The desired inequality follows from (1) and (2). Equality holds if and only if the triangle is equilateral.

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3648. [2011 : 236, 239] *Proposed by Michel Bataille, Rouen, France.*

Find all real numbers x, y, z such that $xyz = 1$ and $x^3 + y^3 + z^3 = \frac{S(S-4)}{4}$ where $S = \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z}$.

I. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

We assume the equation

$$x^3 + y^3 + z^3 = \frac{S(S-4)}{4}(xyz),$$

without the restriction on xyz. This is equivalent to

$$\sum(x^4y^2 + x^3y^3 + x^2y^2z^2) = \sum(x^4yz + 2x^3y^2z),$$

where both sums, taken over the six permutations of the variables, are symmetric.

By the Arithmetic-Geometric Means Inequality,

$$\begin{aligned}\sum x^4y^2 &= x^4(y^2+z^2) + y^4(z^2+x^2) + z^4(x^2+y^2)\\ &\geq 2x^4yz + 2y^4zx + 2z^4xy = \sum x^4yz,\end{aligned}$$

with equality if and only if $x = y = z$.

Recall Schur's Inequality that, for $a, b, c \geq 0$,

$$\begin{aligned}&(a^3+b^3+c^3) + 3abc - (a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)\\ &\qquad = a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0.\end{aligned}$$

Setting $(a, b, c) = (yz, zx, xy)$, we obtain that

$$\sum(x^3y^3 + x^2y^2z^2) \geq 2\sum x^3y^2z,$$

where again each sum is symmetric with six terms.

Therefore

$$\sum(x^4y^2 + x^3y^3 + x^2y^2z^2) \geq \sum(x^4yz + 2x^3y^2z).$$

Since we are assuming that equality holds and that $xyz = 1$, the given equation is satisfied if and only if $x = y = z = 1$.

II. Solution by the proposer.

The given conditions imply that $S = xy^2 + yx^2 + yz^2 + zy^2 + zx^2 + xz^2$. Without loss of generality, let x be the maximum of x, y, z. Define

$$T = 4(xy^2 + yx^2 + yz^2)(zx^2 + xz^2 + zy^2).$$

Then

$$T = S^2 - (y-z)^2(x^2 + xy + xz - yz)^2$$

and also

$$T = 4xyz(x^3 + y^3 + z^3 + S) + 4y^2z^2(x-y)(x-z),$$

whence

$$S^2 = 4(x^3 + y^3 + z^3 + S) + 4y^2z^2(x-y)(x-z) + (y-z)^2(x^2 + xy + xz - yz)^2.$$

Since, by hypothesis, $4(x^3 + y^3 + z^3 + S) = S^2$, we deduce that

$$4y^2z^2(x-y)(x-z) + (y-z)^2(x^2 + xy + xz - yz)^2 = 0.$$

Since $x \geq y, z$, both terms on the left are nonnegative and therefore must vanish. If, say, $x = y$, then $x^2 + xy + xz - yz = 2x^2 \neq 0$, so that $y = z$. Since $xyz = 1$, we must have that $x = y = z = 1$.

No other solutions were received.

3649. [2011 : 236, 239] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, and c be three positive real numbers and let

$$k = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Prove that

$$(a^3 + b^3 + c^3)\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) \geq \frac{k^3 - 15k^2 + 63k - 45}{4},$$

and equality holds if and only if $(a, b, c) = \left(\frac{k - 5 \pm \sqrt{k^2 - 10k + 9}}{4}, 1, 1\right)$ or any of its permutations.

Solution by Oliver Geupel, Brühl, NRW, Germany.

All sums shall be cyclic. Let $x = \sum \frac{a}{b}$, $y = \sum \frac{b}{a}$, $m = \sum \frac{a^2}{bc}$, and $n = \sum \frac{bc}{a^2}$. We have $x + y = k - 3$, hence $4xy \leq (k-3)^2$. Using the relations

$$\begin{aligned}
\sum \frac{a^3}{b^3} &= x^3 - 3(m+n) - 6, \\
\sum \frac{b^3}{a^3} &= y^3 - 3(m+n) - 6,
\end{aligned}$$