József Wildt International Mathematical Competition

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The solution of problems W1. - W60. must be mailed before 26. October 2022, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania,
E-mail: benczemihaly@gmail.com
benczemihaly@yahoo.com

W1. The angles of triangle ABC are related as 1 : 2 : 4, l_a, l_b, l_c there are bisectors and R the radius of the circumscribed circle. Prove that:

$$l_a + l_b + l_c \leq 3R\sqrt{2\cos\frac{\pi}{14}\cos\frac{3\pi}{14}}$$

Pirkuliyev Rovsen

W2. If a is an integer number and n is a positive integer with g.c.d$(a, n) = 1$, prove that

$$a^{\frac{(n^2-n)!}{(n-1)!}} \equiv 1 \left(mod\ n^2\right).$$

Diana Savin

W3. If $A_kB_kC_k,\ k \in \{1, 2, 3\}$ are triangles with areas $F_k,\ k \in \{1, 2, 3\}$, and usual notations, then prove that

$$\frac{x+y}{z}a_1b_2c_3 + \frac{y+z}{x}b_1c_2a_3 + \frac{z+x}{y}c_1a_2b_3 \geq \frac{48}{\sqrt[4]{27}}\sqrt{F_1F_2F_3},\ \forall x, y, z > 0$$

D.M. Bătineţu-Giurgiu, Neculai Stanciu

W4. Prove that in any triangle ABC with usual notations and area F is true the inequality

$$ab + bc + ca \geq 4\sqrt{3}F + \frac{1}{2}\left(a\left(\sqrt{b} - \sqrt{c}\right)^2 + b\left(\sqrt{c} - \sqrt{a}\right)^2 + c\left(\sqrt{a} - \sqrt{b}\right)^2\right)$$

D.M. Bătineţu-Giurgiu, Neculai Stanciu

W5. If $a > 0$ and $(b_n)_{n\geq 1} > 0$ is a positive real sequence such that $\lim\limits_{n\to\infty} \frac{b_{n+1}}{nb_n} = b > 0$, then find

$$\lim_{n\to\infty} \left(\sqrt[n]{a} - 1\right) \sqrt[n]{b_n}$$

D.M. Bătineţu-Giurgiu, Neculai Stanciu

W6. Let A be the set of all integers n such that $1 \le n \le 2025$ and $\gcd(n, 2025) = 1$. For every nonnegative integer j, let $S(j) = \sum_{n \in A} n^j$.
Show that

$$S(j)\ (mod\ 2025) \equiv \begin{cases} 1080 \ \text{ if } \ 4|\ j \\ \\ 0 \ \text{ if } j \text{ is odd} \\ \\ 675 \ \text{ if } \ \ j \equiv 2\ (mod\ 4) \end{cases}$$

Eugen Ionascu

W7. Find an equivalent of the coefficient of X^n in $\left(1 + X + X^2\right)^n$

Moubinool Omarjee

W8. For $n \in N$, consider in R^3 the regular tetrahedron with vertices $O(0,0,0)$, $A(n, 9n, 4n)$, $B(9n, 4n, n)$ and $C(4n, n, 9n)$. Show that the number N of points (x, y, z), $(x, y, z \in Z)$ inside or on the boundary of the tetrahedron $OABC$ is given by

$$N = \frac{343n^3}{3} + \frac{35}{2}n^2 + \frac{7}{3}n + 1.$$

Eugen Ionascu

W9. If

$$H_k = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_k}, \ K_m = \frac{1}{a_1} + \frac{2}{a_2} + \frac{3}{a_3} + ... + \frac{m}{a_m}$$

$k, m \in N^*$, $a_i \in R^*$, $i = \overline{1, n}$, prove that:

a). $\sum\limits_{k=1}^{n-1} H_k = n \cdot H_n - K_n$ b). $H_n^n \ge H_1 \cdot H_2 \cdot ... \cdot H_{n-1} \cdot K_n$

Dorin Mărghidanu

W10. If l_a, l_b, l_c are the bisectors of the triangle ABC of sides a, b, c, prove that

$$\sqrt{l_a^2 \cdot l_b^2 + l_b^2 \cdot l_c^2 + l_c^2 \cdot l_a^2} \le \frac{\sqrt{3}}{4} \cdot \left(a^2 + b^2 + c^2\right)$$

Dorin Mărghidanu

W11. If $S_n = \sum\limits_{m=1}^{n} \frac{1}{\sum\limits_{l=1}^{m} \frac{1}{\sum\limits_{k=1}^{l} k}}$ then prove that

a). $S_n = \frac{n+H_n}{2}$, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$
b). $S_{2022} > 1014$

Dorin Mărghidanu

W12. Calculate

$$\sum_{n=1}^{\infty}(2n-1)\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right).$$

Ovidiu Furdui and Alina Sîntămărian

W13. Let S_n the group of permutation of {1;2;...;n} $\sigma \in S_n$, for any $g \in S_n$, $\Delta_\sigma(g) = \sigma o g$. Find the signature of Δ_σ

Moubinool Omarjee

W14. Evaluate $S \doteq \sum_{n=0}^{\infty} \frac{2^k(2^{3+2k}-1)}{(2^{2+2k}+1)(2^{4+2k}+1)}$

Paolo Perfetti

W15. Let $\{a_k\}_{k\geq 1}$ be a monotonic sequence of real positive numbers such that $\sum\limits_{n=1}^{\infty} a_n < \infty$. Moreover $\{a_k\}$ fulfills the conditions

$$2^{-n+1}a_{2^{n+1}} \geq a_k - a_{k+1} \geq 2^{-n}a_{2^{n+1}}, \quad \forall\ 2^n \leq k \leq 2^{n+1}-1, \quad \forall\ k \geq 1$$

Let α be a quadratic irrational. Prove that the following quantity is bounded,

$$\frac{1}{\ln n}\sum_{k=1}^{n}\frac{a_k}{\sin(k\pi\alpha)}, \qquad n \geq 1$$

Paolo Perfetti

W16. Let $a_k > 0$, $k = 1, 2, \ldots$ be a monotonic strictly increasing sequence such that $\lim\limits_{k\to\infty} a_k = +\infty$. Determine the character of the series $\sum\limits_{k=1}^{\infty}\left(\tan\frac{\pi}{2}\frac{a_k}{a_{k+1}}\right)^{-1}$

Paolo Perfetti

W17. Let $F(z) = \sum\limits_{n=1}^{\infty} n^p \sin\left(A(\ln n)^q\right) z^n$, p, q nonnegative integers, A real, z complex. A point $z = z_0$ is said a "point of regularity" if the Taylor series centered in z_0 exists with positive radius of convergence. Prove or disprove: $z = 1$ is a point of regularity for $F(z)$.

Paolo Perfetti

W18. Prove that both the polynomials

$$n^6 - 2n^4 + n^3 + n^2 - n \text{ and } n^6 - 2n^4 - n^3 + n^2 + n$$

have at least five prime divisors each for $n > 9$.

Amarnath Murthy

W19. Show that

$$\sum_{k=1}^{n} \frac{1}{(2k+r)!} C\left(2k+r,k\right)\left[\left((k)!\right)^2 + \left((k+r)!\right)^2\right] =$$

$$= \left\{(n+r-1)! - (r-1)!\,(n!)\right\} / \left\{(r-1)!\,(n+r-1)!\right\} +$$

$$+\,(n+r)!/\left\{(n-1)!\,(r+1)\right\}$$

Amarnath Murthy

W20. Given an acute angled Triangle. Construct a line segment of length $L = \frac{rs}{R}$. Where r is the radius of in-circle, R is the radius of circum circle and s is semi perimeter. All the constructionsshould lie inside the triangle.

Amarnath Murthy

W21. Consider the circle $C(O,R)$ and $C(I,r)$, $r < R$ and OI = d . Let $P \in C(O,R)$, and denote U and V the intersection points of circle $C(P,PI)$ and $C(O,R)$. Determine a relation between R, r, d such the UV be tangent to the circle $C(I,r)$.

Ion Patrascu

W22. Is considered a, b, c three complex numbers, which have the following properties:
a). $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{b}{a} + \frac{c}{b} + \frac{a}{c} = \frac{10}{3}$
b). $\max\left(\arg a, \arg b, \arg c\right) \le \frac{\pi}{2}$
If $A = \sum\limits_{cyc}^{3} \left(\frac{a-b}{a+b}\right)^3$, show that the inequality

$$|\pi - \arg A| < \arccos\left(\frac{1}{2} \cdot |A|\right)$$

Florin Stanescu

W23. Consider the complex numbers $a, b, c \in C^*$, different , so that their images in the complex plane are in a circle of radius 1, not all located in the same semicircle. Demonstrate that the following inequality is true:

$$\left|\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{a}{c}\right)\right| + 3\sqrt{3}\left|\left(1-\frac{b}{a}\right)\left(1-\frac{c}{b}\right)\left(1e-\frac{a}{c}\right)\right| \le$$

$$\leq 2\left(2\cdot\left|a^2b+b^2c+c^2a\right|-1\right)$$

Florin Stanescu

W24. Let $f:[0,1]\to[0,1]\,,f\left(0\right)=0$, a continous and bijective function. We assumue that function $\frac{f(x)}{x}$, $x\in(0,1]$ has finit limit for $x\searrow 0$. Show that:

a). $\int\limits_0^1 x^t\left(\frac{\int\limits_0^x\left(f(t)dt+f^{-1}(t)\right)dt}{f(x)+f^{-1}(x)}\right)dx\geq\frac{1}{t\cdot 2^t}$, where $t\geq 1$

b). $\int\limits_0^1 x^t\cdot\left(f\left(x\right)+f^{-1}\left(x\right)\right)^{1-t}\left(\int\limits_0^x\left(f\left(t\right)dt+f^{-1}\left(t\right)\right)dt\right)^q dx\geq\frac{1}{(q+1)2^t}$, $(\forall)\,q\geq 0$, $t\geq 1$

Florin Stanescu

W25. Let ABC be a triangle and Γ his circumcircle. Denote ω the circle which trough on A and B, and in A is tangent to AC, and Φ the circle which trough on A and B and is in A tangent to AB. This circle are intersect second time of tangents from B and C to circle Γ in points U and V. Prove that $BU=CV$.

Ion Patrascu

W26. For $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $r>1$. If $x_{00},y_{00},a_{00},b_{00}>0$ and reals $x_{ij},y_{ij},a_{ij},b_{ij},i=1,2,\ldots,n,j=1,2,\ldots,m$, then

$$\frac{\left(\sum_{j=1}^m\sum_{i=1}^n[(x_{ij}+y_{ij})^r+(a_{ij}+b_{ij})^r]\right)^p}{[(x_{00}+y_{00})^r+(a_{00}+b_{00})^r]^{p/q}}\leq$$

$$\leq\left(\frac{\left(\sum_{j=1}^m\sum_{i=1}^n x_{ij}^r\right)^{p/r}}{x_{00}^{p/q}}+\frac{\left(\sum_{j=1}^m\sum_{i=1}^n y_{ij}^r\right)^{p/r}}{y_{00}^{p/q}}\right)^r+$$

$$+\left(\frac{\left(\sum_{j=1}^m\sum_{i=1}^n a_{ij}^r\right)^{p/r}}{a_{00}^{p/q}}+\frac{\left(\sum_{j=1}^m\sum_{i=1}^n b_{ij}^r\right)^{p/r}}{b_{00}^{p/q}}\right)^r. \tag{1.1}$$

with equality if and only if either $x_{ij}=y_{ij}=0$ and $a_{ij}=b_{ij}=0$ for $i=1,\ldots,n$ and $j=1,\ldots,m$ or $x_{ij}=\alpha y_{ij}$ and $a_{ij}=\beta b_{ij}$ for $i=0,1,\ldots,n$ and $j=0,1,\ldots,m$ and some $\alpha,\beta>0$, and

$$\left(\frac{\left(\sum_{j=1}^m\sum_{i=1}^n x_{ij}^r\right)^{p/r}}{x_{00}^{p/q}}+\frac{\left(\sum_{j=1}^m\sum_{i=1}^n y_{ij}^r\right)^{p/r}}{y_{00}^{p/q}}\right):$$

$$:\left(\frac{\left(\sum_{j=1}^m\sum_{i=1}^n a_{ij}^r\right)^{p/r}}{a_{00}^{p/q}}+\frac{\left(\sum_{j=1}^m\sum_{i=1}^n b_{ij}^r\right)^{p/r}}{b_{00}^{p/q}}\right)=$$

$$=(x_{00}+y_{00}):(a_{00}+b_{00}).$$

Chang-Jian Zhao and Mihály Bencze

W27. For $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $r>1$. If $u(x,y), v(x,y), u'(x,y), v'(x,y)>0$, and $f(x,y), g(x,y), f'(x,y), g'(x,y)$ are continuous functions on $[a,b]\times[c,d]$, then

$$\frac{\left(\int_a^b\int_c^d[(f(x,y)+g(x,y))^r+(f'(x,y)+g'(x,y))^r]dxdy\right)^p}{[(u(x,y)+v(x,y))^r+(u'(x,y)+v'(x,y))^r]^{p/q}}\le$$

$$\le\left(\frac{\left(\int_a^b\int_c^d f(x,y)^r dxdy\right)^{p/r}}{u(x,y)^{p/q}}+\frac{\left(\int_a^b\int_c^d g(x,y)^r dxdy\right)^{p/r}}{v(x,y)^{p/q}}\right)^r+$$

$$+\left(\frac{\left(\int_a^b\int_c^d f'(x,y)^r dxdy\right)^{p/r}}{u'(x,y)^{p/q}}+\frac{\left(\int_a^b\int_c^d g'(x,y)^r dxdy\right)^{p/r}}{v'(x,y)^{p/q}}\right)^r. \tag{1.2}$$

with equality if and only if either $f(x,y)=g(x,y)=0$ and $f'(x,y)=g'(x,y)=0$ or $(f(x,y),g(x,y))=\alpha(u(x,y),v(x,y))$ and $(f'(x,y),g'(x,y))=\beta(u'(x,y),v'(x,y))$ and some $\alpha,\beta>0$, and

$$\left(\frac{\left(\int_a^b\int_c^d f(x,y)^r dxdy\right)^{p/r}}{u(x,y)^{p/q}}+\frac{\left(\int_a^b\int_c^d g(x,y)^r dxdy\right)^{p/r}}{v(x,y)^{p/q}}\right):$$

$$:\left(\frac{\left(\int_a^b\int_c^d f'(x,y)^r dxdy\right)^{p/r}}{u'(x,y)^{p/q}}+\frac{\left(\int_a^b\int_c^d g'(x,y)^r\right)^{p/r}}{v'(x,y)^{p/q}}\right)=$$

$$=(u(x,y)+v(x,y)):(u'(x,y)+v'(x,y)).$$

Chang-Jian Zhao and Mihály Bencze

W28. Prove the following inequalities: for $x_k\in(0,\pi]$, $k=1,2,...,n$

a). $\frac{\sum\limits_{k=1}^n ctgx_k}{n}\le ctg\left(\frac{n}{\sum\limits_{k=1}^n\frac{1}{x_k}}\right)$, for $x_k\in(0,\pi]$, $k=1,2,...,n$

b). $ctg\left(\sqrt[n]{\prod\limits_{k=1}^n x_k}\right)\le\frac{\sum\limits_{k=1}^n ctgx_k}{n}$, for $x_k\in\left(0,\frac{\pi}{4}\right]$, $k=1,2,...,n$

Marian Dincă

W29. If $n,p\in N$, $n,p\ge 2$ and $A,B\in M_2(R)-\{O_2\}$ so that

$$(A+B)^n=(A-B)^p=O_2,$$

then prove

$$TrA=TrB=Tr(AB)=0 \text{ and } \det A+\det B=0$$

Nicolae Papacu

W30. Let $p \in N$ be. Prove that

$$\frac{p+1}{4} \leq (n-p-1)\left(n\int\limits_1^2 \frac{x^p}{1+x^n}dx - \ln 2 + 2^{p+1}\ln\left(1+\frac{1}{2^n}\right)\right) \leq p+1,$$

for all $n \in N, n \geq p+2$

Nicolae Papacu

W31. Let a and x_0 be real numbers and let $(x_n)_{n\geq 0}$ be the sequence defined by

$$x_{n+1} = ax_n^2 - (2a-1)\,x_n + a$$

a). Prove that the sequence $(x_n)_{n\geq 0}$ is convergent, if and only if $ax_0 \in [a-1, a]$
b). If $ax_0 \in [a-1, a]$, compute $\lim\limits_{n\to\infty} n\,(x_n - l)$, where $l = \lim\limits_{n\to\infty} x_n$

Nicolae Papacu

W32. Calculate

$$\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\mathrm{d}x,$$

where $\mathrm{Li}_2(z) = \sum\limits_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}\mathrm{d}t$, $|z| \leq 1$, denotes the Dilogarithm function.

Ovidiu Furdui and Alina Sîntămărian

W33. Let p is positive integer such that $p \geq 2$. Find

$$\min_{n\in\mathbb{N}}\left(\frac{1}{n^p + 2^p - 2}\cdot\sum_{k=0}^{[\log_2 n]}\left[\frac{2^k+n}{2^{k+1}}\right]^p\right).$$

Arkady Alt

W34. Let a, b, c be sidelengths of a triangle and let $F(a,b,c) := \sum\limits_{cyc}\sqrt{\frac{b+c-a}{a}}$. Find

$$\inf F(a,b,c)$$

Arkady Alt

W35. Let $p \geq 2$ be positive integer and $a > 1$ be real number. Find

$$\sum_{n=0}^{\infty}\frac{a_n^{p-2} + 2a_n^{p-3} + 3a_n^{p-4} + \ldots + p - 1}{1 + a_n + a_n^2 + \ldots + a_n^{p-1}},$$

where $a_{n+1}=\frac{1}{p}\left(a_n^p+p-1\right), n\in\mathbb{N}\cup\{0\}$ and $a_0=a>1$.

Arkady Alt

W36. Let ABC be a non-obtuse triangle with usual notations. Prove that

$$\frac{2r}{R}+\frac{r^2}{R^2}\leq\sum\frac{a^2\cos A}{bc}\leq 1+\frac{r}{R}.$$

Arkady Alt

W37. Let $\Delta^3\left(\sqrt{n}\right):=\sqrt{n+3}-3\sqrt{n+2}+3\sqrt{n+1}-\sqrt{n}$. Find

$$\lim_{n\to\infty}n^{5/2}\Delta^3\left(\sqrt{n}\right).$$

Arkady Alt

W38. If $F_0=0,F_1=1$ and $F_n=F_{n-1}+F_{n-2}$, $n\geq 2$ is Fibonacci numbers then the following inequality is satisfied:

$$\sum_{n=1}^{\infty}\frac{1}{(F_1F_2...F_n)^{\frac{1}{n}}}<\frac{\sqrt{q}}{\sqrt{q}-1},\ q=\frac{\sqrt{5}+1}{2}$$

Seyran Ibrahimov

W39. Solve for real numbers:

$$\begin{vmatrix}\sin^3 x & \sin^2 x\cos x & \sin x\cos^2 x & \cos^3 x\\ \sin^3 2x & \sin^2 2x\cos 2x & \sin 2x\cos^2 2x & \cos^3 2x\\ \sin^3 3x & \sin^2 3x\cos 3x & \sin 3x\cos^2 3x & \cos^3 3x\\ \sin^3 4x & \sin^2 4x\cos 4x & \sin 4x\cos^2 4x & \cos^3 4x\end{vmatrix}=0$$

Daniel Sitaru

W40. If $A\in M_{3,2}(\mathbb{C}); B\in M_{2,4}(\mathbb{C}); C\in M_{4,3}(\mathbb{C}), A,B,C$ - fixed then find all $x\in\mathbb{C}$ such that:

$$ABC=\begin{pmatrix}3 & 2 & x\\ 2 & x & 3\\ x & 3 & 2\end{pmatrix}$$

Daniel Sitaru

W41. If $x,y,z>0; x+y+z=3, n\in\mathbb{N}^*$ then:

$$(3-2x)^{2n}+(3-2y)^{2n}+(3-2z)^{2n}+3\geq 2(x^{2n}+y^{2n}+z^{2n})$$

Daniel Sitaru

W42. If $F_n; n \in \mathbb{N}$ are Fibonacci numbers $(F_0 = 0; F_1 = 1; F_{n+2} = F_{n+1} + F_n); L_n$ - Lucas' numbers: $(L_0 = 2; L_1 = 1; L_{n+2} = L_{n+1} + L_n)$ then:

$$\frac{1}{(1+F_{4n+1})^2} + \frac{1}{(1+F_{4n+2})^2} > \frac{1}{1+F_{2n+1}F_{2n+2}L_n^2}$$

Daniel Sitaru

W43. Prove that exist $n \in (0, 90) \cap N$ for which

$$(\sin n^\circ)^3 - 3\sin 20^\circ\,(\sin n^\circ)^2 + (\sin 20^\circ)^3 = 0$$

Ionel Tudor

W44. Let n be a positive integer and let $P_1, P_2, \ldots, P_n$ be the n first Pellian numbers. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\mathcal{P}_k\right)^{-2} + \left(\frac{1}{n^2}\prod_{k=1}^{n}\mathcal{P}_k^{-2}\right)\left(\sum_{k=1}^{n}(\mathcal{P}_k^2-1)^{1/2}\right)^2 \leq 1.$$

José Luis Díaz-Barrero

W45. Let $n \geq 2$ be an integer. Prove that

$$\frac{1}{n}\sum_{1\leq i<j\leq n}\frac{F_i^2F_j^2}{F_i^2+F_j^2} \leq \frac{1}{2\,F_nF_{n+1}}\sum_{1\leq i<j\leq n}F_i^2F_j^2,$$

where F_n is the n^{th} Fibonacci number defined by $F_1 = 1, F_2 = 1$ and for all $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$.

José Luis Díaz-Barrero

W46. Let a, b, c, d be the lengths of the sides of a convex quadrilateral. Prove the inequality

$$81\,(-a+b+c+d)\,(a-b+c+d)\,(a+b-c+d)\,(a+b+c-d) \leq$$

$$\leq (3a+b+c+d)\,(a+3b+c+d)\,(a+b+3c+d)\,(a+b+c+3d)$$

Ovidiu Pop

W47. Let $n \in N$, $n \geq 2$, $a_k, b_k \in R$, $k \in \{1, 2, ..., n\}$ such that $a_1^2 > a_2^2 + a_3^2 + ... + a_n^2$ or $b_1^2 > b_2^2 + b_3^2 + ... + b_n^2$. Prove that the inequality

$$\sum_{2\leq k<l\leq n}(a_kb_l-a_kb_k)^2\leq\sum_{l=2}^{n}(a_1b_l-a_lb_1)^2$$

holds.

Ovidiu Pop

W48. For real numbers $x, y, z > 0$ prove the inequality:

$$\frac{x^{55}}{(x^{440}+2y^{440}+5z^{440})(5x^{440}+3y^{440})}+$$

$$+\frac{y^{88}}{(2x^{440}+y^{440}+2z^{440})(x^{440}+3y^{440}+z^{440})}+$$

$$+\frac{z^{40}}{(4x^{440}+5y^{440}+2z^{440})(3x^{440}+2y^{440}+6z^{440})}\leq\frac{12369}{193600}$$

Ovidiu Pop

W49. In any triangle with the lenghts of the sides a, b, c and R the circumradius, we have

$$R^2-\frac{a^2+b^2+c^2}{9}\geq\frac{1}{36}\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

Nicuşor Minculete

W50. Let F_n be the Fibonacci sequence, $n \in N$. Show that

$$1+\sum_{k=1}^{2n}\frac{F_{2k}}{F_k}$$

is a term of the Fibonacci sequence.

Nicuşor Minculete

W51. For all $n \geq 0$ prove that

$$\sum_{k=0}^{n}\binom{2n+1}{2k}\left(\frac{5}{4}\right)^k=\frac{L_{6n+3}}{2^{2n+2}}$$

where L_n is the nth Lucas number.

Ángel Plaza

W52. Prove that if $x_k \in N\setminus\{0,1,2\}$ $(k = 1, 2, ..., n)$ then

$$\prod_{k=1}^{n}\left(\left[\frac{x_k}{2}\right]+1\right)\leq\left[\frac{1}{2}\prod_{k=1}^{n}x_k\right]+1$$

where [·] denote the integer part.

Mihály Bencze and Marius Drăgan

W53. Let $A \in M_n(Q)$ be such that

$$\det\left(A + \sqrt[n]{2}I_n\right) = 0$$

Prove the identity

$$\det(A - I_n) = \det A + (\det(A + I_n))^n$$

Mihály Bencze and Marius Drăgan

W54. Let ABC non-collinear points and $M_1, M_2, M_3 \in Int(ABC)$ such that

$$z_{M_1} = \alpha_1 z_A + \beta_1 z_B + \gamma_1 z_C, z_{M_2} = \alpha_2 z_A + \beta_2 z_B + \gamma_2 z_C,$$

$$z_{M_3} = \alpha_3 z_A + \beta_3 z_B + \gamma_3 z_C,$$

Then M_1, M_2, M_3 are collinear if and only if

$$\frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_1} = \frac{\beta_3 - \beta_1}{\alpha_3 - \alpha_1} = \frac{\gamma_3 - \gamma_1}{\alpha_3 - \alpha_1}$$

Marius Drăgan and Mihály Bencze

W55. Prove that

$$\left[\frac{2}{3t}\left(\sqrt{(n+t)^3} - \sqrt{(n+t)^3}\right)\right] = \left[\sqrt{4n-3}\right]$$

for all $t \in (0,1)$, $n \geq 2$, where [·] denote the integer part.

Marius Drăgan and Mihály Bencze

W56. Prove that there exist an infinitely of natural numbers n for which $\sigma(n) > kn$ where $k \geq 2$ is a given natural number and $\sigma(n)$ represent the sum of all natural divisors of the number n.

Mihály Bencze and Zhao Changjian

W57. Determine every bijective function $f : N^* \to N^*$ for which $f(n) \in \left\{\frac{n}{p}, q^n\right\}$ for every $n \in N^*$, where p, q are prime numbers.

Mihály Bencze and Zhao Changjian

W58. Let be $f : R \to (0,1)$ where $f(x) = \left\{x^n + x^m + x^k\right\}$, when $\{\cdot\}$ denote the fractional part $n, m, k \in N$ such that $\max\{n, m, k\} \geq 2$. Prove that the function f is not periodical.

Mihály Bencze and Ovidiu Bagdasar

W59. Let be $p > 3$ a prime number. Prove that

$$\sum_{k=1}^{w} \binom{p}{k} \equiv 0 \left(\mathrm{mod}\ p^2\right)$$

when $w = \left[\frac{2p}{3}\right]$, $[\cdot]$ denote the integer part.

Mihály Bencze and Ovidiu Bagdasar

W60. In all triangle ABC holds

$$\sum \frac{m_a w_a}{(b+c)\, s_a} \leq \sqrt{3} \left(\frac{s^2 + r^2 + 4Rr}{8sr}\right)^2$$

Mihály Bencze and Themistocles Rassias

W61. Find:

$$L = \lim_{p\to\infty} \left(\lim_{n\to\infty} \left(\sum_{i=1}^{n} \sqrt[p]{1 + \arcsin \frac{i^{p-1}}{n^p}} - n \right) \right)^{p+\sin p}$$

Florică Anastase

W62. In acute ΔABC the following relationship holds:

$$\sum_{cyc} \left(2 + \frac{\sqrt{h_b h_c}}{a} - \frac{2(s-a)^2}{bc}\right) \leq \sum_{cyc} (1 + cscA)^{\frac{1}{1+ctgA}} \cdot (1 + \sec A)^{\frac{a}{1+tgA}}$$

Florică Anastase

W63. If $\lambda > 1, f : [0,1] \to [1, \lambda]$ is continuous and convex function such that $f(0) = 0$ then:

$$\lambda^2 \int_0^{\frac{1}{\lambda}} f(x)\, dx \leq \int_0^1 f(x)\, dx \leq \lambda(\lambda+1) \int_0^1 \frac{dx}{1 + f(x)}$$

Florică Anastase