Commuting Holomorphic Maps of the Unit Disc

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Abstract. Let f and g be two commuting holomorphic self-maps of the open unit disc $\mathbb{D}$ in the complex plane with a common Wolff point $\tau \in \partial\mathbb{D}$: if this two maps agree at τ up to the third order then $f \equiv g$.

1. *Introduction*

The purpose of this paper is to show a connection betweeen iteration theory and the study of commuting holomorphic maps of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

The dynamical properties of a map $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ are well known (see the survey article of Burckel [**6**]): if f is not the identity map then it has at most one fixed point in $\mathbb{D}$ which is attracting provided f is not an elliptic automorphism (a rotation). On the other hand, if f is fixed-point-free then there is still an attracting point τ, called *Wolff point* of f, but it is located on the boundary $\partial\mathbb{D}$ and the sequence of iterates f^n converges to τ uniformly on compact subsets of $\mathbb{D}$.

Now take another map $g \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ and assume that it commutes with f:

$$f \circ g = g \circ f.$$

If f has a fixed point $z_0 \in \mathbb{D}$ then

$$f(g(z_0)) = g(f(z_0)) = g(z_0)$$

and, by uniqueness, $g(z_0) = z_0$, that is z_0 is the fixed point also for the map g. A similar result holds when f is fixed-point-free: in this case f and g have the same Wolff point unless they are two hyperbolic automorphisms with the same fixed points (see [**2**],[**9**]).

The common Wolff point τ contains a lot of interesting information about the two maps. To extract this information we need the following regularity notion: we say that $f \in C^r_K(\tau)$ if it has an expansion at τ of the form

$$f(z) = \tau + f'(\tau)(z-\tau) + \ldots + \frac{1}{r!}f^{(r)}(\tau)(z-\tau)^r + o_K(|z-\tau|^r),$$

where o_K means that the limit is taken non-tangentially (i. e. within an angular region with vertex at τ):

$$\text{K-}\lim_{z\to\tau}\frac{o_K(|z-\tau|^r)}{|z-\tau|^r}=0.$$

Moreover, we say that $f\in C^r(\tau)$ if $o_K(|z-\tau|^r)$ can be replaced by $o(|z-\tau|^r)$ that is the limit is taken in the full disc.

The Julia-Wolff-Carathéodory theorem says that some regularity at τ is expected although τ belongs to the boundary $\partial\mathbb{D}$: $f\in C^1_K(\tau)$ and $0<f'(\tau)\le 1$. If $f'(\tau)<1$ then f is called *hyperbolic*, whereas if $f'(\tau)=1$ then f is called *parabolic*.

What happens if we compare the expansions of f and g at τ? As we will see, to establish that $f\equiv g$ it suffices to check that the two maps agree up to the third order at τ. Note that this "identity principle" is not true neither when the attracting point stays in $\mathbb{D}$ (z^n and z^m commute and their expansions at 0 agree up to the $(\min(n,m)-1)$-order nor when the commuting property does not hold (we will show an example in the last section).

In [**4**], we discussed this problem and we established the following result for the "extreme" cases: when f is hyperbolic or the identity (this is due to Burns and Krantz [**5**])

THEOREM 1.1. *If one of the following conditions holds then $f\equiv g$.*

(1) f is hyperbolic with Wolff point at τ and $f'(\tau)=g'(\tau)$;

(2) $f=Id$, $g\in C^3_K(\tau)$ and $f(\tau)=g(\tau)=\tau$, $f'(\tau)=g'(\tau)=1$, $f''(\tau)=g''(\tau)=0$, $f'''(\tau)=g'''(\tau)=0$.

In this paper we will prove the following theorem for the "middle" case, that is when f is parabolic, improving a previous result appeared in [**4**].

THEOREM 1.2. *If f is parabolic with Wolff point at τ and one of the following conditions holds then $f\equiv g$.*

(1) $f\in C^2(\tau)$, $g\in C^2_K(\tau)$, $f''(\tau)=g''(\tau)\neq 0$ and $\mathrm{Re}\,(f''(\tau)\tau)>0$;

(2) $f,g\in C^2(\tau)$, $f''(\tau)=g''(\tau)\neq 0$ and $\mathrm{Re}\,(f''(\tau)\tau)=0$;

(3) $f\in C^3(\tau)$, $g\in C^3_K(\tau)$, $f''(\tau)=g''(\tau)=0$ and $f'''(\tau)=g'''(\tau)$.

2. *The linear model in $\mathbb{H}$*

The main tool that we are going to use is the construction of a *linear model* for our maps: a "change of coordinates" in a neighborhood of the Wolff point which transforms f in an automorphism of the upper half-plane $\mathbb{H}=\{w\in\mathbb{C}\,:\,\mathrm{Im}\,w>0\}$ or of the entire plane $\mathbb{C}$. To simplify notations, from now on we will work in the upper half-plane which is biholomorphically equivalent to the unic disc $\mathbb{D}$ by the *Cayley transformation* $C(z)=i\frac{\tau+z}{\tau-z}$ that maps τ to ∞. Then, by Julia-Wolff-Carathéodory theorem,

$$\text{K-}\lim_{w\to\infty}\frac{F(w)}{w}=\alpha$$

where

$$\alpha := \inf\left\{ \frac{\operatorname{Im} F(w)}{\operatorname{Im} w} \ : \ w \in \mathbb{H} \right\} \in [0, +\infty).$$

Therefore, if $\alpha > 0$ then $F \in C^1_K(\infty)$, $\alpha = 1/f'(\tau)$ and

$$F(w) = \alpha w + \Gamma(w) \quad \text{with} \quad \Gamma(w) = o_K(|w|).$$

If ∞ is the Wolff point of F then $\alpha \geq 1$. When $\alpha = 1$ and F is not the identity then $\Gamma(w) \in \operatorname{Hol}(\mathbb{H}, \mathbb{H})$ and

$$\begin{array}{ll} \Gamma(w) = \beta + o_K(1) & \text{if } f \in C^2_K(\tau), \\ \Gamma(w) = \beta + \frac{\gamma}{w} + o_K(\frac{1}{|w|}) & \text{if } f \in C^3_K(\tau), \end{array}$$

where $\beta = if''(\tau)\tau$, $\gamma = 2S_f(\tau)\tau^2/3$ and $S_f(\tau) = f'''(\tau) - \frac{3}{2}(f''(\tau))^2$ (the Schwarzian derivative of f at τ). Note that $\operatorname{Re}\beta \geq 0$ and if $\beta = 0$ then $\gamma \leq 0$.

The following result due to Cowen [**7**] gives some precious information about the orbits behaviour.

Theorem 2.1 (Cowen) *Let $F \in \operatorname{Hol}(\mathbb{H}, \mathbb{H})$ with Wolff point at ∞. Then there is an open connected, simply connected set V, called* fundamental set *for F, such that:*

(1) V is F-invariant that is $F(V) \subset V$;

(2) for all compact set K of $\mathbb{H}$, the sequence $F^n(K)$ is evenly contained in V;

(3) F is univalent in V.

The Poincaré distance in $\mathbb{H}$ is defined by

$$d(w, w') = \operatorname{tgh}^{-1}\left|\frac{w - w'}{w - \overline{w'}}\right| \quad \forall w, w' \in \mathbb{H}.$$

We say that F is of *automorphic type* ($F \in \mathcal{A}$) if all orbits are separated in the Poincaré distance:

$$\lim_{n\to\infty} d(w_{n+1}, w_n) > 0 \quad \forall w \in \mathbb{H}$$

where $w_n = x_n + iy_n = F^n(w)$. The above limit exists because F is a d-contraction. If F is hyperbolic then $F \in \mathcal{A}$, on the other hand if F is parabolic then it can be of automorphic type or not. Furthermore, $F \notin \mathcal{A}$ if and only if

$$q_n := \frac{w_{n+1} - w_n}{w_{n+1} - \overline{w_n}} \to 0.$$

Here is the construction of the *linear model* for F due to Baker and Pommerenke (see [**8**],[**1**])

Theorem 2.2 (Baker-Pommerenke) *Let $w_0 \in \mathbb{H}$ and*

$$\sigma_n(w) := \frac{w_n - x_n^0}{y_n^0},$$

then the limit

$$\sigma := \lim_{n\to\infty} \sigma_n$$

exists locally uniformly in $\mathbb{H}$ *and satisfies* $\sigma(w_0) = i$. *Moreover, there is an automorphism* Φ *of* $\mathbb{H}$ *which fixes* ∞ *such that*

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{F} & \mathbb{H} \\ \sigma\big\downarrow & & \big\downarrow\sigma \\ \mathbb{H} & \xrightarrow{\Phi} & \mathbb{H} \end{array}$$

The map $\sigma \not\equiv i$ *if and only if* $F \in \mathcal{A}$.
If $F \notin \mathcal{A}$ *let*

$$\rho_n(w) := \frac{\sigma_n(w) - i}{q_n^0},$$

then the limit

$$\rho := \lim_{n\to\infty} \rho_n$$

exists locally uniformly in $\mathbb{H}$ *and satisfies* $\rho(w_0) = 0$. *Moreover*

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{F} & \mathbb{H} \\ \rho\big\downarrow & & \big\downarrow\rho \\ \mathbb{C} & \xrightarrow{w+2i} & \mathbb{C} \end{array}$$

The following key-lemma will be very useful later.

LEMMA 2.1. *Let* F, G *be two commuting maps with Wolff point at* ∞. *If one of the following conditions holds then* $F \equiv G$.

(1) $F \in \mathcal{A}$ *and* $\sigma \circ F = \sigma \circ G$;

(2) $F \notin \mathcal{A}$ *and* $\rho \circ F = \rho \circ G$.

Proof. Let V be a fundamental set for F. Since F is univalent in V and the set V is F-invariant then also F^n, σ_n, ρ_n are univalent in V for any $n \in \mathbb{N}$. If $F \in \mathcal{A}$ then the limit σ is not constant and, by Hurwitz theorem, it is univalent in V. In the same way, if $F \notin \mathcal{A}$ then the limit ρ (which is not constant) is univalent in V too. Moreover, if G commutes with F and K is a non-empty compact subset of $G(V)$ then $F^n(K) \subset V$ for some $n \in \mathbb{N}$ and

$$F^n(K) \subset F^n(G(V)) = G(F^n(V)) \subset G(V).$$

Therefore the open set $G(V) \cap V$ is non-empty because it contains $F^n(K)$ and it is possible to find a non-empty open set $W \subset V$ such that $G(W) \subset V$. Since σ is injective in V and both sets $F(W)$ and $G(W)$ are contained in V then the condition

$$\sigma(F(w)) = \sigma(G(w)) \quad \forall w \in W$$

implies that $F \equiv G$ in $\mathbb{H}$. In a similar way

$$\rho(F(w)) = \rho(G(w)) \quad \forall w \in W$$

implies that $F \equiv G$ in $\mathbb{H}$. □

As a first application of the previous results we discuss the cases when f is hyperbolic or f is the identity map.

Proof. [Proof of Theorem 1.1.]
Case (1). We know $F(w) = \alpha_F\, w + \Gamma_F(w)$, $G(w) = \alpha_G\, w + \Gamma_G(w)$. Since F and G commute then

$$\begin{aligned}\frac{F^n(G(w)) - x_n^0}{y_n^0} &= \frac{G(w_n) - x_n^0}{y_n^0} = \alpha_G\, \frac{w_n - x_n^0}{y_n^0} \\ &\quad +(\alpha_G - 1)\,\frac{x_n^0}{y_n^0} + \frac{\Gamma_G(w_n)}{w_n}\left(\frac{w_n - x_n^0}{y_n^0} + \frac{x_n^0}{y_n^0}\right)\end{aligned}$$

By a result due to Cowen (see Lemma 2.2 in [**7**]), w_n goes to ∞ non-tangentially and therefore the sequence x_n^0/y_n^0 is bounded and, up to subsequence, we can assume that it converges to the real number M. Moreover

$$\lim_{n\to\infty} \frac{\Gamma_G(w_n)}{w_n} = 0.$$

Thus by Theorem 2.2, taking the limit, we find

$$\sigma(G(w)) = \alpha_G\, \sigma(w) + (\alpha_G - 1)\, M.$$

Similarly

$$\sigma(F(w)) = \alpha_F\, \sigma(w) + (\alpha_F - 1)\, M.$$

If $f'(\tau) = g'(\tau)$ then $\alpha_F = \alpha_G$ and therefore $\sigma \circ F \equiv \sigma \circ G$. Since $F \in \mathcal{A}$, by Lemma 2.1, we find that $F \equiv G$.
Case (2). By hypothesis

$$G(w) = w + \Gamma(w) \quad \text{with} \quad \operatorname*{K-lim}_{w\to\infty} w\Gamma(w) = 0.$$

If G is not the identity map then, by the maximum principle, $T(\mathbb{H}) \subset \mathbb{H}$ where $T(w) := -1/\Gamma(w)$. Furthermore

$$\operatorname*{K-lim}_{w\to\infty} \frac{T(w)}{w} = \operatorname*{K-lim}_{w\to\infty} \frac{-1}{w\Gamma(w)} = \infty$$

and this is a contradiction because, by the Julia-Wolff-Carathéodory theorem applied to the map T, this limit has to be finite. □

3. *The parabolic case*

First we establish a result about the kind of convergence of the orbits of F to ∞. The first case has been proved by Bourdon and Shapiro [**3**].

PROPOSITION 3.1. *Let $F \in \mathrm{Hol}(\mathbb{H}, \mathbb{H})$ be parabolic with Wolff point at ∞.*

(1) If $F \in C^2(\infty)$ and $\beta \neq 0$ then w_n goes to ∞ non-tangentially if and only if $Im(\beta) > 0$.

(2) If $F \in C^3(\infty)$ and $\beta = 0$ then $\gamma < 0$ and w_n has a subsequence that goes to ∞ non-tangentially.

Proof. Case (1): if $F \in C^2(\infty)$ then

$$\frac{w_n}{n} = \frac{w}{n} + \beta + \frac{1}{n}\sum_{j=0}^{n-1}\Gamma(w_j).$$

Therefore, since $\Gamma(w) = o(1)$,

$$\lim_{n\to\infty}\frac{w_n}{n} = \beta.$$

Hence

$$\lim_{n\to\infty}\frac{x_n}{n} = \operatorname{Re}\beta \quad \text{and} \quad \lim_{n\to\infty}\frac{y_n}{n} = \operatorname{Im}\beta$$

and if $\beta \neq 0$ then w_n goes to ∞ nontangentially if and only if $\operatorname{Im}(\beta) > 0$.
Case (2): now $\beta = 0$, $F \in C^3(\infty)$ and $\gamma < 0$ by Theorem 1.1 because F si not the identity map. Therefore

$$w_{n+1} = w_n + \frac{\gamma}{w_n} + o(\frac{1}{|w_n|}) = w_n + \frac{\gamma\,\overline{w_n} + |w_n|\cdot o(1)}{|w_n|^2}.$$

Taking the real part we find that

$$x_{n+1} = x_n \cdot \left(1 + \frac{\gamma\ + \dfrac{|w_n|}{x_n}\cdot \operatorname{Re}(o(1))}{|w_n|^2}\right).$$

Assume by contradiction that the sequence w_n has no subsequence which goes to ∞ nontangentially. Then the sequence $|x_n|/y_n$ is bounded away from zero and the sequence

$$\frac{|w_n|}{x_n} = \frac{|x_n|}{x_n}\sqrt{1+\left(\frac{y_n}{x_n}\right)^2} \text{ is bounded.}$$

Hence evenly

$$c_n := 1 + \frac{\gamma\ + \dfrac{|w_n|}{x_n}\cdot \operatorname{Re}(o(1))}{|w_n|^2} \in (0,1).$$

and

$$|x_{n+1}| = c_n|x_n| \leq |x_n|$$

which means that also the sequence x_n is bounded. Since $|w_n|$ goes to infinity then y_n can not be bounded therefore $|x_n|/y_n$ goes to zero against our assumption. □

The following theorem establishes a necessary and sufficient condition so that two parabolic commuting maps coincide.

THEOREM 3.1. *Let $F, G \in \operatorname{Hol}(\mathbb{H},\mathbb{H})$ be two parabolic commuting maps with Wolff point at ∞. Then the following limit exists locally uniformly in $\mathbb{H}$*

$$H(w) := \lim_{n\to\infty}\frac{\Gamma_G(w_n)}{\Gamma_F(w_n)}.$$

Moreover, $F \equiv G$ if and only if $H \equiv 1$.

Proof. First assume that $F \in \mathcal{A}$. Thus by Theorem 2.2, taking the limit in

$$\sigma_n(F(w)) = \sigma_n(w) + \frac{\Gamma_F(w_n)}{y_n^0}$$

we find that

$$\sigma(F(w)) = \sigma(w) + a_F$$

where

$$a_F = \lim_{n\to\infty} \frac{\Gamma_F(w_n)}{y_n^0} \in \mathbb{R} \setminus \{0\}.$$

It can not be zero otherwise

$$q_n^0 = \frac{1}{2i\dfrac{y_n^0}{\Gamma_F(w_n^0)} + 1} \to 0$$

against the fact that $F \in \mathcal{A}$.

Moreover, since F and G commute

$$\sigma_n(G(w)) = \sigma_n(w) + \frac{\Gamma_F(w_n)}{y_n^0} \cdot \frac{\Gamma_G(w_n)}{\Gamma_F(w_n)}$$

and taking the limit we find

$$H(w) = \frac{\sigma(G(w)) - \sigma(w)}{a_F} = \frac{\sigma(G(w)) - \sigma(w)}{\sigma(F(w)) - \sigma(w)}.$$

Hence $H \equiv 1$ if and only if $\sigma \circ F \equiv \sigma \circ G$ and, by Lemma 2.1, this is equivalent to $F \equiv G$.

Now we consider the case when $F \notin \mathcal{A}$. Thus by Theorem 2.2, taking the limit in

$$\rho_n(F(w)) = \rho_n(w) + \frac{\Gamma_F(w_n)}{y_n^0 q_n^0}$$

we find

$$\rho(F(w)) = \rho(w) + 2i$$

and

$$\lim_{n\to\infty} \frac{\Gamma_F(w_n)}{y_n^0 q_n^0} = 2i.$$

Moreover

$$\rho_n(G(w)) = \rho_n(w) + \frac{\Gamma_F(w_n)}{y_n^0 q_n^0} \cdot \frac{\Gamma_G(w_n)}{\Gamma_F(w_n)}$$

and taking the limit

$$H(w) = \frac{\rho(G(w)) - \rho(w)}{2i} = \frac{\rho(G(w)) - \rho(w)}{\rho(F(w)) - \rho(w)}.$$

Hence $H \equiv 1$ if and only if $\rho \circ F \equiv \rho \circ G$ and, by Lemma 2.1, this is equivalent to $F \equiv G$. □

Note that in the previous theorem the condition that F and G commute is necessary. For example, taking

$$F(w) = w + i \quad \text{and} \quad G(w) = w + i - \frac{1}{(w+i)^N}$$

with $N \geq 1$, then it is simple to verify that these two maps coincide up to the $(N+1)$th-order at ∞ and $H \equiv 1$ even if they are not equal.

Now we are ready to prove our main result.

Proof. [Proof of Theorem 1.2.] First we prove that the regularity conditions for F and G at τ and the Proposition 3.1 imply that H is identically constant.
Case (1): since $\beta_F = if''(\tau)\tau \neq 0$ and $\operatorname{Im}\beta_F = \operatorname{Re}(f''(\tau)\tau) > 0$ then w_n goes to ∞ non-tangentially and

$$H(w) = \lim_{n\to\infty} \frac{\Gamma_G(w_n)}{\Gamma_F(w_n)} = \lim_{w_n\to\infty} \frac{\beta_G + o_K(1)}{\beta_F + o_K(1)} = \frac{\beta_G}{\beta_F}.$$

Case (2): since $\beta_F \neq 0$ and $\operatorname{Im}\beta_F = 0$ then w_n goes to ∞ tangentially and

$$H(w) = \lim_{n\to\infty} \frac{\Gamma_G(w_n)}{\Gamma_F(w_n)} = \lim_{w_n\to\infty} \frac{\beta_G + o(1)}{\beta_F + o(1)} = \frac{\beta_G}{\beta_F}.$$

Case (3): since $\beta_F = 0$ then $\gamma_F = 2/3 f'''(\tau)\tau^2 < 0$, w_n has a subsequence that goes to ∞ non-tangentially and

$$H(w) = \lim_{j\to\infty} \frac{\Gamma_G(w_{n_j})}{\Gamma_F(w_{n_j})} = \lim_{w_{n_j}\to\infty} \frac{\gamma_G + w_{n_j} o_K(1/|w_{n_j}|)}{\gamma_F + w_{n_j} o_K(1/|w_{n_j}|)} = \frac{\gamma_G}{\gamma_F}.$$

By the conditions on the derivatives we obtain that $H \equiv 1$ and therefore $F \equiv G$ by the previous theorem. □

Note that if $F(w_0) = G(w_0)$ for some $w_0 \in \mathbb{H}$ and $F \circ G = G \circ F$ then $F^n(w_0) = G^n(w_0)$ for all $n \in \mathbb{N}$. We have already seen that the regularity conditions for F and G imply that the map H is identically constant. Again this constant is just 1 and therefore $F \equiv G$: if $F \in \mathcal{A}$ then

$$H(w_0) = \frac{\sigma(G(w_0)) - \sigma(w_0)}{\sigma(F(w_0)) - \sigma(w_0)} = 1,$$

if $F \notin \mathcal{A}$ then

$$H(w_0) = \frac{\rho(G(w_0)) - \rho(w_0)}{\rho(F(w_0)) - \rho(w_0)} = 1.$$

This remark is non-trivial provided the sequence w_n^0 is not a Blaschke sequence, that is

$$\sum_{j=0}^{\infty} \operatorname{Im}\left(\frac{-1}{F^j(w_0)}\right) < \infty.$$

This happens for the parabolic map of automorphic type

$$F(w) = w + 1 - \frac{1}{w}.$$

In fact, we have that

$$\lim_{n\to\infty} \frac{w_n}{n} = 1$$

and therefore $|w_n| \approx n$. Moreover

$$\begin{aligned} \operatorname{Im}(w_n) &= \operatorname{Im} w + \sum_{j=0}^{n-1} \operatorname{Im}\left(\frac{-1}{w_j}\right) \\ &\leq \operatorname{Im} w + \sum_{j=0}^{n-1} \frac{1}{|w_j|} \leq C \log n \end{aligned}$$

$$\begin{aligned} \operatorname{Im}(w_n) &= \operatorname{Im} w + \sum_{j=0}^{n-1} \frac{\operatorname{Im}(w_j)}{|w_j|^2} \\ &\leq \operatorname{Im} w + C \sum_{j=0}^{n-1} \frac{\log j}{j^2} < +\infty \end{aligned}$$

Hence

$$\sum_{j=0}^{\infty} \operatorname{Im}\left(\frac{-1}{w_j}\right) = \lim_{n\to\infty} \operatorname{Im}(w_n) - \operatorname{Im} w < +\infty.$$

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