Proposed solution of prob. 1238, ΠME Journal (Spring 2011); solution in "Fall 2011"

Prof. Harold Reiter
Dept.of Math. Univ. of North Carolina Charlotte
9021 University City Boulevard, Charlotte NC 28223–0001

Given $a, b, c, d \in [0,1]$ such that no two of them are simultaneously equal to 0. Prove that

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+d^2}+\frac{1}{d^2+a^2} \geq \frac{8}{3+abcd}$$

Proof By Cauchy–Schwarz we have

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+d^2}+\frac{1}{d^2+a^2} \geq \frac{16}{2(a^2+b^2+c^2+d^2)} \geq \frac{8}{3+abcd}$$

or

$$a^2+b^2+c^2+d^2-3-abcd \leq 0$$

The function $f(a,b,c,d) = a^2+b^2+c^2+d^2-3-abcd$ is convex in each variable ($f_{aa} = f_{bb} = f_{cc} = f_{dd} = 2$) thus the maximum is attained at one of the sixteen vertices of the four–dimensional cube $[0,1]^4$. Since $f(a,b,c,d) \leq 0$ if each coordinate equals 0 or 1, the result is achieved.