József Wildt International Mathematical Competition

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The solution of problems W1. - W58 must be mailed before 26 October 2024, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania,

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W1. Let $n \geq 0$ be an integer number and let $\{S_n\}_{n\geq 0}$ be the integer sequence defined by $S_0 = 1$ and for $n \geq 1$,

$$S_n = \sum_{k=0}^{n} \prod_{j=0}^{k} (n - j + 1).$$

Find a close form for S_n.

José Luis Díaz–Barrero

W2. Let $A(x)$ be a polynomial of positive degree with integer coefficients. Show that for every positive integer number k, there exists an integer n such that the number $A(n)$ has at least k different prime divisors.

José Luis Díaz–Barrero

W3. Fill six natural numbers into the following grids, so that the sum of the three numbers in each row, each column, each diagonal all are equal.

	19	96
1		

Chang-Jian Zhao and Mihály Bencze

W4. Let $n \geq 2$ integer, $(x_1, ..., x_n) \in \mathbb{R}^n$, $(y_1, ..., y_n) \in \mathbb{R}^n$, $P \in \mathbb{R}[X]$, $\deg(P) = n - 1 < n$

Compute

$$Q = \left(P\left(x_i - y_j\right)\right)_{1 \le i,j \le n}.$$

Compute

$$\det\left(P\left(x_i - y_j\right)\right)_{1 \le i,j \le n}$$

Moubinool Omarjee

W5. $T(0) = 0, T(1) = 2, T(2) = 4, T(n+1) = 2T(n) + nT(n-1)$ for $n \ge 2$
Let

$$f(x) = \sum_{n=0}^{+\infty} \frac{T(n)}{n!} x^n$$

for $|x| < R$. Find the limit of

$$\frac{n^n}{2^n} \left(f\left(\frac{1}{n}\right)\right)^n$$

when $n \to +\infty$

Moubinool Omarjee

W6. If $a, b, c, d \in (0, \infty)$ such that $abcd = 1$, prove that

$$\left(1 + \frac{a}{b}\right)^{cd} \left(1 + \frac{b}{c}\right)^{da} \left(1 + \frac{c}{d}\right)^{ab} \left(1 + \frac{d}{a}\right)^{bc} \ge 2^{\frac{16}{a^2+b^2+c^2+d^2}}$$

Dorin Mărghidanu

W7. If $a_1, a_2, ..., a_n > 0$, with the notations:

$$A_n\left[a_1, a_2, ..., a_n\right] := \frac{1}{n} \cdot \sum_{k=1}^{n} a_k$$

$$G_n\left[a_1, a_2, ..., a_n\right] := \sqrt[n]{\prod_{k=1}^{n} a_k}$$

$$H_n\left[a_1, a_2, ..., a_n\right] := \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

Prove the following inequalities:

(a)

$$\left(A_n\left[a_1, a_2, ..., a_n\right]\right)^{A_n\left[a_1, a_2, ..., a_n\right]} \leq G_n\left[a_1^{a_1}, a_2^{a_2}, ..., a_n^{a_n}\right]$$

(b)

$$\left(G_n\left[a_1^{1/a_1}, a_2^{1/a_2}, ..., a_n^{1/a_n}\right]\right)^{H_n\left[a_1, a_2, ..., a_n\right]} \leq H_n\left[a_1, a_2, ..., a_n\right]$$

Dorin Mărghidanu

W8. Evaluate

$$\int_0^{\infty} \frac{\ln\left(1+x+x^2\right)}{1+x^2} dx$$

Paolo Perfetti

W9. Evaluate

$$\int_0^{\infty} \frac{\sqrt{x}\ln\left(1+x+x^2\right)}{1+x^2} dx$$

Paolo Perfetti

W10. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers such that $a_{k+1} \leq a_k(1+c/k)$ where $c > 0$ is a constant. Prove that if $\sum\limits_{k=1}^{\infty} a_k$ diverges the series

$$\sum_{k=1}^{\infty}\left(\sum_{j=k}^{2k} \frac{a_j}{a_{j+1}}\right)^{-1}$$

also diverges

Paolo Perfetti

W11. If M is a point inside the triangle ABC of sides $BC = a, CA = b, AB = c$, area F and d_a, d_b, d_c are the distances from point M to sides BC, CA and AB respectively, show that:

$$\frac{ab^2}{d_a}+\frac{bc^2}{d_b}+\frac{ca^2}{d_c}\geq 24F$$

D.M. Bătineţu-Giurgiu and Ovidiu T. Pop

W12. If M is a point inside the triangle ABC of sides $BC=a, CA=b, AB=c$, area F and d_a, d_b, d_c are the distances from point M to sides BC, CA and AB respectively, show that:

$$\left(\frac{a^6}{d_a^2}+2\right)\left(\frac{b^6}{d_b^2}+2\right)\left(\frac{c^6}{d_c^2}+2\right)\geq 1728F^2$$

D.M. Bătineţu-Giurgiu and Ovidiu T. Pop

W13. The inscribed circle is tangent to the sides AB, AC and BC of the triangle ABC, respectively at the points M, L and K. Prove that:

$$BK^2\cdot KC^2\cdot AM^2\geq\left(\frac{6r^3}{3R-4r}\right)^3$$

Pirkuliyev Rovshan

W14. Calculate the following sum:

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2n-1}-\frac{1}{2n}+\frac{1}{2n+1}-\cdots\right).$$

Ángel Plaza

W15. Let the matrix $A\in M_p(R)$, $n,p\in N$, $p\geq 2$, such that

$$A^{6n+1}=I_p \text{ or } A^{6n+1}=A^2$$

Show that the matrix I_p-A+A^2 is invertible.

Nicolae Papacu

W16. Show that

$$\frac{n}{e^n-1+n}\cdot\frac{(2p+n+2)^2}{(2p+2)^2}\leq\int_0^1\frac{e^{nx^p}}{e^{nx}+1}dx\leq\frac{1}{n}\ln\frac{e^n+1}{2}$$

where $n, p \in N$, $np, \geq 1$

Nicolae Papacu

W17. Let F_n be then nth Fibonacci number defined by $F_1 = 1, F_2 = 1$ and for all $n \geq 3$, $F_n = F_{n-2} + F_{n-1}$. Prove that

$$\sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^{F_{n+2}}$$

is an irrational number but not transcendental.

Toyesh Prakash Sharma and Etisha Sharma

W18. Compute $\int (1 + \ln x) \left(x^x + \frac{1}{x^x}\right) dx$

Etisha Sharma

W19. Find

$$\Omega = \lim_{n\to\infty} \frac{(\int_0^1 e^{x^2} dx \cdot \int_0^1 e^{-x^2} dx)^n}{n(\int_0^1 e^{nx^2} dx)(\int_0^1 e^{-nx^2} dx)}$$

Daniel Sitaru

W20. Find $x, y, z > 0$ such that:

$$\begin{cases} 2x^3 + 2y^3 + 2z^3 = x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 \\ x \tan 9^\circ + y \tan 81^\circ = z(\tan 27^\circ + \tan 63^\circ) + 4 \end{cases}$$

Daniel Sitaru

W21. Find:

$$\Omega = \lim_{n\to\infty} \frac{\sqrt{H_n - 1} + \sqrt{H_{n+1} - 1} + \sqrt{H_{n+2} - 1} + \sqrt{H_{n+3} - 1}}{n\sqrt{H_n H_{n+1} + H_{n+2} H_{n+3}}}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, n \geq 1$

Daniel Sitaru

W22. If $x_n = \sum\limits_{k=1}^{n} \frac{1}{k^2}$, then prove that:

$$\left(\sum_{k=1}^{n} x_k F_{2k-1}\right)\left(\sum_{k=1}^{n} \frac{F_{2k-1}}{x_k}\right) \leq \frac{\left(\pi^2+6\right)^2}{24\pi^2} F_{2n}^2$$

for any positive inyteger n.

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W23. Let $(a_n)_{n\geq 1}$, $a_1 = 1$, $a_{n+1} = (n+1)!a_n$ for all $n \in N^*$

1). Compute $\lim\limits_{n\to\infty} \frac{n!\sqrt[2n]{n!F_n^2}}{\sqrt[n^2]{a_n}}$ 2). Compute $\lim\limits_{n\to\infty} \frac{n!\sqrt[2n]{n!L_n^2}}{\sqrt[n^2]{a_n}}$

when F_n and L_n denote the nth Fibonacci respective Lucas numbers.

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W24. Let $f : [0,1] \to [0,1]$, $f(0) = 0$ continuous and bijective function. We assume that the function $\frac{x^t}{(f(x)+f^{-1}(x))^{t-1}}$, $x \in (0,1]$ has a limit for $x \searrow 0$, where $t \geq 1$. Show that:

a).

$$\int_0^1 u(x) \frac{x^t}{\left(f(x)+f^{-1}(x)\right)^{t-1}} \geq \frac{\int\limits_0^1 u(x) \cdot \left(f(x)+f^{-1}(x)\right) dx}{2^t}$$

where $u : [0,1] \to [0,\infty]$ it is differentiable function, with derivate integrable and positive on $[0,1]$

Florin Stănescu

W25. Show that:

$$\sum_{n=1}^{\infty} \sum_{k=\left[\frac{n}{2}\right]}^{n-1} \sum_{m=1}^{n-k} \frac{(-1)^{m-1}}{n} \cdot \frac{k+m}{k+1} \cdot \binom{k+1}{m-1} \cdot 2^{k-m+1} x^{2k+1-n} = \frac{\pi - \arccos x}{2\sqrt{1-x^2}}$$

with $|x| < 1$

Florin Stănescu

W26. We consider a prime number $n \geq 3$, two matrices $A, B \in M_{n-1}(Q)$, $AB = BA$, and

$$\varepsilon = \cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n}$$

such that:
a). $\det(A + \varepsilon B) = 0$
b). $\sum\limits_{k=0}^{n-1} \varepsilon^{n-ki}\left(1 - \varepsilon^{-k}\right) \det\left(A + \varepsilon^k B\right) \geq 0$, for all $i \in \{0, 1, ..., n-2\}$
c). $\det B > 0$
Show that whatever $m \geq 1$ is the natural number and whatever the rational numbers $b_0, b_1, ..., b_m$, while

$$\sum_{k=0}^{m} b_k A^k B^{m-k} \neq O_n$$

then

$$\det\left(\sum_{k=0}^{m} b_k A^k B^{m-k}\right) \neq 0$$

Florin Stănescu

W27. Let S be the sphere $x^2 + y^2 + z^2 = R^2$ and let C_{z_0} be the family of cardioids $\{x^2 + y^2 = \alpha(z_0)(\sqrt{x^2 + y^2} - x), z = z_0, -R \leq z_0 \leq R\}$. Each element of C_{z_0} belongs on a plane orthogonal to the z axis and its point with the largest abscissa belongs to the sphere. For any cardioid consider the conic surface with vertex in $P = (0, 0, R)$ and generatrices the segments joining P and the points of the cardioid. Find the maximum in C_{z_0} of the volume enclosed by the conic surface

Paolo Perfetti

W28. In $\triangle$ABC, I-incenter and D, E, F the points of contact of the cevians AI, BI, CI with the circle, then the following relationship holds:

$$ID + IE + IF \leq \frac{2\left(R^2 - Rr + r^2\right)}{r}$$

Marian Ursărescu

W29. In acute $\triangle$ABC, A', B', C'-simmetrics of the points A, B, C to the sides BC, CA, AB respectively, the following relationship holds:

$$\frac{[A'B'C']}{[ABC]} \geq 4\left(\frac{r}{R}\right)^2 + 8 \cdot \frac{r}{R} - 1$$

where $[*]-$ area.

Marian Ursărescu

W30. In $\triangle$ABC, A', B', C' -middle points of the arcs $\widehat{BC}$, $\widehat{CA}$, $\widehat{AB}$ respectively made with the circumscrible circle of the triangle ABC, the following relationship holds:

$$\frac{6r}{R} \leq \frac{AB}{A'B'} + \frac{BC}{B'C'} + \frac{CA}{C'A'} \leq 3$$

Marian Ursărescu

W31. Let $z_1, z_2, z_3 \in C^*$ different in pairs such that $|z_1| = |z_2| = |z_3| = 1$. If

$$\sum_{cyc} \frac{1}{\left(\left|(z_1 - z_2)\,|z_1 - z_3| + (z_1 - z_3)\,|z_1 - z_2|\right|\right)^2} =$$

$$= \frac{3}{\left(|z_1 - z_2| + |z_2 - z_3| + |z_3 - z_1|\right)^2}$$

then z_1, z_2, z_3 are affixses of an equilateral triangle.

Marian Ursărescu

W32. In $\triangle$ABC, K-Lemoine's point, F-area of the triangle ABC and a, b, c the lengths sides, the following relationship holds:

$$\frac{1}{\sin(AKB)} + \frac{1}{\sin(BKC)} + \frac{1}{\sin(CKA)} \geq \frac{24F}{a^2 + b^2 + c^2}$$

Marian Ursărescu

W33. Prove that in any acute angled triangle ABC holds inequality

$$\left(a^2 + b^2 + c^2\right)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq \frac{2\left(9R^2 + 4Rr + r^2 - s^2\right)}{R^2}.$$

Arkady Alt

W34. For any real $a, b, c, d > 0$ prove inequality

$$\frac{a}{\sqrt[3]{a^3+63bcd}}+\frac{b}{\sqrt[3]{b^3+63acd}}+\frac{c}{\sqrt[3]{c^3+63abd}}+\frac{d}{\sqrt[3]{d^3+63abc}}\geq 1.$$

Arkady Alt

W35. For any positive a, b, c such that $a+b+c=ab+bc+ca$ prove that

$$\frac{2}{3}\left(a^3+b^3+c^3\right)\geq abc+\frac{a^4+b^4+c^4}{a^2+b^2+c^2}.$$

Arkady Alt

W36. a). If $a_1, a_2, a_3, , , , a_{n+1}, n \geq 2$ are positive real numbers that satisfy inequality

$$\left(a_1^2+a_2^2+...+a_{n+1}^2\right)^2 > n\left(a_1^4+a_2^4+...+a_{n+1}^4\right)$$

then any n of them, let it be $a_1, a_2, a_3, , , , a_n$, satisfies inequality

$$\left(a_1^2+a_2^2+...+a_n^2\right)^2 > (n-1)\left(a_1^4+a_2^4+...+a_n^4\right)$$

b). If $a_1, a_2, a_3, , , , a_{n+1}, n \geq 2$ are positive real numbers that satisfy

$$\left(a_1^2+a_2^2+...+a_{n+1}^2\right)^2 > n\left(a_1^4+a_2^4+...+a_{n+1}^4\right)$$

then for any $1 \leq k_1 < k_2 < k_3 \leq n+1$ holds inequality

$$\left(a_{k_1}^2+a_{k_2}^2+a_{k_3}^2\right)^2 > 2\left(a_{k_1}^4+a_{k_2}^4+a_{k_3}^4\right)$$

that is $a_{k_1}, a_{k_2}, a_{k_3}$ be side lengths of some triangle.

Arkady Alt

W37. Let ABC be a non-obtuse angles triangle with usual notations. Prove that

$$\frac{2r}{R}+\frac{r^2}{R^2}\leq\sum\frac{a^2\cos A}{bc}\leq 1+\frac{r}{R}.$$

Arkady Alt

W38. Let $p \geq 2$ be positive integer and $a > 1$ be real number. Find

$$\sum_{n=0}^{\infty} \frac{a_n^{p-2} + 2a_n^{p-3} + 3a_n^{p-4} + ... + p - 1}{1 + a_n + a_n^2 + ... + a_n^{p-1}},$$

where $a_{n+1} = \frac{1}{p}\left(a_n^p + p - 1\right), n \in \mathbb{N} \cup \{0\}$ and $a_0 = a > 1$.

Arkady Alt

W39. Find the following limits:

a). $\lim\limits_{n\to\infty} \dfrac{\left(1 + \dfrac{1}{n(n+a)}\right)^{n^3}}{\left(1 + \dfrac{1}{n+b}\right)^{n^2}}$

b). $\lim\limits_{n\to\infty} (n+1)\left(\left(1 + \dfrac{1}{n(n+1)}\right)^{(n+2)n} - \left(1 + \dfrac{1}{n}\right)^n\right).$

c). $\lim\limits_{n\to\infty} \left((n+2)\left(1 + \dfrac{1}{n(n+1)}\right)^{(n+1)n} - (n+1)\left(1 + \dfrac{1}{n}\right)^n\right).$

Arkady Alt

W40. Let F be area of a triangle ABC and l_a, l_b, l_c be angle bisectors, respectively, from vertices A, B, C. Prove the following double inequality

$$\frac{1}{\sqrt{3}} \min\left\{l_a^2, l_b^2, l_c^2\right\} \leq F \leq \frac{1}{\sqrt{3}} \max\left\{l_a l_b, l_b l_c, l_c l_a\right\}.$$

Arkady Alt

W41. Prove that

$$\binom{n}{0}^3 + \binom{n}{1}^3 + ... + \binom{n}{n}^3 \geq \frac{27}{4}\left(\binom{2n}{n} - 2^{n+1} + n + 1\right)$$

for all $n \in N^*$

Ionel Tudor

W42. Solve in R the equation

$$(x-1)^{\log_{2023} 2024} - x^{\log_{2024} 2023} = 1$$

Ionel Tudor

W43. Let a, b and c be nonegative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$(1-a^2)a^3 + (1-b^2)b^3 + (1-c^2)c^3 \geq 2abc.$$

Eugen Ionaşcu

W44. (a) Solve the following differential equation

$$(y'^2(y'-y) = 1 \tag{1}$$

(b) Show that the initial value problem

$$(y'^2(y'-y) = 1, y(0) = -\frac{1}{4^{1/3}} \tag{2}$$

has at least 5 different solutions on any interval around the origin.

Eugen Ionaşcu

W45. Let be $a_1 = 6$ and

$$n(n+1)a_{n+1} = 3(n+2)(a_n + n(n+1)3^n)$$

for all $n \geq 1$. Prove that:

1).

$$\sum_{k=1}^{n} \frac{a_k}{k(k+1)} = \frac{3}{4}((2n-1)3^n + 1)$$

2).

$$\lim_{n\to\infty} \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n^2}} = \sqrt{3}$$

Mihály Bencze

W46. 1). Determine all functions $f, g : (0, +\infty) \to R$ for which $f(x) e^{-x}$ is a primitive function of $g(x)$ and $g(x) e^{-x}$ is a primitive function of $f(x)$ for all $x > 0$
2). Prove that for all $n \in N$ holds:

$$\begin{cases} x\left(f\left(\ln x\right)\right)^{(2n+1)} = g\left(\ln x\right) \\ x\left(g\left(\ln x\right)\right)^{(2n+1)} = f\left(\ln x\right) \end{cases}$$

for all $x > 0$

Mihály Bencze

W47. In all acute triangle ABC holds

$$\sum \left(\frac{e}{\cos A}\right)^{\frac{\cos^2 A - e^2}{e \cos A}} + 3\left(2e\right)^{\frac{1-4e^2}{2e}} \geq 2 \sum \left(\frac{e}{\sin \frac{A}{2}}\right)^{\frac{\sin^2 \frac{A}{2} - e^2}{e \sin \frac{A}{2}}}$$

Mihály Bencze

W48. If $a, b, c > 0$ then

$$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} \geq 2$$

then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{567}{64} + abc$$

Mihály Bencze and Chang-Jian Zhao

W49. In all triangle ABC holds

$$\sum \cos \frac{A}{2} \geq \frac{(R+2r)\, s}{2R^2}$$

Mihály Bencze and Nicuşor Minculete

W50. Let $z_1, z_2, z_3, z_4, z_5, z_6$ be the affixes of vertices $A_1, A_2, A_3, A_4, A_5, A_6$ of a regular hexagon. Prove that

$$\left(\frac{z_3 - z_1}{z_6 - z_1}\right)^n + \frac{1}{\sqrt{3}}\left(\frac{z_4 - z_2}{z_1 - z_2}\right)^{n+1} + \frac{1}{3}\left(\frac{z_5 - z_3}{z_2 - z_3}\right)^{n+2} + \frac{1}{3\sqrt{3}}\left(\frac{z_6 - z_4}{z_3 - z_1}\right)^{n+3} +$$

$$+\frac{1}{9}\left(\frac{z_1-z_5}{z_4-z_5}\right)^{n+4}=\left(\sqrt{3}i\right)^n$$

for all $n\in N$

Marius Drăgan

W51. Let $A,B\in M_n\left(C\right)$ such that

$$\det\left(A+\left(ctg^2\frac{k\pi}{2n+1}\right)B\right)=0$$

for all $k\in\{1,2,...,n\}$. Prove that

$$\det A\det B=(-1)^n\left(2n+1\right)$$

Marius Drăgan

W52. In all triangle ABC holds $16\sum\frac{b}{a}\geq 11\sum\frac{a}{b}+15$

Marius Drăgan

W53. Let be $f:N\to[0,1]$ where $f\left(n\right)=\left\{p^{n+\frac{1}{p}}\right\}$ where $\{\cdot\}$ denote the fractional part and p is a prime number.
1). Prove that the function f is injective
2). Prove that the function f is not surjective

Mihály Bencze and Ovidiu Bagdasar

W54. Let $A_1A_2...A_n$ be a convex polygon. Prove that

$$\sum_{k=1}^{n}\frac{1}{\sqrt{\cos\frac{A_k}{2}}}\geq\frac{n}{\sqrt{\cos\frac{(n-2)\pi}{2n}}}$$

Mihály Bencze and Ovidiu Bagdasar

W55. Let be the set $M=\{x\in R|[x]\cdot\{x\}=1\}$. Prove that if
$x_i\in M, i=\overline{1,n}$ then holds:

$$\left\{\sum_{k=1}^{n}x_k^2\right\}\geq\frac{1}{n}\left(\sum_{k=1}^{n}\{x_k\}\right)^2,$$

where $\{*\}$ denote the fractional part, and $[*]$ denote the integer part.

Florică Anastase

W56. Find:
1).

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\cdot\sum_{m=1}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{n+k}}$$

2).

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{p=0}^{m}\cdot\left(n^{p}\sum_{k=0}^{np}\frac{\binom{m}{k}}{\binom{(p+1)m}{p+k}}\right)^{-1}$$

Florică Anastase

W57. Let $n\geq 1$ be an odd integer. Solve in $\mathcal{M}_2(\mathbb{R})$ the equation

$$A^n=\begin{pmatrix}0 & 1\\ 1 & 0\\ & \end{pmatrix}$$

Ovidiu Furdui and Alina Sîntămărian

W58. Let $n\geq 1$ be an odd integer. Solve in $\mathcal{M}_2(\mathbb{R})$ the equation
$A^n=AA^T$, where A^T denotes the transpose of A.

Ovidiu Furdui and Alina Sîntămărian