Sets of Periods for Expanding Maps on Flat Manifolds

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Abstract. It is proven that the sets of periods for expanding maps on n-dimensional flat manifolds are *uniformly cofinite*, i.e. there is a positive integer m_0, which depends only on n, such that for any integer $m \geq m_0$, for any n-dimensional flat manifold $\mathcal{M}$ and for any expanding map F on $\mathcal{M}$, there exists a periodic point of F whose least period is exactly m.

Expanding maps were first introduced in a differentiable setting by M. Shub in [**12**], and then studied by D. Ruelle in [**11**] who proposed a more general definition based on a simple metric property: they are open continuous maps which locally expand distances. In general, it is rather difficult to prove the existence of at least an expanding map on a metric space, but there is a class of connected compact manifolds where the set of expanding maps is always non-empty: flat manifolds. The term *flat* derives from the fact that flat manifolds are connected Riemannian compact manifolds whose Levi-Civita connection has curvature that identically vanishes (e.g. the n-torus, the Klein bottle...).

Due to the strong topological properties of expanding maps on flat manifolds, in this note, I am able to determine the uniform cofiniteness of their sets of periods. This work has been inspired by the paper [**7**] where B. Jiang and J. Llibre studied the sets of periods for generic continuous maps of the n-torus and obtained a similar result in the expanding case.

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1. Preliminaries.

Let $\mathcal{M}$ be a compact connected topological n-dimensional manifold.

Definition 1.1 An open continuous map $F : \mathcal{M} \to \mathcal{M}$ is *expanding* if there exist a metric d compatible with the topology of $\mathcal{M}$ and constants $\epsilon_0 > 0$, $\lambda > 1$ such that for $x, x' \in \mathcal{M}$

$$d(x, x') \leq \epsilon_0 \quad \text{implies} \quad d(F(x), F(x')) \geq \lambda d(x, x'). \tag{1}$$

We will denote by $\mathcal{E}(\mathcal{M})$ the set of all maps expanding on $\mathcal{M}$.

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We briefly summarize the properties of expanding maps which will be useful later (see [**12**] for more details). Let $F \in \mathcal{E}(\mathcal{M})$ then

(i) F is a self-covering map, N-to-1 with $N \geq 2$;

(ii) $F^k \in \mathcal{E}(\mathcal{M})$ for all $k \geq 1$;

(iii) the set of fixed points $\mathrm{Fix}_{\mathcal{M}}(F) \stackrel{\text{def}}{=} \{x \in \mathcal{M} \ : \ F(x) = x\}$ is non-empty and finite, and the set of periodic points $\bigcup_{k\geq 1} \mathrm{Fix}_{\mathcal{M}}(F^k)$ is countable and dense in $\mathcal{M}$;

(iv) the homomorphism $\widetilde{F}^\sharp$ induced by F on the deck transformation group of the universal covering space of $\mathcal{M}$ is injective and characterizes the topological properties of F. This means that expanding maps which induce the same homomorphism are topologically conjugate: if $\Phi \in \mathcal{E}(\mathcal{M})$ and $\widetilde{F}^\sharp = \widetilde{\Phi}^\sharp$ then there exists a homeomorphism α_0 of $\mathcal{M}$ such that

$$F = \alpha_0^{-1} \circ \Phi \circ \alpha_0.$$

In this note we are interested in a particular class of manifolds: flat manifolds (see [**4**] as a general reference).

Definition 1.2 A cocompact, torsion free, discrete subgroup Γ of $O(n) \ltimes \mathbb{R}^n$, the group of the affine isometries of $\mathbb{R}^n$, is called a *Bieberbach group* and $\mathcal{M} = \mathbb{R}^n/\Gamma$ is the *flat manifold* associated to Γ.

The following statements will allow us to find a better representation of a given flat manifold and of its expanding maps. Let Γ be a Bieberbach group then

(v) ([**2**]) the *holonomy group* of Γ, i. e. $\Psi \stackrel{\text{def}}{=} \Gamma/(\Gamma \cap (\{\mathbb{I}\} \ltimes \mathbb{R}^n))$, has finite order $|\Psi|$;

(vi) ([**3**]) there is an element (B, b) of the affine group $\mathrm{Aff}(\mathbb{R}^n)$, which conjugates Γ to a subgroup $\Gamma' \subset \mathrm{Aff}(\mathbb{R}^n)$, called *affine Bieberbach group*, such that for any $\gamma \in \Gamma'$:

$$\gamma = (U, u) \quad \text{with} \quad U \in \mathrm{GL}(n, \mathbb{Z}) \ \text{ and } \ |\Psi| u \in \mathbb{Z}^n.$$

Note that $|\det(U)| = 1$. Moreover, $\Gamma' \cap (\{\mathbb{I}\} \ltimes \mathbb{R}^n) = \{\mathbb{I}\} \ltimes \mathbb{Z}^n$ and the holonomy group becames $\Psi' = \Gamma'/(\{\mathbb{I}\} \ltimes \mathbb{Z}^n)$;

(vii) ([**8**]) if $\varphi : \Gamma' \to \Gamma'$ is an injective homomorphism of the affine Bieberbach group Γ', there exists $(A, a) \in \mathbb{Z}^{n \times n} \ltimes \mathbb{R}^n \subset \mathrm{Aff}(\mathbb{R}^n)$ such that, for all $\gamma = (U, u) \in \Gamma'$:

$$\varphi(\gamma) = (A, a)^\sharp(\gamma) = (A, a)\gamma(A, a)^{-1} = (AUA^{-1}, Au + (\mathbb{I} - AUA^{-1})a).$$

Let $\mathcal{M} = \mathbb{R}^n/\Gamma$ be a flat manifold and let $\mathcal{M}'$ be the quotient space $\mathbb{R}^n/\Gamma'$, where $\Gamma' = (B, b)\Gamma(B, b)^{-1}$ is the affine Bieberbach group given by (vi). Then, (B, b) induces a homeomorphism from $\mathcal{M}$ onto $\mathcal{M}'$. For this reason, from now on, the flat manifold $\mathcal{M}$ will be considered as the quotient space of $\mathbb{R}^n$ by the affine Bieberbach group Γ' rather than the Bieberbach group Γ.

Let F be an expanding map of $\mathcal{M}$ then, by (iv), F induces an injective homomorphism φ on the deck transformation group Γ' of the universal covering $\mathbb{R}^n$. By (vii), there is an affine map (A, a), that is a lifting to $\mathbb{R}^n$ of a map $\Phi_{(A,a)} \in \mathcal{E}(\mathcal{M})$, which induces on Γ' a homomorphism $\Phi^\sharp_{(A,a)}$ equal to φ. Therefore, again by (iv), F and $\Phi_{(A,a)}$ are topologically conjugate and we will say that $\Phi_{(A,a)}$ is the endomorphism associated to F. Note that, by (1), the map $\Phi_{(A,a)}$ is expanding iff all the eigenvalues of the integer matrix A are outside the closed unit disc in $\mathbb{C}$.

Now we are ready to establish a result, proved by D. Epstein and M. Shub in [**5**], which really motivates the study of expanding maps just on flat manifolds.

Theorem 1.3 *If $\mathcal{M}$ is a flat manifold, then $\mathcal{E}(\mathcal{M})$ is not empty.*

Proof. The flat manifold $\mathcal{M}$ can be represented as quotient space $\mathbb{R}^n/\Gamma'$ where Γ' is an affine Bieberbach group. The affine map $((|\Psi'| + 1)\mathbb{I}, 0)$ induces on Γ' a homomorphism φ such that, for all $\gamma = (U, u) \in \Gamma'$,

$$\varphi(\gamma) = ((|\Psi'| + 1)\mathbb{I}, 0)\gamma((|\Psi'| + 1)\mathbb{I}, 0)^{-1} = (U, (|\Psi'| + 1)u) = (\mathbb{I}, |\Psi'|u)(U, u).$$

By (vi), $|\Psi'|u \in \mathbb{Z}^n$, hence $\varphi(\gamma) \in \Gamma'$.

Therefore, the affine map $((|\Psi'|+1)\mathbb{I}, 0)$ is the lifting of the map $\Phi_{(|\Psi'|+1)\sigma I,0)}$ which belongs to $\mathcal{E}(\mathcal{M})$ because $(|\Psi'| + 1) \geq 2$. □

2. Fixed points.

Let $\mathcal{M}$ be a flat manifold and let $F \in \mathcal{E}(\mathcal{M})$. We know by (iii) that the number of fixed points of F is finite. Now, we want to compute exactly the number $\mathcal{N}(F) \stackrel{\text{def}}{=} \text{card}(\text{Fix}_{\mathcal{M}}(F))$. The following remarks and the next lemma will be of value for this purpose.

Since, by (vi), $\mathbb{I}\ltimes\mathbb{Z}^n$ is a subgroup of the affine Bieberbach group Γ', then $\mathcal{M}$ is always covered by the *torus* $\mathbf{T}^n \stackrel{\text{def}}{=} \mathbb{R}^n/(\mathbb{I}\ltimes\mathbb{Z}^n)$. When this covering is not trivial, i. e. when $\Psi' \neq \{(\mathbb{I}, 0)\}$, the manifold is called an *infra-torus*. If $\Phi_{(A,a)}$ is the endomorphism associated to F then the following commutative diagram holds

$$\begin{array}{ccc} \mathbb{R}^n & \stackrel{(A,a)}{\longrightarrow} & \mathbb{R}^n \\ \pi' \downarrow & & \downarrow \pi' \\ \mathbf{T}^n & \stackrel{R_a \circ \Phi_A}{\longrightarrow} & \mathbf{T}^n \\ \pi'' \downarrow & & \downarrow \pi'' \\ \mathcal{M} & \stackrel{\Phi_{(A,a)}}{\longrightarrow} & \mathcal{M} \end{array}$$

where: $R_a : \mathbf{T}^n \to \mathbf{T}^n$ is a *toral rotation*, $R_a(x) = x + a$ and $\Phi_A : \mathbf{T}^n \to \mathbf{T}^n$ is a *toral linear endomorphism* $\Phi_A(x) = Ax$.

Lemma 2.1 *Let A be a matrix in $\mathbb{Z}^{n\times n}$ then:*

(a) if A is non-singular then Φ_A is a self-covering of $\mathbf{T}^n$ with degree equal to $|\det(A)| \geq 1$;

(b) if the spectrum of A has no roots of unity, i. e. $\det(A^k - \mathbb{I}) \neq 0$ for all $k \geq 1$ then

$$\mathrm{card}(\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A^k)) = |\det(A^k - \mathbb{I})| \qquad \forall k \geq 1.$$

Proof. (a) Since A is non-singular, Φ_A is a self-covering of $\mathbf{T}^n$. Moreover there exist two matrices $P, Q \in \mathrm{GL}(n, \mathbb{Z})$ such that $A = PDQ$ where $D \in \mathbb{Z}^{n\times n}$ is diagonal (see [**6**] p. 384). This means that $\Phi_A = \Phi_P \circ \Phi_D \circ \Phi_Q$ where Φ_P and Φ_Q are homeomorphisms of $\mathbf{T}^n$. Hence

$$\mathrm{card}(\Phi_A^{-1}(0)) = \mathrm{card}(\Phi_D^{-1}(0)) = |\det(D)| = |\det(A)| \geq 1.$$

(b) A point $x \in \mathrm{Fix}_{\mathbf{T}^n}(\Phi_A^k)$ iff there exists $y \in \mathbb{R}^n$ such that $(A^k - \mathbb{I})y \in \mathbb{Z}^n$. Therefore, since $\det(A^k - \mathbb{I}) \neq 0$ for all $k \geq 1$,

$$\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A^k) = \mathrm{Ker}(\Phi_{(A^k - \sigma I)}) = \Phi^{-1}_{(A^k - \sigma I)}(0),$$

and, by (a), $\mathrm{card}(\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A^k)) = |\det(A^k - \mathbb{I})|$. □

Here is the theorem which gives the explicit formula for $\mathcal{N}(F)$.

Theorem 2.2 *Let $\mathcal{M}$ be a flat manifold. If $F \in \mathcal{E}(\mathcal{M})$ and $\Phi_{(A,a)}$ is the endomorphism associated to F, then*

$$\mathcal{N}(F) = \frac{1}{|\Psi'|} \sum_{U \in r(\Psi')} |\det(A - U)|. \tag{2}$$

where r is the map that assigns to each $(U, u) \in \Psi'$ its rotational part U.

Proof. Since the maps F and $\Phi_{(A,a)}$ are topologically conjugate, $\mathcal{N}(F) = \mathcal{N}(\Phi_{(A,a)})$ and it is enough to compute the number of fixed points of $\Phi_{(A,a)}$. Since $x \in \mathrm{Fix}_{\mathcal{M}}(\Phi_{(A,a)})$ iff there exist $y \in \mathbb{R}^n$ and $(U, u) \in \Gamma'$ such that $(A, a)(y) = (U, u)(y)$ and $\pi''(\pi'(y)) = x$,

$$\mathrm{Fix}_{\mathcal{M}}(\Phi_{(A,a)}) = \pi''\circ\pi'\Big(\bigcup_{(U,u)\in\Gamma'} (A-U)^{-1}(u-a)\Big) = \pi''\Big(\bigcup_{(U,u)\in\Psi'} \Phi^{-1}_{(A-U)}(\pi'(u-a))\Big). \tag{3}$$

Now, we show that if (U, u) and (V, v) are two different elements of Ψ' then

$$\Phi^{-1}_{(A-U)}(\pi'(u-a)) \cap \Phi^{-1}_{(A-V)}(\pi'(v-a)) = \emptyset.$$

Otherwise there exist $y \in \mathbb{R}^n$ and $p, q \in \mathbb{Z}^n$ such that

$$\begin{cases} (A-U)y = u - a + p \\ (A-V)y = v - a + q. \end{cases}$$

These equations yield $(V,v+q)^{-1}(U,u+p)y=y$. Since the action of Γ' on $\mathbb{R}^n$ is properly discontinuous, $(V,v+q)^{-1}(U,u+p)=(\mathbb{I},0)$ and $U=V$ contradicting the hypothesis. By (3), since the degree of the covering π'' is equal to $|\Psi'|$,

$$\mathcal{N}(\Phi_{(A,a)})=\frac{1}{|\Psi'|}\sum_{U\in r(\Psi')}\operatorname{card}(\Phi^{-1}_{(A-U)}(\pi'(u-a))).$$

To complete the proof, it is enough to remark that $\operatorname{card}(\Phi^{-1}_{(A-U)}(\pi'(u-a)))=|\det(A-U)|$ by the preceding lemma. $\square$

3. Sets of periods and uniform cofiniteness.

Definition 3.1 For $m\geq 1$, the number of periodic points of least period m for F is denoted by

$$p_F(m)\stackrel{\text{def}}{=}\operatorname{card}\left(\operatorname{Fix}_{\mathcal{M}}(F^m)\setminus\bigcup_{k=1}^{m-1}\operatorname{Fix}_{\mathcal{M}}(F^k)\right).$$

The *set of periods* $\mathcal{P}(F)$ of the map F is the set of positive integers m such that $p_F(m)>0$.

By (iii), we know that $p_F(m)$ is finite for all $m\geq 1$ and $\mathcal{P}(F)$ is infinite. But some periods may be missing: for example, $\Phi_{-2\sigma I}\in\mathcal{E}(\mathbf{T}^n)$ has no points of period 2:

$$p_{\Phi_{-2\mathrm{I}}}(2)=\mathcal{N}(\Phi^2_{-2\sigma I})-\mathcal{N}(\Phi_{-2\sigma I})=|\det(4\mathbb{I}-\mathbb{I})|-|\det(-2\mathbb{I}-\mathbb{I})|=3^n-3^n=0.$$

However, B. Jiang and J. Llibre have proven in [**7**] that there is a positive integer m_0 such that for any integer $m\geq m_0$ and for any expanding map F of $\mathbf{T}^n$ there exists a periodic point of F whose least period is exactly m. In the next theorem we state that the above property is verified not only for $\mathbf{T}^n$ but for each n-dimensional flat manifold $\mathcal{M}$. The following lemma on algebraic numbers (see [**7**] and [**10**]) is needed.

Lemma 3.2 *Let α be a nonzero algebraic number with minimal polynomial $Q\in\mathbb{Z}[x]$ of degree d. If $|\alpha|\neq 1$ then*

$$||\alpha|-1|\geq\frac{1}{2^{d^2}M(\alpha)^d}$$

with $M(\alpha)\stackrel{\text{def}}{=}|a|\prod_{i=1}^d\max\{1,|\alpha_i|\}$ where a is the leading coefficient of Q and $\alpha_1,\dots,\alpha_d$ the roots of Q.

Here is the main result of this note.

Theorem 3.3 *Let n be a positive integer. Then the sets of periods for expanding maps on n-dimensional flat manifolds are* uniformly cofinite, *i.e. there is a positive integer m_0, which depends only on n, such that for any integer $m \geq m_0$, for any n-dimensional flat manifold $\mathcal{M}$ and for any expanding map F on $\mathcal{M}$, there exists a periodic point of F whose least period is exactly m.*

Proof. Let $\mathcal{M}$ be a n-dimensional flat manifold and let $F \in \mathcal{E}(\mathcal{M})$ with $\Phi_{(A,a)}$ the associated endomorphism. Suppose that $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of A and let $\varrho(A) \stackrel{\text{def}}{=} \max\{|\alpha_1|, \ldots, |\alpha_n|\}$.

First observe that if $U \in r(\Psi')$ and $k \geq 1$ then

$$(A^k U^{-1})^j = (A^k U^{-1} A^{-k})(A^{2k} U^{-1} A^{-2k}) \ldots (A^{jk} U^{-1} A^{-jk}) A^{jk} \quad \forall j \geq 1. \quad (4)$$

Since, by (v) and (vi), $Ar(\Psi')A^{-1} \subset r(\Psi')$ and $r(\Psi')$ is a finite group of order $|\Psi'|$, there is an integer $1 \leq j_0 \leq |\Psi'|$ such that $A^{j_0 k} U^{-1} A^{-j_0 k} = U^{-1}$. Let

$$V = (A^k U^{-1} A^{-k})(A^{2k} U^{-1} A^{-2k}) \ldots (A^{j_0 k} U^{-1} A^{-j_0 k}).$$

Then $V \in r(\Psi')$ and therefore $V^{|\Psi'|} = \mathbb{I}$. Hence, by (4),

$$(A^k U^{-1})^{j_0 |\Psi'|} = V^{|\Psi'|} A^{k j_0 |\Psi'|} = (A^k)^{j_0 |\Psi'|}.$$

This means that, in absolute value, the eigenvalues of $A^k U^{-1}$ and A^k are the same: $|\alpha_1|^k, \ldots, |\alpha_n|^k$.

Since, by (vi), $|\det(U)| = 1$ for all $U \in r(\Psi')$, it follows from (2) that

$$\mathcal{N}(F^k) = \frac{1}{|\Psi'|} \sum_{U \in r(\Psi')} |\det(A^k - U)| = \frac{1}{|\Psi'|} \sum_{U \in r(\Psi')} |\det(A^k U^{-1} - \mathbb{I})|.$$

Therefore, for $m, k \geq 1$

$$\frac{\mathcal{N}(F^m)}{\mathcal{N}(F^k)} = \frac{\sum_{U \in r(\Psi')} |\det(A^m U^{-1} - \mathbb{I})|}{\sum_{U \in r(\Psi')} |\det(A^k U^{-1} - \mathbb{I})|} \geq \prod_{i=1}^{n} \frac{|\alpha_i|^m - 1}{|\alpha_i|^k + 1}. \quad (5)$$

The eigenvalues $\alpha_1, \ldots, \alpha_n$ are algebraic numbers greater than 1 in absolute value: the minimal polynomial of each α_i is monic, has degree $d_i \leq n$ and therefore $2 \leq M(\alpha_i) \leq \varrho(A)^n$. Hence, by the previous lemma,

$$|\alpha_i| - 1 \geq \frac{1}{2^{d_i^2} M(\alpha_i)^{d_i}} \geq \frac{1}{2^{n^2} \varrho(A)^{n^2}}.$$

Let $1 \leq k \leq \frac{m}{2}$. Then

$$\frac{|\alpha_i|^m - 1}{|\alpha_i|^k + 1} \geq |\alpha_i|^k \frac{|\alpha_i|^{m-k} - 1}{|\alpha_i|^k + 1} \geq |\alpha_i|^k \frac{|\alpha_i| - 1}{|\alpha_i|^k + 1} \geq \frac{|\alpha_i| - 1}{2} \geq \frac{1}{2^{n^2+1} \varrho(A)^{n^2}},$$

and it follows from (5) that

$$\frac{\mathcal{N}(F^m)}{\mathcal{N}(F^k)} \geq (\frac{1}{2^{n^2+1}\varrho(A)^{n^2}})^{n-1}\frac{\varrho(A)^m - 1}{\varrho(A)^k + 1} \geq \frac{\varrho(A)^{\frac{m}{2}} - 1}{2^{n^3}\varrho(A)^{n^3}}.$$

The right member of the above inequality is an increasing function with respect to $\varrho(A)$ for $m \geq 2n^3$. Thus there is $m_0 \geq 2n^3$, which depends only on the dimension n, such that the inequality

$$\frac{\mathcal{N}(F^m)}{\mathcal{N}(F^k)} \geq \frac{\varrho(A)^{\frac{m}{2}} - 1}{2^{n^3}\varrho(A)^{n^3}} \geq \frac{2^{\frac{m}{2n}} - 1}{2^{n^3+n^2}} > \frac{m}{2} \tag{6}$$

holds for all $m \geq m_0$.

Let $x \in \mathcal{M}$ be a fixed point of F^m. Then it has a least period k with $1 \leq k \leq m$. Moreover k divides m: indeed $m = qk + r$ with $q \geq 0$ and $0 \leq r < k$, so $x = F^m(x) = F^r(F^{qk}(x)) = F^r(x)$, which implies that $r = 0$ by the minimality of k. Therefore

$$p_F(m) = \mathrm{card}(\mathrm{Fix}_{\mathcal{M}}(F^m) \setminus \bigcup_{k|m, k<m} \mathrm{Fix}_{\mathcal{M}}(F^k)),$$

and, since the conditions $k|m$ and $k < m$ imply that $1 \leq k \leq \frac{m}{2}$, we obtain by inequality (6)

$$p_F(m) \geq \mathcal{N}(F^m) - \sum_{k|m, k<m} \mathcal{N}(F^k) > \mathcal{N}(F^m)(1 - \sum_{1\leq k\leq \frac{m}{2}} \frac{2}{m}) = 0,$$

that is $m \in \mathcal{P}(F)$ for $m \geq m_0$. □

As a final remark, we give the complete list of all the missing periods for expanding maps on flat manifolds up to dimension 3 (for higher dimensions there are no results). As regards the n-torus, the situation is summarized in the following table (see [**1**] and [**7**]).

Torus	Characteristic Polynomial of A	$\mathbb{N}^* \setminus \mathcal{P}(\Phi_A)$
$\mathbf{T}^1$	$x+2$	2
$\mathbf{T}^2$	x^2+2x+2	2, 3
	x^2+2	4
$\mathbf{T}^3$	x^3+2	2, 6
	x^3-2	3
	x^3+x^2+x+2	2, 4
	x^3+x^2+2	5

On the other hand, if we consider an n-infra-torus $\mathcal{M}$ then $\mathcal{P}(F) = \mathbb{N}^*$ for all $F \in \mathcal{E}(\mathcal{M})$ and $n \leq 3$ (see [**9**] for $n = 2$ and [**13**] for $n = 3$).

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