frac{a^{m+1}\cdot b}{d_b^m}+\frac{b^{m+1}\cdot c}{d_c^m}+\frac{c^{m+1}\cdot a}{d_a^m} \ge 2^{m+2}\cdot F\cdot\left(\sqrt{3}\right)^{m+1}$$

D.M. Bătineţu-Giurgiu and Ovidiu T. Pop

**W8.** If $m \ge 0$, the following inequality in triangle $ABC$

$$\left(\frac{a+b}{h_c}\right)^{m+1}+\left(\frac{b+c}{h_a}\right)^{m+1}+\left(\frac{c+a}{h_b}\right)^{m+1} \ge 4^{m+1}\cdot\left(\sqrt{3}\right)^{1-m}$$

holds.

D.M. Bătineţu-Giurgiu and Ovidiu T. Pop

**W9.** Calculate

$$\sum_{n=2}^{\infty} \frac{H_n H_{n+1}}{(n-1)n},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the $n$th harmonic number.

Ovidiu Furdui and Alina Sîntămărian

**W10.** The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1$$

$$L_{n+2} = L_{n+1} + L_n, L_0 = 2, L_1 = 1$$

Also, $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}$, and $\beta = \frac{1-\sqrt{5}}{2}$.

Prove that

$$\sum_{n=0}^{\infty} \frac{L_{2F_{n+2}} - L_{2F_{n+1}}}{L_{2F_n} + L_{2F_{n+3}}} = \frac{1}{3}.$$

Ángel Plaza

**W11.** Evaluate

$$\sum_{n=0}^{\infty}(n+1) \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(-1)^n}{2^i \; i!(n-2i)!}.$$

Ángel Plaza

**W12.** Calculate

$$\int_0^1 \int_0^1 \frac{\ln\left[\left(x^0+y^0\right)+\left(x^1+y^1\right)+\left(x^2+y^2\right)+\cdots\right]}{(x^0+y^0)+(x^1+y^1)+(x^2+y^2)+\cdots} \mathrm{dxdy}$$

Ankush Kumar Parcha

**W13.** Evaluate

$$\int_0^{\infty} \frac{\tan^2 x}{x^2\left(\sec^2 x + \csc^2 x\right)}$$

Toyesh Prakash Sharma

**W14.** Let $a, b, c$ be positive real numbers. Prove that

$$4\sqrt{9b^2+16c^2} + 4\sqrt{9c^2+16a^2} + 4\sqrt{9a^2+16b^2} \geq$$

$$\geq 5\sqrt{b^2+15c^2} + 5\sqrt{c^2+15a^2} + 5\sqrt{a^2+15b^2}$$

Paolo Perfetti

**W15.** Find those positive values $\alpha$ and $\beta$ for which

$$\sum_{n=1}^{+\infty} \prod_{k=3}^{n} \frac{\alpha + k \ln k \ln(\ln k)}{\beta + (k+1)\ln(k+1)\ln(\ln(k+1))},$$

converges. For those values of $\alpha$ and $\beta$, evaluate the sum

Paolo Perfetti

**W16.** Let $a, b, c$ be positive real numbers. Prove that

$$4\sqrt{\frac{c}{a+15b}} + 4\sqrt{\frac{a}{b+15c}} + 4\sqrt{\frac{b}{c+15a}} \geq 3$$

Paolo Perfetti

**W17.** If $x, y, z \in R_+$ and $m, n, p; r, s, t \in N^*$ find the minimum of expression

$$E(x, y, z) = mx + ny + pz$$

if $x^r y^s z^t = k$ and specify the values for which the minimum is reached

Dorin Mărghidanu

**W18.** If $f : I \subset R \longrightarrow R$ is differentiable function, with continuous derivative and $a, b \in I$, $a < b$, then prove that:

$$\int\limits_a^b \left[f(x) + \varphi(x) \cdot f'(x)\right] dx = (b-a) \cdot f\left(\frac{a+b}{2}\right) + (f(b) - f(a)) \cdot \frac{a+b}{2}$$

where

$$\varphi(x) = \begin{cases} x + \frac{b-a}{2} & if \quad x \in \left[a, \frac{a+b}{2}\right] \\ x - \frac{b-a}{2} & if \quad x \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

Dorin Mărghidanu

**W19.** If $f_1, f_2, ..., f_p : R \to (0, \infty)$ are integrable functions, prove that:

$$e^{\frac{1}{x} \cdot \int\limits_0^x \ln(f_1(t) + f_2(t) + ... + f_p(t))dt} \geq e^{\frac{1}{x} \cdot \int\limits_0^x \ln f_1(t)dt} + e^{\frac{1}{x} \cdot \int\limits_0^x \ln f_2(t)dt} + ... + e^{\frac{1}{x} \cdot \int\limits_0^x \ln f_p(t)dt}$$

for any $x > 0$ and $p \in N^*$.

Dorin Mărghidanu

**W20.** Let $M$ be a subset of $\{1, 2, 3, \ldots, 2023\}$ such that for any three elements (not necessarily distinct) $a, b, c$ of $M$ we have $|a + b - c| > 12$. Determine the largest possible number of elements of $M$.

José Luis Díaz–Barrero

**W21**. Let $f$ be a nonnegative and non increasing function on $[0, +\infty)$ and let $g$ be a function defined on $[0, +\infty)$ such that $0 \leq g(x) \leq 2023$ with $\int_0^{+\infty} g(x)\, dx = 6069$. Prove that

$$\int_0^{+\infty} f(x)g(x)\, dx \leq 2023 \int_0^3 f(x)\, dx.$$

José Luis Díaz–Barrero

**W22**. If $0 < x, y < z$ and $a, b, c > 0$ then:

$$\left(1 + a\sqrt{\frac{z}{y}}\right)\left(1 + b\sqrt{\frac{z}{z-y}}\right) + \left(1 + b\sqrt{\frac{y}{x}}\right)\left(1 + c\sqrt{\frac{y}{y-x}}\right) +$$

$$+\left(1+c\sqrt{\frac{z}{x}}\right)\left(1+a\sqrt{\frac{z}{z-x}}\right)\geq 3\left(1+\sqrt[3]{2\sqrt{2}abc}\right)^2$$

Mihály Bencze and Florică Anastase

**W23**. For $a,b,c\in\left(0,\frac{1}{2}\right)$ and $k>0$ let $S_k=a^k+b^k+c^k$. If $S_{n+1}=1$, then:

$$(2S_1-1)^{S_1}(1+a^n)^a(1+b^n)^b(1+c^n)^c\leq(1+S_1)^{S_1}\prod_{cyc}(a+b^{n+1}+c^{n+1})$$

Mihály Bencze and Florică Anastase

**W24**. In $\Delta ABC$ the following relationship holds:

$$\sum_{cyc}\frac{w_a^n}{h_a^n+m_a^n}\leq\frac{3}{2}\cdot\left(\frac{R}{2r}\right)^{\frac{2n}{3}}, n\in\mathbb{N}$$

Mihály Bencze and Florică Anastase

**W25**. In $\Delta ABC$ the following relationship holds:

$$\prod_{cyc}\left(1+\frac{s^2+r^2ctg^2\frac{A}{2}}{ctg\frac{A}{2}}\right)\geq\left(1+\frac{10rs}{3}\right)^3$$

Mihály Bencze and Florică Anastase

**W26**. Let be the function $f:[0,1]\to R$ integrable such that $f(1)=1$ and $\int_x^y f(t)\,dt=\frac{1}{2}(yf(y)-xf(x))$, $\forall x,y\in[0,1]$. Find

$$I=\int_0^{\frac{\pi}{4}}f(x)\cdot tg^2x\,dx.$$

Mihály Bencze and Florică Anastase

**W27**. If $n\in\mathbb{N}; n\geq 1$ then:

$$F_{n-1}^4+F_n^4+F_{n+1}^4>F_{3n}\left(F_{n+1}+\frac{F_{n-1}F_n}{F_{n+1}}\right)$$

Daniel Sitaru

**W28**. If in $\Delta ABC, N$ - ninepoint center, $I$ - incenter then:

$$\frac{NA \cdot IA}{bc} + \frac{NB \cdot IB}{ca} + \frac{NC \cdot IC}{ab} > 1$$

Daniel Sitaru

**W29**. If $F_n$ - Fibonacci numbers $(F_0 = 0; F_1 = 1; F_{n+2} = F_{n+1} + F_n; n \in \mathbb{N})$ then:

$$\int_{F_n^2}^{F_{n+2}^2} \int_{F_n^2}^{F_{n+2}^2} \frac{dxdy}{1+xy} < \frac{2F_{2n+2}^2}{(1+F_n^2)(1+F_{n+2}^2)}$$

Daniel Sitaru

**W30**. Let $ABC$ be a triangle, where $AB = AC$ and $m\left(\widehat{BAC}\right) = \alpha$, $60^\circ < \alpha < 120^\circ$. The point M is considered with in the triangle ABC, such that $m\left(\widehat{MBC}\right) = 30^\circ$ and $m\left(\widehat{MBC}\right) = \beta$, where $2\beta + 60^\circ = \alpha$. Calculate $m\left(\widehat{AMC}\right)$.

Ion Pătraşcu

**W31**. If $F(n)_{n\geq 1}$ are Fibonacci numbers with $F_1 = 1, F_2 = 1$ then find all solutions of the following equation:

$$F_1 F_2 \cdot ... \cdot F_n = F_m$$

Seyran Ibrahimov

**W32**. Let $a_0 = a > 1$ and $a_{n+1} = \frac{1}{2}\left(a_n^2 + 1\right), n \in \mathbb{N} \cup \{0\}$ .Prove inequality

$$\prod_{k=0}^{n} (a_n + 1) \geq 2^{n+1} + 2^n (n+1)(a-1), n \in \mathbb{N} \cup \{0\}.$$

Arkady Alt

**W33**. Let $f(m) := m + [\sqrt{m}]$ and $f_n(m)$ defined recursively by

$$f_n(m) = f(f_{n-1}(m)),\ n \in \mathbb{N}$$

and $f_0(m) = m$. Prove that for any $m \in \mathbb{N}$ sequence

$f_0(m)$, $f_1(m), f_2(m), ..., f_n(m), ...$contain at least one perfect square.

Arkady Alt

**W34**. Using only elementary technique (without derivatives or methods of linear algebra) find greatest value of function

$$h(x_1, x_2, ..., x_n) := x_1x_2 + x_2x_3 + ... + x_{n-1}x_n,$$

for any real $x_1, x_2, ..., x_n$ such that

$$x_1^2 + x_2^2 + ... + x_n^2 = 1, n \geq 3.$$

Arkady Alt

**W35**. Let $S_n(x)$ be polynomial defined by recurrence

$$S_{n+1} - 2(x+1)S_n + S_{n-1} = 2x, n \in \mathbb{N}$$

with initial conditions $S_0 = 0, S_1 = x$.Prove that

$$S_n(x) \leq (1 + nx)^n - 1, x \geq 0, n \in \mathbb{N};$$

Arkady Alt

**W36**. Calculate $\lim\limits_{n\to\infty} \dfrac{n!\left(1+\dfrac{1}{n}\right)^{n^2+n}}{n^{n+1/2}}$.

Arkady Alt

**W37**. Let $n \in N^*$. Prove that the function $f : (0, +\infty) \to R$ where

$$f^n(x) = \frac{\prod\limits_{k=1}^{n} \ln(x+k)}{\ln \prod\limits_{k=1}^{n}(x+k)}$$

is concave.

Marius Drăgan

**W38**. Let $n \in N^*$, it exist $t \in (0,1)$ such that

$$\left\{ \sum_{k=-1}^{n} \frac{t^{k+2}}{k(k+1)(k+2)} \right\} < \frac{1}{18}$$

where $\{\cdot\}$ represents the fractional part.

Marius Drăgan

**W39**. If $1 < a \leq b$ then

$$\int_a^b \frac{dx}{x \ln x} \geq \ln \frac{(b-1)(2a+1)}{(a-1)(2b+1)}$$

Mihály Bencze and Ionel Tudor

**W40**. If $a, b > 1$ then compute:

$$\int \frac{x^{2n}(ab)^x \ln \frac{b}{a} + nx^{n-1}(b^x - a^x) + x^n(b^x \ln b - a^x \ln a)}{x^{2n}(ab)^x + x^n(a^x + b^x) + 1}$$

Mihály Bencze and Ovidiu Bagdasar

**W41**. If $a_k \in R$ $(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} \sqrt{a_1^2 + (1-a_2)^2} \geq \frac{1}{\sqrt{2}} \left( \left| \sum_{k=1}^{n} a_k \right| + \left| n - \sum_{k=1}^{n} a_k \right| \right)$$

Mihály Bencze and Sorin Rădulescu

**W42**. If $1 < a \leq b$ then

$$\int_a^b \frac{(\ln x)^2 dx}{2x^2 - x - 1} \leq \frac{1}{6} \ln ab \ln \frac{b}{a}$$

Mihály Bencze and Ovidiu Bagdasar

**W43**. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} a_k = 1$ then

$$\sum_{cyclic} \left( (n-1) \frac{a_1}{a_2} + (n-2) \frac{a_2}{a_3} + ... + \frac{2a_{n-2}}{a_{n-1}} + \frac{a_{n-1}}{a_n} \right) \sqrt[n-1]{a_1^{n-3}} \geq$$

$$\geq \frac{n(n-1)}{2} \sum_{k=1}^{n} a_k$$

Mihály Bencze and Sorin Rădulescu

**W44**. Determine all two times differentiable functions $f : [-3, +\infty) \to R$ for which $f(0) = 2$, $f'(0) = \frac{3}{2}$ and

$$\left(f'(x)\right)^2 + f(x) f''(x) = 2(1 - \sin x - \cos x) + x(\sin x - \cos x)$$

for all $x \in [-3, +\infty)$.

Mihály Bencze and Florică Anastase

**W45**. If $0 < a \leq b$ and $0 < c \leq d$ then

$$\int\limits_a^b \int\limits_c^d \left( \frac{(x^2+1)(y^2+1)}{(2x+y)(x+2y)} \right)^2 dxdy \geq \frac{9}{16} arctg \frac{b-a}{1+ab} arctg \frac{d-c}{1+cd}$$

Mihály Bencze and Florică Anastase

**W46**. If $a_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} \left( \left( \frac{a_1}{a_2} \right)^2 + \left( \frac{a_2}{a_1} \right)^2 \right) + 4n \geq 3 \sum_{cyclic} \frac{a_1^2 + a_2^2}{a_1 a_2}$$

Mihály Bencze and Eugen Ionaşcu

**W47**. In all triangle ABC holds:

1). $\sum \frac{a^4+4a^2b^2+b^4}{(a^2+b^2)r_a r_b} \geq 3\left(1 + \left(\frac{4R+r}{s}\right)^2\right)$ 2). $\sum \frac{r_a^4+4r_a^2r_b^2+r_b^4}{\left(r_a^2+r_b^2\right)ab} \geq 3\left(2 + \frac{r}{2R}\right)$

Mihály Bencze and Gabriel Prăjitură

**W48**. If $a_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} \frac{a_1}{(a_2 + a_3 + ... + a_n)^r} \geq \left(\frac{n}{n-1}\right)^r$$

where $r \geq 1$.

Mihály Bencze and Rovsen Pirguliev

**W49**. If $x_k \in R$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^2 = n$ then

$$\sum_{k=1}^{n} |x_k| - \prod_{k=1}^{n} x_k \leq n + 1$$

Mihály Bencze and Marius Olteanu

**W50**. If $a_k \geq 1$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} a_k = e^n$ then

$$\sum_{k=1}^{n} \left(2a_k - \frac{1}{a_k}\right) \geq 4n$$

Mihály Bencze and Nicuşor Minculete

**W51**. Find the volume of the ellipsoid

$$x^2 + y^2 + z^2 + xy + yz + zx = 1$$

Eugen J. Ionaşcu

**W52**. Let $n \geq 2$ be an integer. Show that for all $x, y, z \geq 0$, we have

$$\frac{x^{n+1} + n}{n + x + y + z^{n+1}} + \frac{y^{n+1} + n}{n + y + z + x^{n+1}} + \frac{z^{n+1} + n}{n + z + x + y^{n+1}} \geq \frac{3\,(n+1)}{n+3}$$

Eugen J. Ionaşcu

**W53**. Let $\triangle ABC$ be an acute triangle with $A, B, C$ in counterclockwise order, and let $a, b, c$ denote the side-lengths in the usual order (a opposite A, etc.).

(i). Prove that there are unique points $D, E, F$ on $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively, such that $\overline{DE} \perp \overline{CA}$, $\overline{EF} \perp \overline{AB}$ and $\overline{FD} \perp \overline{BC}$, (see Figure 1).

(ii). Prove that $\triangle DEF$ from (i) is similar to $\triangle ABC$, with similarity ratio $\frac{4S}{a^2+b^2+c^2}$

(iii). Iterate the operation of (i) to get a nested sequence of triangles; the next one would be $\triangle KLM$ with $K, L, M$ opposite $E, F, D$, respectively.

Prove that the vertices of these triangles converge to the point $I$ with barycentric coordinates $\left[\frac{1}{c^2}, \frac{1}{a^2}, \frac{1}{b^2}\right]$

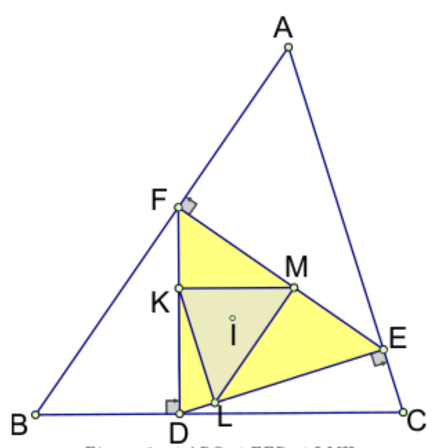


*Figure* 1, $\triangle ABC \sim \triangle EFD \sim \triangle LMK$

Eugen J. Ionaşcu

**W54**. If $x \in \left(0, \frac{\pi}{2}\right)$, then prove

$$\frac{\sqrt{\sin x}}{\cos^2 x} + \frac{\sqrt{\cos x}}{\sin^2 x} \geq \sqrt[8]{\frac{3^{18}}{2^{19}} \sin^3 2x}$$

Pirkuliyev Rovsen

**W55**. Solve on $R_+$ the equation

$$\ln \frac{1+\arcsin^2 x}{1+\arccos^2 x} = arctg\left(\arccos x\right) - arctg\left(\arcsin x\right)$$

Pirkuliyev Rovsen

**W56**. Let $(a_n)_{n\in N}$ a sequence of real numbers, defined as follows:

$$a_1 \in (0,1]\,, a_{n+1} = \frac{1-e^{-n^2 a_n}}{(n+1)^2},\ n \geq 1$$

Calculate:

$$\lim_{n\to\infty} \frac{n^4\left(\left(n^2-1\right)\left(1-e^{a_n}\right) - n\left(e^{-na_n}-1\right)\right)}{\ln n}$$

Stănescu Florin

**W57**. Let $(A,+,\cdot)$ a finite ring, with $1+1$ invertible, $S$ a subset of the $A$, $a\in A$, $b\in S$ two invertible elements, which have the following properties:
a). The orders of the two elements in the group $(A,\cdot)$ there are two odd numbers;
b). There is a divisor $p$ of the number $|S|!+1$, such that:
$\sum\limits_{k=0}^{p-1} \binom{p-1}{k} a^k\left(ab+ba\right)a^{p-1-k} = b$
(we note $x^0=1$, $x\in A$). Show that $ax+xa\in S$ for all $x\in S$ if and only if $a=(1+1)^{-1}$

Stănescu Florin

**W58**. Let the function $f:[0,A]\to[0,\infty)$, $A>0$, continuous, strictly increasing, three times differentiable in 0, such that:

$$f'(0)=1,\ f''(0)\neq 0,\ 0<f(x)<x,\ \ (\forall)\,x\in(0,A]$$

Consider the sequence of real numbers $(x_n)_{n\in N^*}$ defined by:

$$x_1\in(0,A]\,, x_n = f\left(\frac{x_1+x_2+...+x_{n-1}}{n-1}\right),\ n\geq 2$$

a). Show that $\lim\limits_{n\to\infty} x_n\cdot\ln n = -\frac{2}{f^n(0)}$
b). Claculate: $\lim\limits_{n\to\infty} n\ln n\left(x_{n+1}\cdot\ln(n+1) - x_n\cdot\ln(n)\right)$

Stănescu Florin