Proposed solution to problem 902

Given a sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers, let $s_n = \sum_{k=1}^{n} a_k$ and

$$I = \sum_{k=1}^{\infty} \frac{a_k^p}{s_k^q}$$

Find all real values of p and q such that the infinite series I is convergent

Answer: If $q > p \geq 1$ the series defining I converges regardless the convergence of $\sum_k a_k$. All the other cases do not yield convergence regardless that one of $\sum_k a_k$

Proof We will make use of the celebrated Abel–Dini theorem which can be found in many textbooks, for example the famous G.H.Hardy, J.E.Littlewood, G.Pólya, Inequalities, Cambridge Univ. Press, pp.120–121. Since it it is a well known theorem we omit the proof.

Theorem 1. *If $a_n > 0$, $n = 1, 2, \ldots$ and $s_n \doteq \sum_{k=1}^{n} a_k$ then: i) $\sum_{n=1}^{\infty} a_n/s_n$ converges $\Longleftrightarrow$ $\sum_{n=1}^{\infty} a_n$ converges. ii) $\sum_{n=1}^{\infty} a_n/s_n^{1+\delta}$ converges for any $\delta > 0$ regardless the convergence of $\sum_{n=1}^{\infty} a_n$*

Another theorem we are going to use is the integral comparison theorem

Theorem 2. *If $f(x) \searrow 0$, $\int^{+\infty} f(x)dx$ converges or diverges with $\sum_k f(k)$.*

First we examine the values of p and q yielding convergence.

1) $p = 1$, $q > 1$. Theorem 1 assures convergence of the series defining I.

2) $p > 1$ and $q > p$. Let $\sum_k a_k^p$ converge. $\sum_k a_k$ may converge or diverge but in both cases

$$0 < \sum_{k=1}^{\infty} \frac{a_k^p}{s_k^q} \leq \sum_{k=1}^{\infty} \frac{a_k^p}{a_1^q} < +\infty$$

Let $\sum_k a_k^p$ diverge. $s_n^q = s_n^{p+\delta} s_n^{q-p-\delta}$ where $\delta > 0$ is so small that $q - p - \delta > 0$. Let $s_k^{(r)} \doteq \sum_{j=1}^{k} a_j^r$. Now $s_k^{p+\delta} = (s_k^p)^{\frac{p+\delta}{p}} \geq (s_k^{(p)})^{\frac{p+\delta}{p}}$. Then we have

$$0 < \sum_{k=1}^{\infty} \frac{a_k^p}{s_k^q} \leq \sum_{k=1}^{\infty} \frac{a_k^p}{(s_k^{(p)})^{\frac{p+\delta}{p}} s_n^{q-p-\delta}} \leq C \sum_{k=1}^{\infty} \frac{a_k^p}{(s_k^{(p)})^{\frac{p+\delta}{p}}}$$

where $C > 0$ comes from the bound $(s_k^{(p)})^{\frac{-(p+\delta)}{p}} \leq C$. Theorem 2 assures the convergence of I.

Now we examine the values of p and q for which a series $\sum_k a_k$ may be provided such that $\sum_k a_k^p/s_k^q$ does not converge.

- $p < 0$. For $a_n = 2^{-n}$ we have that $\sum a_n^p / s_n^q$ diverges since s_n tends to a finite value.

- $q < 0$. For $a_n = 2^n$ we have that $\sum a_n^p / s_n^q$ diverges since s_n^q tends to 0

Now we consider only nonnegative values of p and q.

- $p = q > 0$. If $a_n = 2^n$ we have $s_n = 2^{n+1} - 2$ so that $a_n^p / s_n^p = 2^{np}/(2^{n+1} - 2)^p \geq 2^{-p}$ yielding the divergence of I.

- $p = q = 0$. Trivially $\sum a_n^p / s_n^q = \sum 1 = +\infty$

- $p > 1$, $q < p$. If $a_n = 2^n$ we have $s_n \sim 2^{n+1} - 2$ and then $a_n^p / s_n^q \geq 2^{n(p-q)}$ and the divergence of $\sum_k a_n^p / s_n^q$ follows

- $p = 1$, $q < 1$. Taking $a_n = 1/n$ it is well know that $s_n = \ln n + O(1)$ hence asymptotically $a_n / s_n^q \sim (n \ln^q n)^{-1}$ and then $\sum a_n / s_n^q$ diverges by the Theorem 2 since $f(x) = (\ln^{1-q}(x))/(1 - q)$ is the primitive of $(x \ln^q x)^{-1}$

- $p < 1$. If $a_n = n^{-\alpha}$, $\alpha \neq 1$, we have $a_n^p / s_n^q \sim n^{-\alpha p + \alpha q - q}$ and for $q > p$ we take $\alpha > (q - 1)/(q - p)$ while for $q < p$ we take $\alpha < (q - 1)/(q - p)$. In both cases $\sum a_n^p / s_n^q$ diverges.