PROBLEMS AND SOLUTIONS

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Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before July 31, 2008. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11348. *Proposed by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA.* A polynomial f over a field K is *powerful* if every irreducible factor of f has multiplicity at least 2. When q is a prime or a power of a prime, let $P_q(n)$ denote the number of monic powerful polynomials of degree n over the finite field $\mathbb{F}_q$. Show that for $n \geq 2$,

$$P_q(n) = q^{\lfloor n/2 \rfloor} + q^{\lfloor n/2 \rfloor - 1} - q^{\lfloor (n-1)/3 \rfloor}.$$

11349. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania.* In triangle ABC, let h_a denote the altitude to the side BC and let r_a be the *exradius* relative to side BC, which is the radius of the circle that is tangent to BC and to the extensions of AB beyond B and AC beyond C. Define h_b, h_c, r_b, and r_c similarly. Let p, r, R, and S be the semiperimeter, inradius, circumradius, and area of ABC. Let ν be a positive number. Show that

$$2(h_a^\nu r_a^\nu + h_b^\nu r_b^\nu + h_c^\nu r_c^\nu) \leq r_a^\nu r_b^\nu + r_b^\nu r_c^\nu + r_c^\nu r_a^\nu + 3S^\nu \left(\frac{3p}{4R+r}\right)^\nu.$$

11350. *Proposed by Bhavana Deshpande, Poona College of Arts, Science & Commerce Camp, Pune, India, and M. N. Deshpande, Institute of Science, Nagpur, India.* Given a positive integer n and an integer k with $0 \leq k \leq n$, form a permutation $a = (a_1, \dots, a_n)$ of $(1, \dots, n)$ by choosing the first k positions at random and filling the remaining $n - k$ positions in ascending order. Let $E_{n,k}$ be the expected number of left-to-right maxima. (For example, $E_{3,1} = 2$, $E_{3,2} = 11/6$, and $E_{4,2} = 13/6$.) Show that $E_{n+1,k} - E_{n,k} = 1/(k+1)$. (A left-to-right maximum occurs at k when $a_j < a_k$ for all $j < k$.)

11351. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bârlad, Romania.* Given positive integers p and q, find the least positive integer m such that among any m distinct integers in $[-p, q]$ there are three that sum to zero.

11352. *Proposed by Daniel Reem, The Technion-Israel Institute of Technology, Haifa, Israel.* Let I be an open interval containing the origin, and let f be a twice-differentiable function from I into $\mathbb{R}$ with continuous second derivative. Let T_2 be the Taylor polynomial of order 2 for f at 0, and let R_2 be the corresponding remainder. Show that

$$\lim_{\substack{(u,v)\to(0,0)\\ u\neq v}} \frac{R_2(u)-R_2(v)}{(u-v)\sqrt{u^2+v^2}} = 0.$$

11353. *Proposed by Ernst Schulte-Geers, BSI, Bonn, Germany.* For $s > 0$, let $f(s) = \int_0^\infty (1+x/s)^s\, e^{-x}\, dx$ and $g(s) = f(s) - \sqrt{s\pi/2}$. Show that g maps $\mathbb{R}^+$ onto $(2/3, 1)$ and is strictly decreasing on its domain.

11354. *Proposed by Matthias Beck, San Francisco State University, San Francisco, CA, and Alexander Berkovich, University of Florida, Gainesville, FL.* Find a polynomial f in two variables such that for all pairs (s, t) of relatively prime positive integers,

$$\sum_{m=1}^{s-1}\sum_{n=1}^{t-1} |mt - ns| = f(s, t).$$

SOLUTIONS

Unsolved in 1990

6576 [1986, 1036]. *Proposed by Hans V. Gerber, University of Lausanne, Switzerland.* Suppose $X_1, X_2, \ldots$ are independent identically distributed real random variables with $E(X_k) = \mu$. Put $S_k = X_1 + X_2 + \cdots + X_k$ for $k = 1, 2, \ldots$.

(**a**) If $\rho < \mu < 1$, where $\rho = -0.278465\ldots$ is the real root of $xe^{1-x} = -1$, show that the series

$$\sum_{k=1}^{\infty} S_k^k e^{-S_k}/k!$$

converges with probability one.

(**b**) If $X_1, X_2, \ldots$ are positive and if $\mu < 1$, show that the expectation of

$$\sum_{k=1}^{\infty} S_k^k e^{-S_k}/k!$$

is $\mu/(1-\mu)$.

*(**c**) In (**b**) is it possible to relax the condition that the random variables are positive? For example, would it suffice to assume $E(|X_k|) < \infty$ and $\rho < \mu < 1$?

Solution to (c) by Daniel Neuenschwander, Université de Lausanne, Lausanne, and Universität Bern, Bern, Switzerland. We prove the validity of (**c**) under the additional assumption that supp (X_1) [the support of X_1, i.e., the intersection of all closed subsets A of the real line for which $P(X_1 \in A) = 1$] is contained in the open interval $(\rho, -\rho)$. Let $\lambda = \sup\{|x| : x \in \text{supp}\,(X_1)\}$. Note that $\lambda < -\rho$. By Stirling's formula, one sees that

$$\sum_{k,n\geq 0} \frac{(k\lambda)^k}{k!}\frac{(n\lambda)^n}{n!} < \infty. \tag{1}$$

Thus by Lebesgue's Dominated Convergence Theorem, the expectation of the series displayed in (**a**) is given by the absolutely convergent sum

$$\sum_{k\geq 1,\ n\geq 0} \frac{(-1)^n}{k!n!} E(S_k^{k+n}).$$

By approximation, we may assume that X_1 has finite support:

$$P(X_1 = z_j) = p_j \qquad (j = 1, 2, \dots, h)$$

where $p_1, \dots, p_h$ are positive, $\sum_{j=1}^h p_j = 1$, and $z_j \in (\rho, -\rho)$ for $1 \leq j \leq h$. Now let $p_1, \dots, p_h$ be fixed. The required equation

$$E(\dots) = \frac{\mu}{1-\mu} \tag{2}$$

[where $(\dots)$ is the series of (**a**) and $\mu = \sum_{j=1}^h p_j z_j$] can be viewed as an equality of two functions in h real variables $z_1, \dots, z_h$ on the open cube $(\rho, -\rho)^h$. By (1), the left side of (2) extends as a complex analytic function in h variables to the domain D^h, where $D = \{z \in \mathbb{C}\colon |z| < -\rho\}$. The same holds for the right side of (2). By (**b**), (2) holds on the subcube $(0, -\rho)^h$ of C, and by standard methods of complex analysis, it thus holds also on D^h. This proves (**c**) in the asserted case.

Editorial comment. Solutions for (**a**) and (**b**) were published in the December, 1990, issue of this MONTHLY (pages 930–932).

A Determinant Identity

11242 [2006, 656 & 848]. *Proposed by Gerd Herzog and Roland Lemmert, Universität Karlsruhe, Karlsruhe, Germany.* (**corrected**) Let f and g be entire holomorphic functions of one complex variable, and let A and B be complex $n \times n$ matrices. If the application of such a function to a matrix means applying the power series of this function to the matrix, prove that

$$\det\left(f(A)f(B) + g(A)g(B)\right) = \det\left(f(B)f(A) + g(B)g(A)\right).$$

Solution by Roger A. Horn, University of Utah. The crucial property of these matrix functions is that $f(Z)$ and $g(Z)$ commute whenever both are defined. The following result is key. (See D. Carlson et al., *Linear Algebra Gems*, MAA, 2002, p. 13.)
Lemma. Let C, D, X, Y be $n \times n$ complex matrices. If C commutes with X, then

$$\det\begin{bmatrix} C & Y \\ X & D \end{bmatrix} = \det(CD - XY).$$

Proof. If C is nonsingular, then

$$\det\begin{bmatrix} C & Y \\ X & D \end{bmatrix} = \det\left(\begin{bmatrix} I & 0 \\ -XC^{-1} & I \end{bmatrix}\begin{bmatrix} C & Y \\ X & D \end{bmatrix}\right) = \det\begin{bmatrix} C & Y \\ 0 & D - XC^{-1}Y \end{bmatrix}$$

$$= \det C \cdot \det(D - XC^{-1}Y) = \det(CD - CXC^{-1}Y)$$

$$= \det(CD - XCC^{-1}Y) = \det(CD - XY).$$

(If C is singular, we invoke the foregoing result for $C + \varepsilon I$ and take limits as $\varepsilon \to 0$.)

Let

$$M = \begin{bmatrix} f(A) & g(B) \\ -g(A) & f(B) \end{bmatrix}, \quad N = \begin{bmatrix} f(B) & -g(A) \\ g(B) & f(A) \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $M = PNP$, $\det M = (\det P)^2 \det N = \det N$. Because $f(A)$ commutes with $-g(A)$ and $f(B)$ commutes with $g(B)$, the lemma ensures that

$$\det(f(A)f(B) + g(A)g(B)) = \det M = \det N = \det(f(B)f(A) + g(B)g(A)).$$

Also solved by S. Amghibech (Canada), R. Chapman (U. K.), K. Dale (Norway), G. Dospinescu (France), H. Flanders, J. Grivaux (France), E. A. Herman, J. H. Lindsey II, O. P. Lossers (Netherlands), K. Schilling, R. Stong, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposers.

A Bessel Function Identity

11246 [2006, 760]. *Proposed by Lee Goldstein, Wichita, KS.* Let J_n be the nth Bessel function of the first kind, and let K_n be the nth modified Bessel function of the second kind (also known as a Macdonald function), defined by

$$K_n(z) = \frac{\Gamma(|n| + 1/2)(2z)^{|n|}}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{(t^2 + z^2)^{|n|+1/2}} \, dt.$$

Show that, for any positive b and any real λ,

$$\frac{\sqrt{\pi}}{2\sqrt{b}} e^{\lambda^2 - 2b} = \sum_{-\infty}^{\infty} J_{2n}(4\lambda\sqrt{b}\,) K_{n+1/2}(2b).$$

Solution by Mourad E. H. Ismail, University of Central Florida, Orlando, FL. First we note formula (4.10.2) from M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable,* Cambridge University Press, 2005:

$$K_{n+1/2}(x) = \sqrt{\pi} e^{-x} x^{-1/2} \sum_{k=0}^{n} \frac{(-n)_k \, (n+1)_k}{k!} (-2x)^{-k}, \qquad n = 0, 1, \cdots,$$

where $(a)_0 := 1$ and $(a)_n := \prod_{k=0}^{n-1}(a+k)$ for $n > 0$. Since $K_\nu(x) = K_{-\nu}(x)$ and $J_{-n}(x) = (-1)^n J_n(x)$, the series on the right side of the identity to be proved is

$$\frac{\sqrt{\pi}}{\sqrt{b}} \, e^{-2b} \sum_{n=0}^{\infty} J_{2n}(4\lambda\sqrt{b}\,) \sum_{k=0}^{n} \frac{(-n)_k \, (n+1)_k}{k!} (4b)^{-k}.$$

Formula (9.0.1) in the same reference is

$$\sum_{m=0}^{\infty} a_m b_m \frac{(zw)^m}{m!} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!(\gamma + n)_n} \sum_{r=0}^{\infty} \frac{b_{n+r} z^r}{r!(\gamma + 2n + 1)_r} \sum_{s=0}^{n} \frac{(-n)_k (n+\gamma)_s}{s!} a_s w^s \ .$$

This, together with

$$J_{2n}(2x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s+2n}}{s!(2n+s)!},$$

proves the identity claimed.

Also solved by R. Chapman (U. K.), J. Grivaux (France), F. Holland (Ireland), G. Lamb, A. R. Miller, V. Schindler (Germany), A. Stadler (Switzerland), V. Stakhovsky, R. Stong, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

A Strip Problem

11247 [2006, 760]. *Proposed by Jürgen Eckhoff, University College London, London, U. K.* Let A, B, C, and D be distinct points in the plane with the property that any three of them can be covered by some strip of width 1. Show that there is a strip of width $\sqrt{2}$ covering all four points, and demonstrate that if no strip of width less than $\sqrt{2}$ covers all four, then the points are the corners of a square of side $\sqrt{2}$. (A strip of width w is the closed set of points bounded by two parallel lines separated by distance w.)

Solution by Li Zhou, Polk Community College, Winter Haven, FL. Let S be the set $\{A, B, C, D\}$. If the convex hull of S is a triangle, then that triangle is covered by a strip of width 1, and so is S. It thus suffices to assume that $ABCD$ is a convex quadrilateral. Given an arbitrary triangle XYZ, let h_X denote the altitude of XYZ at vertex X. For $X, Y, Z \in S$, by the assumed property of $ABCD$, $\min\{h_X, h_Y, h_Z\} \le 1$.

Lemma. *Consider a triangle XYZ with $X, Y, Z \in S$. If $h_X \ge \sqrt{2}$, then $\angle X \le 45°$. Equality holds if and only if $h_X = \sqrt{2}$, $h_Y = 1$, and $\angle Y = 90°$, or $h_X = \sqrt{2}$, $h_Z = 1$, and $\angle Z = 90°$.*

Proof. Either $h_Y \le 1$ or $h_Z \le 1$, say $h_Y \le 1$. By the Law of Sines, $\sin\angle X \le \sin\angle X/\sin\angle Y = YZ/XZ = h_Y/h_X \le 1/\sqrt{2}$. Thus $\angle X \le 45°$.

Now assume that $ABCD$ is labeled clockwise. Without loss of generality, assume that rays $\overrightarrow{BA}$ and $\overrightarrow{CD}$ do not intersect, and similarly $\overrightarrow{CB}$ and $\overrightarrow{DA}$ do not intersect. The minimum width of all strips that cover $ABCD$ is the minimum length of the perpendiculars from A to BC, A to CD, B to AD, and D to AB. Suppose this minimum exceeds $\sqrt{2}$. Applying the lemma repeatedly, taking XYZ to be ABC, ACD, BAD, and DAB, we conclude that all of $\angle BAC$, $\angle CAD$, $\angle ABD$, and $\angle ADB$ are less than $45°$. This contradicts the fact that these angles sum to $180°$ as internal angles of triangle ABD. Thus $ABCD$ may be covered by a strip of width $\sqrt{2}$. If the minimum equals $\sqrt{2}$, then each of the four angles must equal $45°$. By the conditions for equality in the lemma, it follows that $ABCD$ is a square.

Also solved by E. A. Herman, J. H. Lindsey II, B. Schmuland (Canada), R. Stong, M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), and the proposer.

Double Sum Inequality

11250 [2006, 847]. *Proposed by Sun Wen Cai, Pinggang Middle School, Shenzhen, Guangdong Province, China.* Show that if n is a positive integer and $x_1, \dots, x_n$ are nonnegative real numbers that sum to 1, then

$$\sum_{j=1}^{n} \sqrt{x_j} \sum_{k=1}^{n} \frac{1}{1+\sqrt{1+2x_k}} \le \frac{n^2}{\sqrt{n}+\sqrt{n+2}}.$$

Solution by Vitaly Stakhovsky, Redwood City, CA. Let $n > 1$, $0 \le x \le 1$, $x_0 = 1/n$, $\beta = \sqrt{1+2x_0}$, $f(x) = \sqrt{nx}$, and

$$g(x) = \frac{\sqrt{n}+\sqrt{n+2}}{\sqrt{n}+\sqrt{n+2nx}} = \frac{1+\beta}{1+\sqrt{1+2x}}.$$

We want to prove that

$$\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right) \le 1.$$

Expand $f(x)$ and $g(x)$ by Taylor's theorem: $f(x) = 1 + f'(x_0)(x - x_0) - \phi(x)/2$, where $f'(x_0) = n/2$ and $\phi(x) = (\sqrt{nx} - 1)^2$; and $g(x) = 1 + g'(x_0)(x - x_0) + \psi(x)/2$, where

$$\psi(x) = \frac{(\sqrt{1+2x} - \beta)^2}{1+\beta}\left(\frac{2}{1+\sqrt{1+2x}} + \frac{1}{\beta}\right) \le \left(\sqrt{1+2x} - \beta\right)^2$$

$$\le \left(\sqrt{1+2x} - \beta\right)^2 \left(\frac{\sqrt{1+2x} + \sqrt{1+2x_0}}{\sqrt{2x} + \sqrt{2x_0}}\right)^2 = \frac{2}{n}\left(\sqrt{nx} - 1\right)^2 \le \phi(x).$$

Using $\sum_{i=1}^{n}(x_i - x_0) = 0$, we obtain:

$$\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right) = \left(1 - \frac{1}{2n}\sum_{i=1}^{n}\phi(x_i)\right)\left(1 + \frac{1}{2n}\sum_{i=1}^{n}\psi(x_i)\right)$$

$$\le \left(1 - \frac{1}{2n}\sum_{i=1}^{n}\phi(x_i)\right)\left(1 + \frac{1}{2n}\sum_{i=1}^{n}\phi(x_i)\right) \le 1.$$

Also solved by D. R. Bridges, G. Crandall, P. P. Dályay (Hungary), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), B. Schmuland (Canada), A. Stenger, R. Stong, J. Sun, L. Zhou, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

An Inequality Proved Without Computer Assistance

11251 [2006, 847]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bârlad, Romania.* Suppose that a, b, and c are positive real numbers, two of which are less than or equal to 1, and $ab + ac + bc = 3$. Show that

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} - \frac{3}{4} \ge \frac{3(a-1)(b-1)(c-1)}{2(a+b)(a+c)(b+c)}.$$

Solution by Vitaly Stakhovsky, Redwood City, CA. The inequality follows from

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{3}{2}\,\frac{(a-1)(b-1)(c-1)}{(a+b)(b+c)(c+a)} + \frac{3}{2} \tag{1}$$

and $x^2 + 1/4 \ge x$ with $x = 1/(a+b)$, $1/(b+c)$, and $1/(c+a)$. Multiplying by the positive factor $(a+b)(b+c)(c+a)$, we see that (1) is equivalent to

$$(b+c)(c+a) + (a+b)(c+a) + (a+b)(b+c)$$

$$\ge \frac{3}{2}\,(a-1)(b-1)(c-1) + \frac{3}{2}\,(a+b)(b+c)(c+a). \tag{2}$$

Using S_1, S_2, and S_3 for the symmetric expressions $a+b+c$, $ab+bc+ca$, and abc, inequality (2) becomes

$$S_1^2 + S_2 \ge \frac{3}{2}\,(S_3 - S_2 + S_1 - 1) + \frac{3}{2}(S_1 S_2 - S_3) = \frac{3}{2}\,(S_1 - 1)(S_2 + 1). \tag{3}$$

Now by hypothesis $S_2 = 3$, so finally (3) is equivalent to $S_1^2 + 3 \geq 6(S_1 - 1)$. This is equivalent to $(S_1 - 3)^2 \geq 0$, which is always true.

Also solved by D. Beckwith, P. Bracken, J.-P. Grivaux (France), J. H. Lindsey II, K. McInturff, T.-L. Rădulescu & V. Rădulescu (Romania), V. Schindler (Germany), R. Stong, S. Wagon, T. R. Wilkerson, L. Zhou, GCHQ Problem Solving Group (U. K.), Microsoft Research Problem Solving Group, Northwestern University Math Problem Solving Group, and the proposer.

A Productive Inequality

11252 [2006, 847]. *Proposed by Ovidiu Bagdasar, Babeş Bolyai University, Cluj-Napoca, Romania.* Let n be an integer greater than 2 and let $a_1, \ldots, a_n$ be positive numbers. Let $S = \sum_{i=1}^{n} a_i$. Let $b_i = S - a_i$, and let $S' = \sum_{i=1}^{n} b_i$. Show that

$$\frac{\prod_{i=1}^{n} a_i}{\prod_{i=1}^{n}(S - a_i)} \leq \frac{\prod_{i=1}^{n} b_i}{\prod_{i=1}^{n}(S' - b_i)}.$$

Solution by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bârlad, Romania. We begin with two lemmas.

Lemma 1. Let m be a positive integer, and let $a, a_1, \ldots, a_m$ be positive. Then

$$\frac{\left[(a + a_1) \cdots (a + a_m)\right]^{1/m}}{ma + a_1 + \cdots + a_m} \geq \frac{(a_1 \cdots a_m)^{1/m}}{a_1 + \cdots + a_m}$$

Proof. Apply Jensen's inequality to the concave function $f(x) = \ln(x/(a + x))$ to get

$$\ln \frac{(a_1 + \cdots + a_m)/m}{a + (a_1 + \cdots + a_m)/m} \geq \frac{1}{m} \sum_{k=1}^{m} \ln \frac{a_k}{a + a_k}$$

which is equivalent to the claim.

Lemma 2. Let $a, a_1, \ldots, a_m$ be positive and let $A = a_1 + \cdots + a_m$. Then

$$(a + A - a_1) \cdots (a + A - a_m) \geq \big(a + (m - 1)a_1\big) \cdots \big(a + (m - 1)a_m\big)$$

Proof. By the AM-GM inequality we have

$$(a + A - a_1)^{m-1} \geq \big(a + (m - 1)a_2\big) \cdots \big(a + (m - 1)a_m\big)$$

and $m - 1$ similar inequalities. Multiply them together to get the claimed result.

Now for the problem proposed, using Lemma 2 and then Lemma 1 gives

$$\begin{aligned}
\frac{\left[(S - a_1) \cdots (S - a_{n-1})\right]^{1/(n-1)}}{S' - b_n} &= \frac{\left[(S - a_1) \cdots (S - a_{n-1})\right]^{1/(n-1)}}{(n - 1)a_n + (n - 2)a_1 + \cdots + (n - 2)a_{n-1}} \\
&\geq \frac{\left[(a_n + (n - 2)a_1) \cdots (a_n + (n - 2)a_{n-1})\right]^{1/(n-1)}}{(n - 1)a_n + (n - 2)a_1 + \cdots + (n - 2)a_{n-1}} \\
&\geq \frac{\left[((n - 2)a_1) \cdots ((n - 2)a_{n-1})\right]^{1/(n-1)}}{(n - 2)a_1 + \cdots + (n - 2)a_{n-1}} = \frac{\left[a_1 \cdots a_{n-1}\right]^{1/(n-1)}}{a_1 + \cdots + a_{n-1}}.
\end{aligned}$$

Thus we have

$$\frac{\left[(S - a_1) \cdots (S - a_{n-1})\right]^{1/(n-1)}}{S' - b_n} \geq \frac{\left[a_1 \cdots a_{n-1}\right]^{1/(n-1)}}{a_1 + \cdots + a_{n-1}}$$

and $n - 1$ similar inequalities. Multiply these to get the required result.

Also solved by P. P. Dályay (Hungary), O. P. Lossers (Netherlands), B. Schmuland (Canada), R. Stong, and the proposer.

A Myth About Infinite Products

11257 [2006, 939]. *Proposed by Raimond Struble, Santa Monica, CA.* Let $\langle z_n \rangle$ be a sequence of complex numbers, and let $s_n = \sum_{k=1}^{n} z_k$. Suppose that all s_n are nonzero.
(**a**) Given that s_n does not tend to zero, show that $\sum_{n=1}^{\infty} z_n/s_n$ converges if and only if $\lim_{n\to\infty} s_n$ exists.
(**b**) Show that if s_n tends to a limit s, and $s - s_n$ is never zero, then $\sum_{k=1}^{\infty} z_n/(s - s_{n-1})$ diverges.

Solution by Jean-Pierre Grivaux, Paris, France. We reduce both parts to a commonly-believed (but false) myth about infinite products. In fact, both parts are false, in general.

(**a**) Write

$$\lambda_n = \frac{z_n}{s_n} = \frac{s_n - s_{n-1}}{s_n} = 1 - \frac{s_{n-1}}{s_n}. \tag{1}$$

Since the sums s_n are nonzero, $\lambda_n \neq 1$ and

$$s_n = \frac{s_1}{(1-\lambda_2)(1-\lambda_3)\cdots(1-\lambda_n)}. \tag{2}$$

So the question has been reduced to:

$\sum \lambda_n$ *converges if and only if* $\lim_{n\to\infty} \prod_{k=2}^{n}(1-\lambda_k)$ *exists and is nonzero.*

This is not true. Let $\lambda_n = (-1)^n/\sqrt{n}$. By (2), this defines s_n, and by (1) z_n. The series $\sum \lambda_n$ converges, but $\prod_{k=2}^{n}(1-\lambda_k) \to 0$, since

$$\log\left(1 - \frac{(-1)^k}{\sqrt{k}}\right) = -\frac{(-1)^k}{\sqrt{k}} - \frac{1}{2k} + O\left(\frac{1}{k^{3/2}}\right).$$

Alternatively, if $\lambda_n = i(-1)^n/\sqrt{n}$, then $\sum \lambda_n$ converges, but $\prod_{k=2}^{n}(1-\lambda_k) \to \infty$. If $\lambda_n = (1+i)(-1)^n/\sqrt{n}$, then $\sum \lambda_n$ converges, but $\prod_{k=2}^{n}(1-\lambda_k)$ diverges while remaining bounded.

(**b**) Let $r_n = s - s_{n-1}$. If

$$\mu_n = \frac{z_n}{r_n} = \frac{r_n - r_{n+1}}{r_n} = 1 - \frac{r_{n+1}}{r_n}, \tag{3}$$

then

$$r_{n+1} = r_1(1-\mu_1)(1-\mu_2)\cdots(1-\mu_n). \tag{4}$$

Since $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} r_n = 0$. This reduces the problem to:

If $\lim_{n\to\infty} \prod_{k=1}^{n}(1-\mu_k) = 0$, *then* $\sum \mu_k$ *diverges.*

This is not true, as $\mu_n = (-1)^n/\sqrt{n}$ shows.

Also solved by D. Borwein (Canada), J. H. Lindsey II, O. P. Lossers (The Netherlands), P. Perfetti (Italy), A. Stadler (Switzerland), R. Stong, BSI Problems Group (Germany), and GCHQ Problem Solving Group (U.K.).