OLYMPIAD SOLUTIONS

OC121. Prove that for all positive real numbers x, y, z we have

$$\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx).$$

Originally question 2 from the 2012 Balkan Mathematical Olympiad.

We present two solutions.

Solution 1, composed of similar solution of David Manes and Paolo Perfetti.

By Cauchy-Schwarz we have $(z+x)(z+y) \geq (\sqrt{z}\sqrt{z}+\sqrt{x}\sqrt{y})^2$. Moreover, AM-GM gives $x+y \geq 2\sqrt{xy}$. Thus

$$\begin{aligned}\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} &\geq \sum_{cyc}(x+y)(z+\sqrt{xy}) \\ &= \sum_{cyc}(x+y)z + \sum_{cyc}(x+y)\sqrt{xy} \\ &\geq \sum_{cyc} xz+yz + 2\sum_{cyc} xy \\ &= 4(xy+yz+zx)\,.\end{aligned}$$

Solution 2, composed of similar solutions by Arkady Alt and Šefket Arslanagić.

Let

$$a := \sqrt{y+z},\ b := \sqrt{z+x},\ c := \sqrt{x+y}\,.$$

Then $a, b,$ are the side lengths of an acute triangle, because

$$\frac{b^2+c^2-a^2}{2} = x > 0, \quad \frac{c^2+a^2-b^2}{2} = y > 0, \quad \frac{a^2+b^2-c^2}{2} = z > 0.$$

Moreover, we have

$$\begin{aligned}4\,(xy+yz+zx) &= \sum_{cyclic}\left(b^2+c^2-a^2\right)\left(c^2+a^2-b^2\right) \\ &= 2a^2b^2+b^2c^2+c^2a^2-a^4-b^4-c^4 = 16F^2,\end{aligned}$$

where F is the area of the triangle.

Let R, r, s be circumradius, inradius and semiperimeter of the triangle. Then, original inequality becomes

$$abc\,(a+b+c) \geq 16F^2 \iff 8FRs \geq 16F^2 \iff Rs \geq 2F \iff Rs \geq 2sr \iff R \geq 2r,$$

where latter inequality is the well known Euler's Inequality.