Centralizers of polynomials

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ABSTRACT. - We prove that the elements of an open dense subset of the non-linear polynomials' set have trivial centralizers, i. e. they commute only with their own iterates.

0. Let $\mathbb{C}[z]$ be the set of complex polynomials endowed with the topology induced by the norm $||P|| = \sup_{0 \leq i \leq n}\{|a_i|\}$ where n is the degree of P and

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0.$$

Given a non-linear polynomial $P \in \mathbb{C}[z]$, we define the *centralizer* $\mathcal{Z}(P)$, as the set of all non-linear polynomials Q which commute with P:

$$\mathcal{Z}(P) \stackrel{\text{def}}{=} \{Q \in \mathbb{C}[z] \ : \ P \circ Q = Q \circ P \ \text{ and } \ \deg(Q) \geq 2\}.$$

If $n = \deg(P) \geq 2$ then the number of polynomials in $\mathcal{Z}(P)$ of fixed degree is at most $n-1$ (see [Bo]), hence $\mathcal{Z}(P)$ is always countable.

The purpose of this paper is to investigate when the centralizer $\mathcal{Z}(P)$ contains only the iterates of P. The following result is motivated by the fact that the same problem has been already studied for other dynamical systems such as the diffeomorphisms on the circle (see [Ko]), the expanding maps on the circle (see [Ar]), and the Anosov diffeomorphisms on the torus (see [PaYo]).

Theorem 0.1 *There exists an open dense subset of the set of all non-linear polynomials whose elements P have trivial centralizer:*

$$\mathcal{Z}(P) = \{P^k \ : \ k \geq 1\}.$$

The question arises whether it is possible to generalize this result for the set of rational functions of degree at least two.

1. For a polynomial P of degree $n \geq 2$, the *Julia set* $\mathcal{J}(P)$ is defined as the set of all points $z \in \widehat{\mathbb{C}}$ such that the family of iterates $\{P^k\}_{k \geq 1}$ is not *normal* in any neighborhood of z. We recall that $\mathcal{J}(P)$ is a non-empty bounded perfect set, which is *completely invariant*, i. e. $P(\mathcal{J}(P)) = P^{-1}(\mathcal{J}(P)) = \mathcal{J}(P)$.

Moreover, if $\mathcal{J}(P)$ is the unit circle $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$ or the interval $[-1, 1]$ then P is conjugate to a *Tchebycheff polynomial* which is $e^{i\theta} T_n$ for S^1 where $T_n(z) = z^n$ and is T_n or $-T_n$ for $[-1, 1]$ where T_n is defined inductively in the following way $T_n(z) = 2zT_{n-1}(z) - T_{n-2}(z)$ and $T_0 = 1$, $T_1(z) = z$. A Tchebycheff polynomial has a very big centralizer because $T_n \circ T_m = T_m \circ T_n = T_{nm}$ for $n, m \geq 0$, and this is the only kind of non-linear polynomial whose centralizer has at least a polynomial for any degree (see [BT] and [Ber]).

Recently, G. M. Levin, has recovered in a modern way a very old result of J. F. Ritt. This is its reformulation:

Theorem 1.1 [Le], [Ri2] *If two non-linear polynomials P and Q commute then one of the following conditions is necessary:*

(a) P and Q have a common iterate, i. e. there exist integers $i, j \geq 1$ such that $P^i = Q^j$;

(b) the common Julia set is either a circle or an interval.

A. F. Beardon, starting from the work of I. N. Baker and A. Eremenko ([BE]), has succeded to characterize all pairs of non-linear polynomials P which have the same Julia set (e. g. when they commute) in the term of the group $\Sigma(P)$ of symmetries of the Julia set of P:

$$\Sigma(P) \stackrel{\text{def}}{=} \{\sigma \in \mathcal{E} \ : \ \sigma(\mathcal{J}(P)) = \mathcal{J}(P)\}$$

where $\mathcal{E}$ is the group of the conformal Euclidean isometries of $\mathbb{C}$, $z \xrightarrow{\sigma} e^{i\theta}z + c$. Since the Julia set of a non-linear polynomial P is bounded, then $\Sigma(P)$ can not contain any translation $z \to z + c$ with $c \neq 0$. Moreover, if $\sigma_1, \sigma_2 \in \Sigma(P)$ then their commutator $\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$ belongs also to $\Sigma(P)$ and it is a translation, therefore it is the identity map. Hence, σ_1 and σ_2 commute and it follows that $\Sigma(P)$ is a group of rotations about a common fixed point $\zeta \in \mathbb{C}$. The next theorem gives a complete description of this group:

Theorem 1.2 *Let P be a non-linear polynomial then the following facts hold:*

(a) [Be1] *$\Sigma(P)$ is a group of rotations around the point $\zeta = -\frac{a_{n-1}}{na_n}$, called centroid of P (it is the barycentre of the zeros of P). If $\Sigma(P)$ is infinite then $\mathcal{J}(P)$ is a circle. Otherwise $\Sigma(P)$ is finite and, if we put the centroid in 0, the order of $\Sigma(P)$ is the largest integer $d \geq 1$ such that P can be written in the form $P(z) = z^a\widetilde{P}(z^d)$ for some polynomial $\widetilde{P}$ with $0 \leq a < n$.*

(b)[Be2] *If Q is a polynomial which has the same degree of P and $\mathcal{J}(P) = \mathcal{J}(Q)$ then there is a symmetry $\sigma \in \Sigma(P) = \Sigma(Q)$ such that $P = \sigma Q$.*

These facts shed a new light on another result of J. F. Ritt which allow us to be more precise when two commuting polynomials happen to have a common iterate.

Theorem 1.3 [Ri1] *If two non-linear polynomials P and Q have a common iterate then there exist a non-linear polynomial R, two integers $s, t \geq 1$ and two symmetries $\sigma_1, \sigma_2 \in \Sigma(P) = \Sigma(Q)$ such that:*

$$P(z) = \sigma_1 R^s(z) \ \text{ and } \ Q(z) = \sigma_2 R^t(z) \quad \forall z \in \mathbb{C}.$$

Proof. We follow the Ritt's proof emphasizing the steps where the theory of Beardon is useful to semplify the reasoning.

Since P and Q have the same Julia set, there is a map Φ, called *Böttcher function*, univalent in some neighborhood U of ∞ such that

$$\Phi \circ P \circ \Phi^{-1}(z) = a_n z^n \text{ and } \Phi \circ Q \circ \Phi^{-1}(z) = b_m z^m \ \ \forall z \in U$$

where $b_m z^m$ is the leading term of Q. By hypothesis, P and Q have a common iterate, hence there exist integers r, u, v such that $n = r^u$ and $m = r^v$. If we denote with (u, v) the G.C.D. of u and v then the polynomial R can be chosen of the form:

$$R(z) = \Phi^{-1}(c[\Phi(z)]^{r^{(u,v)}})$$

where $c \in \mathbb{C}$ is such that $\mathcal{J}(R) = \mathcal{J}(P) = \mathcal{J}(Q)$. For two suitable positive integers s and t, the degrees of P, R^s and R^t are equal and, by (b) of Theorem 1.2, there exist two symmetries $\sigma_1, \sigma_2 \in \Sigma(P)$ such that $P = \sigma_1 R^s$ and $Q = \sigma_1 R^t$. Q.E.D.

2. Now, we give the proof of Theorem 0.1.

Proof. Define the set $\mathcal{S}$ of all non-linear polynomials P such that $\mathrm{Fix}_{\widehat{\mathbb{C}}}(P) \stackrel{\text{def}}{=} \{z \in \widehat{\mathbb{C}} : P(z) = z\}$ has $n+1$ different points where $n \geq 2$ is the degree of P, and such that the following property holds

$$\text{if } x, y \in \mathrm{Fix}_{\widehat{\mathbb{C}}}(P) \text{ and } x \neq y \text{ then } P'(x) \neq P'(y). \tag{1}$$

It is clear that $\mathcal{S}$ is open and dense in the set of all non-linear polynomials. Let $P \in \mathcal{S}$ then, since $P(\infty) = \infty$ and $P'(\infty) = 0$, by the property (1), at any finite fixed point z of P, $P'(z) \neq 0$. Conjugating P by an affine transformation we can assume that the centroid is 0 and that $\mathcal{J}(P)$ becomes S^1 if it is a circle or $[-1, 1]$ if it is an interval. Note that the conjugation preserves the property (1).

$\mathcal{J}(P)$ can not be S^1 because otherwise $P = e^{i\theta} T_n$ and at the fixed point 0, $P'(0) = 0$. If $\mathcal{J}(P) = [-1, 1]$ then $P = T_n$ or $P = -T_n$ and all the finite fixed points of P are contained in $[-1, 1]$ because $\widehat{\mathbb{C}} \setminus [-1, 1]$ is the basin of attraction of ∞. Moreover, for $x \in [-1, 1]$, $T_n(x) = \cos(n\alpha)$ with $\alpha = \cos^{-1}(x)$ and therefore the derivative of T_n at $x \in]-1, 1[$ is

$$T_n'(x) = (\frac{d}{d\alpha}\cos(n\alpha))\frac{d\alpha}{dx} = \frac{-n\sin(n\alpha)}{-\sin(\alpha)} = n\frac{\sin(n\alpha)}{\sin(\alpha)}.$$

Hence, if $x \in \mathrm{Fix}_{\mathbb{C}}(P) \setminus \{1, -1\}$ then $|\cos(n\alpha)| = |\cos(\alpha)|$ and $|P'(x)| = n$ because $|\sin(n\alpha)| = |\sin(\alpha)|$. By hypothesis P has n different finite fixed points and, by property (1), the degree n has to be less than 5. Moreover, one can easily check that $\pm T_3$ and $\pm T_4$ do not belong to $\mathcal{S}$ whereas $\pm T_2 \in \mathcal{S}$.

So, if $P \in \mathcal{S} \setminus \{T_2, -T_2\}$, by Theorem 1.1, if $Q \in \mathcal{Z}(P)$ then P and Q must have a common iterate and, by Theorem 1.3, there exists a non-linear polynomial R such that:

$$P(z) = \sigma_1 R^s(z) \text{ and } Q(z) = \sigma_2 R^t(z) \qquad \forall z \in \mathbb{C} \tag{2}$$

where $s, t \geq 1$ and $\sigma_1, \sigma_2 \in \Sigma(P)$. Since $\mathcal{J}(P)$ is not a circle, by (a) of Theorem 1.2, the group $\Sigma(P)$ is finite of order $d \geq 1$.

Now we distinguish two cases:

(i) If $0 \in \mathrm{Fix}_{\mathbb{C}}(P)$ then $d = 1$ and therefore $P = R^s$ and $Q = R^t$.

In fact, by (a) of Theorem 1.2, since $\mathcal{J}(R) = \mathcal{J}(P)$, $P(z) = z^a \widetilde{P}(z^d)$ for some polynomial $\widetilde{P}$. Assume that $d \geq 2$, then, since $P'(0) \neq 0$, $a = 1$ and computing the derivative in a point $z \in \mathbb{C}$ we obtain

$$P'(z) = \widetilde{P}(z^d) + dz^d \widetilde{P}'(z^d).$$

Let $\sigma \in \Sigma(P)$ be different from the identity. Let z_1 be a finite fixed point of P different from 0 then $z_2 = \sigma z_1$ is another finite fixed point of P because $\sigma^d = 1$ and

$$P(z_2) = \sigma z_1 \widetilde{P}(\sigma^d z_1^d) = \sigma P(z_1) = \sigma z_1 = z_2.$$

Since $z_1^d = z_2^d$, if we compute the derivative in these points, we obtain

$$P'(z_1) = \widetilde{P}(z_1^d) + dz_1^d \widetilde{P}'(z_1^d) = P'(z_2).$$

This contradicts the property (1) and $d = 1$.

(ii) If $0 \notin \mathrm{Fix}_{\mathbb{C}}(P)$ then $\sigma_1 = \sigma_2$ and $P = (\sigma_1 R)^s$ and $Q = (\sigma_1 R)^t$.

In fact, by (a) of Theorem 1.2, $R(z) = z^a \widetilde{R}(z^d)$ for some polynomial $\widetilde{R}$. Since $P(0) \neq 0$, then, by (2), $a = 0$ and $\widetilde{R}(0) \neq 0$. If $z \in \mathrm{Fix}_{\mathbb{C}}(R)$ then

$$P^i(z) = \sigma_1 R^{si}(z) = \sigma_1 z \text{ and } Q^j(z) = \sigma_2 R^{tj}(z) = \sigma_2 z$$

because $\sigma_1^d = \sigma_2^d = 1$. Since $z \neq 0$ and $P^i = Q^j$, we can conclude by the above equation that $\sigma_1 = \sigma_2$. Moreover, $P = \sigma_1 R^s = (\sigma_1 R)^s$ and $Q = \sigma_2 R^t = (\sigma_2 R)^t$.

In both cases (i) and (ii), we have found a non-linear polynomial G such that $P = G^s$ and $Q = G^t$; now we show that $s = 1$, i. e. Q is an iterate of P. If $z \in \mathrm{Fix}_{\mathbb{C}}(P)$ then $P'(z) \neq 0$ and therefore also $G'(z) \neq 0$. Since P and G commute, we have

$$P(G(z)) = G(P(z)) = G(z),$$

and deriving $P \circ G = G \circ P$ we obtain

$$P'(G(z))G'(z) = G'(P(z))P'(z) = G'(z)P'(z).$$

These equations yield that also $G(z) \in \mathrm{Fix}_{\mathbb{C}}(P)$ is a fixed point of P and $P'(G(z)) = P'(z)$. By property (1), $G(z) = z$ and therefore G has as many fixed points as P. But, by hypothesis, the number of finite fixed points of P is exactly n and therefore the degree of G is at least n. This is possible only when $s = 1$.

Hence we can conclude that the wanted open dense set is $\mathcal{S} \setminus \{T_2, -T_2\}$.

Q.E.D.

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