Common fixed points of commuting holomorphic maps of the polydisc which are expanding on the torus

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Let F and G be two holomorphic maps of the unit polydisc

$$\Delta^n \stackrel{\text{def}}{=} \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \ : \ |z_i| < 1 \ \text{ for } \ i = 1, \dots, n\}$$

which are continuous on the closure $\overline{\Delta}^n$ of Δ^n. According to A. L. Shields [17] (for $n = 1$), D. J. Eustice [4] (for $n = 2$) and L. F. Heath and T. J. Suffridge [8] (for any finite $n \geq 1$), if F and G commute under composition, they have a common fixed point in $\overline{\Delta}^n$. See T. Kuczumow [11] and I. Shafrir [16] for the infinite dimensional case.

Several questions arise concerning the cardinality and the location in $\overline{\Delta}^n$ of the set of all common fixed points. Some of these questions are investigated in this article, under the additional hypothesis that F and G map into itself the Šilov boundary of $\overline{\Delta}^n$, which is the n-dimensional torus

$$\mathbf{T}^n \stackrel{\text{def}}{=} \{x = (x_1, \dots, x_n) \in \mathbb{C}^n \ : \ |x_i| = 1 \ \text{ for } \ i = 1, \dots, n\},$$

and their restrictions to $\mathbf{T}^n$ are both expanding.

Some of the results of this paper are summarized by the following theorem, in which F and G denote also the restrictions of these maps to $\mathbf{T}^n$:

Theorem. *Let F and G be two holomorphic maps of Δ^n which are continuous on $\overline{\Delta}^n$, map $\mathbf{T}^n$ in itself and are expanding on $\mathbf{T}^n$. If these maps commute on $\mathbf{T}^n$ then they commute on $\overline{\Delta}^n$ and have a unique fixed point in Δ^n.*

If moreover the numbers $n(F)$ and $n(G)$, respectively of the fixed points of F and G on $\mathbf{T}^n$, are relatively prime, then F and G have a unique common fixed point also on $\mathbf{T}^n$.

The proof of this theorem is a consequence of some results of independent interest concerning the behaviour of F and G on $\mathbf{T}^n$, which extend to any dimension a theorem established by A. S. A. Johnson and D. J. Rudolph [10] in the one-dimensional case. The main technique will be that of replacing, via conjugation by a suitable homeomorphism, the two maps F and G of $\mathbf{T}^n$ by two hyperbolic linear endomorphisms A and B of the universal covering space $\mathbb{R}^n$. The existence of a common fixed point of F and G will then be shown to be equivalent to the solvability of a diophantine matrix equation. In the second part, as a consequence of the existence of a smooth invariant measure on $\mathbf{T}^n$ for F, we show that this map have one and only one fixed point inside Δ^n hence it is the unique internal common fixed point. Some of the results established in this part extend to the n-dimensional case some facts obtained by J. H. Neuwirth [13] and myself [20].

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§1. Let $\mathbf{M}$ be a closed smooth manifold, i.e., a compact connected smooth manifold without boundary. In differentiable dynamics two of the more studied classes of maps are: the Anosov diffeomorphisms and the expanding maps.

A C^1 diffeomorphism $F : \mathbf{M} \to \mathbf{M}$ is called an *Anosov diffeomorphism* if for some (and hence any) Riemannian metric on M there are constants $c > 0$, $\lambda > 1$ such that at any point $x \in \mathbf{M}$ there is a splitting of the tangent space $T_xM = E^s \oplus E^u$ which is preserved by the differential D_xF and for all integers $k \geq 1$

$$||D_xF^k(v)|| \leq c\lambda^k||v|| \;\forall v \in E^s \;\text{ and }\; ||D_xF^{-k}(v)|| \leq c\lambda^k||v|| \;\forall v \in E^u.$$

A C^1 map $F : \mathbf{M} \to \mathbf{M}$ is *expanding* if for some (and hence any) Riemannian metric on M there are constants $c > 0$, $\lambda > 1$ such that at any point $x \in \mathbf{M}$ and for all integers $k \geq 1$

$$||D_xF^k(v)|| \geq c\lambda^k||v|| \;\forall v \in T_xM.$$

Let $\mathcal{A}(X)$ be the set of Anosov diffeomorphisms of X and call $\mathcal{E}(X)$ the set of the expanding maps on X. For the main properties of these maps we refer to the papers of Franks ([5],[6]), Manning ([12]) and Shub ([18], [19]) and the recent book of Aoki and Hiraide ([2]).

Not all smooth closed manifolds are acted upon by Anosov diffeomorphisms or expanding maps: one class of manifolds which have been investigated with reference to this problem, see [14] and [3], are the compact *flat manifolds* (Riemannian manifold with sectional curvature identically zero) and in particular the n-dimensional torus $\mathbf{T}^n$. Note that $\mathcal{A}(\mathbf{T}^n) = \emptyset$ iff $n = 1$, whereas $\mathcal{E}(\mathbf{T}^n)$ is always not empty.From now on we will assume that $\mathbf{M} = \mathbf{T}^n$.

We recall some properties of the affine maps of $\mathbf{T}^n$ that we will need in the following. Let $\pi : \mathbb{R}^n \to \mathbf{T}^n \simeq \mathbb{R}^n/\mathbb{Z}^n$ be the natural projection. Then if $\omega \in \mathbb{R}^n$, we define the *toral rotation* $R_\omega : \mathbf{T}^n \to \mathbf{T}^n$:

$$R_\omega(\pi(y)) = \pi(y + \omega) \ \ \forall y \in \mathbb{R}^n.$$

If $S \in \mathbb{Z}^{n\times n}$, we define the *toral endomorphism* $\Phi_S : \mathbf{T}^n \to \mathbf{T}^n$:

$$\Phi_S(\pi(y)) = \pi(Sy) \ \ \forall y \in \mathbb{R}^n.$$

The composition of a rotation with an endomorphism, $R_\omega \circ \Phi_S$, will be called an *affine toral map*. The main properties of these endomorphisms are:

1) Φ_S is a homomorphism of the group $(\mathbf{T}^n, \cdot)$. It is surjective iff $\det(S) \neq 0$ and in this case it is a self covering map with a constant number of sheets equals to $|\det(S)|$. Φ_S is an automorphism iff $\det(S) = \pm 1$.

2) $\mathrm{Fix}_{\mathbf{T}^n}(\Phi_S) \stackrel{\text{def}}{=} \{x \in \mathbf{T}^n : \Phi_S(x) = x\}$ is a subgroup of $(\mathbf{T}^n, \cdot)$. If S does not have 1 as an eigenvalue, then the number of fixed points $n(\Phi_S)$ is finite and equals to $|\det(S - I)|$ (see [7]).

3) $\Phi_S \in \mathcal{A}(\mathbf{T}^n)$ iff Φ_S is an automorphism and S is *hyperbolic* i.e., all the eigenvalues of S have absolute value different from 1. $\Phi_S \in \mathcal{E}(\mathbf{T}^n)$ iff S has all the eigenvalues of S of absolute value greater than 1 (see [2]).

If $F : \mathbf{T}^n \to \mathbf{T}^n$ is a continuous map, there exists a lifting $\tilde{F}$ of F such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \stackrel{\tilde{F}}{\longrightarrow} & \mathbb{R}^n \\ \pi \downarrow & & \downarrow \pi \\ \mathbf{T}^n & \stackrel{F}{\longrightarrow} & \mathbf{T}^n \end{array}$$

The lifting $\tilde{F}$ determines a unique homomorphism of $\mathbb{Z}^n$, the group of the covering transformations, represented by the integral matrix A which satisfies the equation

$$A(u) = \tilde{F}(y+u) - \tilde{F}(y) \ \ \forall u \in \mathbb{Z}^n \text{ and } \forall y \in \mathbb{R}^n. \tag{1}$$

Thus the map F and the toral endomorphism Φ_A induce the same homomorphism of $\pi_1(\mathbf{T}^n) \simeq \mathbb{Z}^n$, the fundamental group of $\mathbf{T}^n$, and since the universal covering space $\mathbb{R}^n$ is contractible, this means that they are homotopic (see [7]). Any $F \in \mathcal{A}(\mathbf{T}^n) \cup \mathcal{E}(\mathbf{T}^n)$ is topologically conjugated to the endomorphism Φ_A i.e., there exists a homeomorphism h of $\mathbf{T}^n$ such that $h \circ F \circ h^{-1} = \Phi_A$. Since $\mathcal{A}(\mathbf{T}^n)$ and $\mathcal{E}(\mathbf{T}^n)$ are invariant under this topological conjugation the structure of the spectrum of the linear map A is described in 3). Hence all topological properties of F can be recovered by those of the associated endomorphism Φ_A: by 1) N, the degree of F, is equal to $|\det(A)| \geq 1$ and by 2) the cardinality of the set $\mathrm{Fix}_{\mathbf{T}^n}(F)$ is equal to $n(F) = n(\Phi_A) = |\det(A-I)| \geq 1$. Note that $N = 1$ iff $F \in \mathcal{A}(\mathbf{T}^n)$.

Let F, G be two continuous map which commute on $\mathbf{T}^n$, that is

$$(F \circ G)(x) = (G \circ F)(x) \quad \forall x \in \mathbf{T}^n$$

and let $\tilde{F}$ and $\tilde{G}$ be two of their liftings. By the commutativity of F and G there exists $r \in \mathbb{Z}^n$ such that

$$\tilde{F}(\tilde{G}(y)) = \tilde{G}(\tilde{F}(y)) + r \quad \forall y \in \mathbb{R}^n. \tag{2}$$

As in (1), let Φ_A and Φ_B be the toral endomorphisms associated respectively to F and G. We denote by $\mathcal{S}$ the set of all continuous maps $\tilde{\alpha} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\tilde{\alpha}(y+u) = y + \tilde{\alpha}(u) \ \ \forall u \in \mathbb{Z}^n \text{ and } \forall y \in \mathbb{R}^n. \tag{3}$$

Now assume that $F, G \in \mathcal{A}(\mathbf{T}^n) \cup \mathcal{E}(\mathbf{T}^n)$. Then the linear maps A and B are invertible and we can define for $\tilde{\alpha} \in \mathcal{S}$ the maps

$$\mathcal{F}(\tilde{\alpha})(\cdot) \stackrel{\text{def}}{=} (A^{-1} \circ \tilde{\alpha} \circ \tilde{F})(\cdot) \text{ and } \mathcal{G}_\theta(\tilde{\alpha})(\cdot) \stackrel{\text{def}}{=} (B^{-1} \circ \tilde{\alpha} \circ \tilde{G})(\cdot) - B^{-1}\theta$$

where $\theta \in \mathbb{R}^n$ is a parameter that will be chosen in a suitable way. The maps $\mathcal{F}$ and $\mathcal{G}_\theta$ send $\mathcal{S}$ into itself because for all $u \in \mathbb{Z}^n$ and $y \in \mathbb{R}^n$

$$\begin{aligned} \mathcal{F}(\tilde{\alpha})(y+u) &= (A^{-1} \circ \tilde{\alpha} \circ \tilde{F})(y+u) = (A^{-1} \circ \tilde{\alpha})(\tilde{F}(y) + Au) = \\ &= A^{-1}(\tilde{\alpha}(\tilde{F}(y)) + Au) = \mathcal{F}(\tilde{\alpha})(y) + u \end{aligned}$$

that is $\mathcal{F}(\tilde{\alpha}) \in \mathcal{S}$. The same kind of arguments holds for $\mathcal{G}_\theta$. It can be proved that the map $\mathcal{F}$ has exactly one fixed point $\tilde{\alpha}$ in $\mathcal{S}$ and its projection α is a homeomorphism of $\mathbf{T}^n$ (see [2] page 244). Now, it will be shown that the parameter θ can be chosen in such a way that the maps $\mathcal{G}_\theta$ and $\mathcal{F}$ have the same fixed point. The first step is the following lemma:

Lemma 1.1 *The matrices A and B commute.*

Proof. Let $u \in \mathbb{Z}^n$. Since $B(u) \in \mathbb{Z}^n$, (1) implies that

$$A(B(u)) = \tilde{F}(y + B(u)) - \tilde{F}(y) = \tilde{F}(y + \tilde{G}(u) - \tilde{G}(0)) - \tilde{F}(y) \; \forall y \in \mathbb{R}^n.$$

Choosing $y = \tilde{G}(0)$ yields

$$A(B(u)) = \tilde{F}(\tilde{G}(u)) - \tilde{F}(\tilde{G}(0)) \tag{4}$$

and, in the same way, since $A(u) \in \mathbb{Z}^n$,

$$B(A(u)) = \tilde{G}(\tilde{F}(u)) - \tilde{G}(\tilde{F}(0)). \tag{5}$$

Hence subtracting (5) from (4) we have, by (2), that $A(B(u)) = B(A(u))$ for all $u \in \mathbb{Z}^n$.

Q.E.D.

The second step is to make the maps $\mathcal{F}$ and $\mathcal{G}_\theta$ commute:

Proposition 1.2 *If we choose $\theta = (A - I)^{-1} r$ then the maps $\mathcal{F}$ and $\mathcal{G}_\theta$ commute in $\mathcal{S}$.*

Proof. First of all, for every $\tilde{\alpha} \in \mathcal{S}$ and $y \in \mathbb{R}^n$

$$\begin{aligned}
\mathcal{F}(\mathcal{G}_\theta(\tilde{\alpha}))(y) &= (A^{-1} \circ \mathcal{G}_\theta(\tilde{\alpha}) \circ \tilde{F})(y) = \\
&= A^{-1}(B^{-1} \circ \tilde{\alpha} \circ \tilde{G})(\tilde{F}(y)) - A^{-1}B^{-1}\theta = \\
&= A^{-1}B^{-1}\tilde{\alpha}(\tilde{G}(\tilde{F}(y))) - A^{-1}B^{-1}\theta
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathcal{G}_\theta(\mathcal{F}(\tilde{\alpha}))(y) &= (B^{-1} \circ \mathcal{F}(\tilde{\alpha}) \circ \tilde{G})(y) - B^{-1}\theta = \\
&= B^{-1}(A^{-1} \circ \tilde{\alpha} \circ \tilde{F})(\tilde{G}(y)) - B^{-1}\theta = \\
&= B^{-1}A^{-1}\tilde{\alpha}(\tilde{F}(\tilde{G}(y))) - B^{-1}\theta
\end{aligned}$$

Then (2) and (3) yield

$$\tilde{\alpha}(\tilde{F}(\tilde{G}(y))) = \tilde{\alpha}(\tilde{G}(\tilde{F}(y)) + r) = \tilde{\alpha}(\tilde{G}(\tilde{F}(y))) + r \tag{6}$$

and by (6) and lemma 1.1, we find that $\mathcal{F}$ and $\mathcal{G}_\theta$ commute iff

$$\mathcal{F}(\mathcal{G}_\theta(\tilde{\alpha})(y) - \mathcal{G}_\theta(\mathcal{F}(\tilde{\alpha})(y) = -A^{-1}B^{-1}\theta - A^{-1}B^{-1}r + B^{-1}\theta \equiv 0$$

that is iff $\theta = (A - I)^{-1} r$.

Q.E.D.

From now on we choose $\theta = (A - I)^{-1} r$ and we prove the following theorem.

Theorem 1.3 *If $F, G \in \mathcal{A}(\mathbf{T}^n) \cup \mathcal{E}(\mathbf{T}^n)$ commute, there exists a homeomorphism α of $\mathbf{T}^n$ onto itself such that $\tilde{\alpha} \in \mathcal{S}$ and*

$$\begin{cases} \alpha \circ F \circ \alpha^{-1} = \Phi_A \\ \alpha \circ G \circ \alpha^{-1} = R_\theta \circ \Phi_B \end{cases} \tag{7}$$

Proof. We know that the map $\mathcal{F}$ has a unique fixed point $\tilde{\alpha} \in \mathcal{S}$ and by proposition 1.2 the maps $\mathcal{F}$ and $\mathcal{G}_\theta$ commute in $\mathcal{S}$.

Hence $\mathcal{F}(\mathcal{G}_\theta(\tilde{\alpha})) = \mathcal{G}_\theta(\mathcal{F}(\tilde{\alpha})) = \mathcal{G}_\theta(\tilde{\alpha})$ and the uniqueness of α yields $\mathcal{G}_\theta(\tilde{\alpha}) = \tilde{\alpha}$. Thus

$$\begin{cases} \tilde{\alpha} \circ \tilde{F} = A \circ \tilde{\alpha} \\ \tilde{\alpha} \circ \tilde{G} = B \circ \tilde{\alpha} + \theta \end{cases} \tag{8}$$

and, since α is a homeomorphism of $\mathbf{T}^n$, the projection of (8) onto $\mathbf{T}^n$ gives (7).

Q.E.D.

If $F, G \in \mathcal{A}(\mathbf{T}^n) \cup \mathcal{E}(\mathbf{T}^n)$ commute then, by theorem 1.3, they may be conjugated to two affine toral maps which turn out to offer a better view of the set of their common fixed points. The following theorem yields an existence criterion.

Theorem 1.4 *Two commuting maps $F, G \in \mathcal{A}(\mathbf{T}^n) \cup \mathcal{E}(\mathbf{T}^n)$ have a common fixed point on $\mathbf{T}^n$ iff the following equation in $p, q \in \mathbb{Z}^n$ is solvable:*

$$(A - I)q - (B - I)p = r. \tag{9}$$

If that is the case, for every solution (p, q), $\alpha^{-1}(\pi((A - I)^{-1}p)$ is a common fixed point.

Proof. The point $w = \pi(y) \in \mathbf{T}^n$ is a common fixed point iff there exist $p, q \in \mathbb{Z}^n$ such that

$$\begin{cases} \tilde{F}(y) = y + p \\ \tilde{G}(y) = y + q \end{cases}$$

Then, by (8),

$$\begin{cases} (A \circ \tilde{\alpha})(y) = (\tilde{\alpha} \circ \tilde{F})(y) = \tilde{\alpha}(y + p) = \tilde{\alpha}(y) + p \\ (B \circ \tilde{\alpha})(y) = (\tilde{\alpha} \circ \tilde{G})(y) - \theta = \tilde{\alpha}(y + q) - \theta = \tilde{\alpha}(y) + q - \theta \end{cases}$$

that is

$$\begin{cases} (A - I) \circ \tilde{\alpha}(y) = p \\ (B - I) \circ \tilde{\alpha}(y) = q - (A - I)^{-1}r \end{cases}$$

which yields (9).

On the other hand, the solvability of (9) with respect to $p, q \in \mathbb{Z}^n$ implies that $\alpha^{-1}(\pi((A - I)^{-1}p)) \in \mathbf{T}^n$ is fixed point of F and G.

Q.E.D.

The existence of a common fixed point semplifies the equations (7):

Corollary 1.5 *If $F, G \in \mathcal{A}(\mathbf{T}^n) \cup \mathcal{E}(\mathbf{T}^n)$ are two commuting maps with at least one common fixed point on $\mathbf{T}^n$, there exists a homeomorphism h of $\mathbf{T}^n$ such that*

$$\begin{cases} h \circ F \circ h^{-1} = \Phi_A \\ h \circ G \circ h^{-1} = \Phi_B \end{cases}$$

Proof. By theorem 1.4, the existence of a common fixed point, implies that (9) is solvable. If (p, q) is a solution we can choose $h = R_\varphi \circ \alpha$ with $\varphi = -(A - I)^{-1}p$. In fact, by (7), since $(I - A)\varphi = p \in \mathbb{Z}$,

$$h \circ F \circ h^{-1} = R_\varphi \circ \Phi_A \circ R_{-\varphi} = R_{(I-A)\varphi} \circ \Phi_A = \Phi_A$$

and, by (9), $(I - B)\varphi + \theta = q \in \mathbb{Z}$,

$$h \circ G \circ h^{-1} = R_{\varphi+\theta} \circ \Phi_B \circ R_{-\varphi} = R_{(I-B)\varphi+\theta} \circ \Phi_B = \Phi_B.$$

Q.E.D.

As a consequence of theorem 1.4 we are able to establish a sufficient condition for the existence of a common fixed point, depending only on the number of fixed points of the two commuting maps:

Theorem 1.6 *Let $F, G \in \mathcal{A}(\mathbf{T}^n) \cup \mathcal{E}(\mathbf{T}^n)$ be two commuting maps, if $n(F)$ and $n(G)$ are relatively prime then there exists one and only one common fixed point on $\mathbf{T}^n$.*

Proof. Existence: $(A - I), (B - I) \in \mathbb{Z}^{n \times n}$ have a *left greatest common divisor* $D \in \mathbb{Z}^{n \times n}$ (i.e., D is a left divisor of $(A - I)$ and $(B - I)$ and every other left divisor of $(A - I)$ and $(B - I)$ is a left divisor of D). Moreover there exist $P, Q \in \mathbb{Z}^{n \times n}$ such that

$$(A - I)Q - (B - I)P = D \tag{10}$$

(see chapter 14 in [9]). Since D is a common left divisor, $\det(D) \in \mathbb{Z}$ divides $n(F) = |\det(A - I)| \geq 1$ and $n(G) = |\det(B - I)| \geq 1$. But by hypothesis $n(F), n(G)$ are relatively prime, hence $\det(D) = \pm 1$ i.e., $D \in \mathrm{GL}(n, \mathbb{Z})$.

Multiplying both sides of (10) by $D^{-1}r \in \mathbb{Z}^n$ on the right, we can write

$$(A - I)Q(D^{-1}r) - (B - I)P(D^{-1}r) = D(D^{-1}r) = r$$

and (9) is solvable with $p = P(D^{-1}r) \in \mathbb{Z}^n$ and $q = Q(D^{-1}r) \in \mathbb{Z}^n$.

As for uniqueness, since there exists at least one common fixed point, we obtain from corollary 1.5 that

$$\mathrm{Fix}_{\mathbf{T}^n}(F) \cap \mathrm{Fix}_{\mathbf{T}^n}(G) = h^{-1}(\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A) \cap \mathrm{Fix}_{\mathbf{T}^n}(\Phi_B)).$$

The order of the subgroup $\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A) \cap \mathrm{Fix}_{\mathbf{T}^n}(\Phi_B)$ divides $n(F)$ and $n(G)$, so $\mathrm{card}(\mathrm{Fix}_{\mathbf{T}^n}(F) \cap \mathrm{Fix}_{\mathbf{T}^n}(G)) = 1$.

Q.E.D.

§2. Let $\mathrm{Hol}(\Delta^n, \Delta^n)$ be the set of all holomorphic maps of Δ^n into Δ^n and

$$\mathcal{I}(\Delta^n) \overset{\mathrm{def}}{=} \mathrm{Hol}(\Delta^n, \Delta^n) \cap \mathrm{C}(\overline{\Delta}^n, \overline{\Delta}^n) \cap \mathrm{C}(\mathbf{T}^n, \mathbf{T}^n).$$

If $F = (f_1, \ldots, f_n) \in \mathcal{I}(\Delta^n)$ then every component map $f_i : \overline{\Delta}^n \to \overline{\Delta}$ is inner in Δ^n and continuous on $\overline{\Delta}^n$; hence every f_i for $i = 1, \ldots, n$ is a rational function with the following form (see chapter 5 in [15]):

$$f_i(z) = \frac{M_i(z)\overline{Q}_i(\frac{1}{z})}{Q_i(z)} \tag{11}$$

where Q_i is a polynomial in $\mathbb{C}[z_1, \ldots, z_n]$ which has no zero in $\overline{\Delta}^n$, $\overline{Q}_i$ is the polynomial Q_i with the conjugated coefficients; $\frac{1}{z}$ stands for $(\frac{1}{z_1}, \ldots, \frac{1}{z_n})$. M_i is a monomial whose coefficient has absolute value 1 and is such that $P_i(z) \overset{\mathrm{def}}{=} M_i(z)\overline{Q}_i(\frac{1}{z})$ is a polynomial in $\mathbb{C}[z_1, \ldots, z_n]$.

If $f \in L^1(\mathbf{T}^n, \mathbb{C})$, then the value of the *Poisson integral* $\mathcal{P}[f]$ computed in a point z of the polydisc Δ^n is

$$\mathcal{P}[f](z) = \int_{\mathbf{T}^n} f(x) P_z(x) d\nu(x) \tag{12}$$

where ν is the normalized Lebesgue measure on $\mathbf{T}^n$ and the *Poisson kernel* $P_z(x) \overset{\mathrm{def}}{=} \prod_{i=1}^n \frac{1-|z_i|^2}{|x_i - z_i|^2}$ for all $x \in \mathbf{T}^n$. Now, we recall two of its main properties that we will use later (see chapter 2 in [15]).

1) If f is continuous on $\overline{\Delta}^n$ and n-harmonic in Δ^n, that is f is harmonic in each variable, then

$$\mathcal{P}[f](z) = f(z) \ \ \forall z \in \Delta^n. \tag{13}$$

2) If $f \in L^1(\mathbf{T}^n, \mathbb{C})$ then

$$\lim_{r\to 1^-} \mathcal{P}[f](rw) = f(w) \ \text{ for almost every } w \in \mathbf{T}^n. \tag{14}$$

The following lemma will be useful later

Lemma 2.1 *Let $r, s \geq 1$ such that $r + s = n$ and*

$$f(u,v) = \frac{M(u,v)\overline{Q}(\frac{1}{u},\frac{1}{v})}{Q(u,v)} = \frac{P(u,v)}{Q(u,v)} : \overline{\Delta}^r \times \overline{\Delta}^s \to \overline{\Delta}$$

be a rational function of the form (11) *(where we identify z with (u,v)). If there exists $(\tilde{u},\tilde{v}) \in \mathbf{T}^r \times \Delta^s$ such that $|f(\tilde{u},\tilde{v})| = 1$ then f does not depend on the variable v.*

Proof. Assuming $f(\tilde{u},\tilde{v}) = 1$, by the maximum principle $f(\tilde{u},\cdot) \equiv 1$. Write $Q(u,v)$ in the form $a(u)v^d + \ldots + b(u) \in \mathbb{C}[v_1,\ldots,v_s]$ where $v^d = v_1^{d_1}\ldots\ v_s^{d_s}$, with $d_j \in \mathbb{N}$, and $|d| \stackrel{\text{def}}{=} d_1 + \ldots\ + d_s$ is the degree of $Q(u,v)$ with respect to the variable v. By the hypothesis on Q, $Q(u,0) = b(u) \neq 0$ for all $u \in \mathbf{T}^r$.

Let $M(u,v) = e^{i\theta}u^{d''}v^{d'}$ then $d'_i \geq d_i$ for $i = 1,\ldots,s$. If $|d'| > |d|$ then we should have $P(u,0) = 0$ for all $u \in \mathbf{T}^r$ whereas

$$P(\tilde{u},0) = f(\tilde{u},0)Q(\tilde{u},0) = b(\tilde{u}) \neq 0. \tag{15}$$

Hence $d' = d$ and $P(u,0) = e^{i\theta}u^{d''}\overline{a}(\frac{1}{u}) = e^{i\theta}u^{d''}\overline{a(u)}$ for all $u \in \mathbf{T}^r$.

Let $v(\zeta) \stackrel{\text{def}}{=} (\zeta,\ldots,\zeta) \in \overline{\Delta}^s$ for $\zeta \in \overline{\Delta}$, then

$$q(\zeta) \stackrel{\text{def}}{=} Q(\tilde{u},v(\zeta)) = a(\tilde{u})\zeta^{|d|} + \ldots + b(\tilde{u}) \in \mathbb{C}[\zeta].$$

If $|d| > 0$ the polynomial q has $|d|$ zeros and the absolute value of the product of such zeros is, by (15)

$$\left|\frac{b(\tilde{u})}{a(\tilde{u})}\right| = \left|\frac{P(\tilde{u},0)}{a(\tilde{u})}\right| = \left|\frac{e^{i\theta}\tilde{u}^{d''}\overline{a(\tilde{u})}}{a(\tilde{u})}\right| = 1.$$

Hence at least one zero, say $\tilde{\zeta}$ belongs to $\overline{\Delta}$ that is $q(\tilde{\zeta}) = Q(\tilde{u}, v(\tilde{\zeta})) = 0$ and we have a contradiction because $(\tilde{u}, v(\tilde{\zeta})) \in \overline{\Delta}^n$. So $d = 0$, and f does not depend on the variable v.

Q.E.D.

The following proposition generalizes a result proved for $n = 1$: see [13] and [20].

Theorem 2.2 *If $F \in \mathcal{I}(\Delta^n)$ and expanding on the Šilov boundary $\mathbf{T}^n$ then F has one and only one fixed point in Δ^n.*

Proof. Assume that F does not have any fixed point in Δ^n. Then, the sequence $\{F^k\}$ of the iterates of F is divergent on the compact sets of Δ^n and there exists a subsequence $\{F^{k_j}\}$ converging to a map in $\mathrm{Hol}(\Delta^n, \partial(\overline{\Delta}^n))$ (see [1]). By the maximum principle, at least one component of this limit map, say the first, is identically equal to $c \in \mathbf{T}^1$. Hence the subsequence $\{(F^{k_j})_1\}$ converges uniformly on the compact sets of Δ^n to the constant c.

Put $p_m(\zeta) \stackrel{\text{def}}{=} \zeta^m$ with $m \in \mathbb{N}$ and $\zeta \in \overline{\Delta}$; then $p_m \circ (F^{k_j})_1$ is holomorphic in Δ^n and continuous on $\overline{\Delta}^n$. Therefore we can apply (13) and for all $z \in \Delta^n$ we have

$$\int_{\mathbf{T}^n} p_m((F^{k_j})_1(x)) P_z(x) d\nu(x) = \mathcal{P}[p_m \circ (F^{k_j})_1](z) = p_m((F^{k_j})_1(z)). \qquad (16)$$

Now, the limit of (16), for $j \to \infty$ equals $p_m(c)$. Since the complex vector space generated by the functions p_m, $\overline{p_m}$ for all $m \in \mathbb{N}$ is dense in $C(\mathbf{T}^1, \mathbb{C})$ (it is the space of the trigonometric polynomials, see [15]), for all $g \in C(\mathbf{T}^1, \mathbb{C})$

$$\lim_{j\to\infty} \int_{\mathbf{T}^n} g((F^{k_j})_1(x)) P_z(x) d\nu(x) = g(c). \qquad (17)$$

Note that the complex vector space generated by the Poisson kernels P_z for all $z \in \Delta^n$ is dense in $L^1(\mathbf{T}^n, \mathbb{C})$, because, if $f \in L^\infty(\mathbf{T}^n, \mathbb{C})$ is such that

$$\int_{\mathbf{T}^n} f(x) P_z(x) d\nu(x) = 0 \ \ \forall z \in \Delta^n$$

then, by (14), $f(x) = 0$ for almost every $x \in \mathbf{T}^n$.

Therefore, by (17) and the identity $\int_{\mathbf{T}^n} P_z(x)d\nu(x) = 1$, the following equation holds for all $h \in L^1(\mathbf{T}^n, \mathbb{C})$ and $g \in C(\mathbf{T}^1, \mathbb{C})$:

$$\lim_{j\to\infty} \int_{\mathbf{T}^n} g((F^{k_j})_1(x))h(x)d\nu(x) = g(c) \int_{\mathbf{T}^n} h(x)d\nu(x). \qquad (18)$$

Since $F_{|\mathbf{T}^n}$ is C^∞-expanding, there exists an invariant probability measures μ_F equivalent to the Lebesgue measure ν (see for example [23]). Therefore, by the Radon-Nikodym theorem, it is possible to find $h \in L^1(\mathbf{T}^n, \mathbb{R})$ such that $d\mu_F = hd\nu$; thus, by (18),

$$\lim_{j\to\infty} \int_{\mathbf{T}^n} g((F^{k_j})_1(x))d\mu_F(x) = g(c) \int_{\mathbf{T}^n} h(x)d\nu(x) = g(c)\mu_F(\mathbf{T}^n) = g(c). \quad (19)$$

By the invariance of the measure μ_F, for all $g \in C(\mathbf{T}^1, \mathbb{C})$ and all $j \in \mathbb{N}$, then

$$\int_{\mathbf{T}^n} g((F^{k_j})_1(x))d\mu_F(x) = \int_{\mathbf{T}^n} g(x_1)d\mu_F(x), \qquad (20)$$

and, by (19), this implies

$$\int_{\mathbf{T}^n} g(x_1)d\mu_F(x) = g(c) \ \ \forall g \in C(\mathbf{T}^1, \mathbb{C}). \qquad (21)$$

Let $\{g_i\}$ be a bounded sequence in $C(\mathbf{T}^1, \mathbb{C})$ that converges pointwise to I_c, the characteristic function of the set $\{c\}$, and let $E = \{c\} \times \mathbf{T}^{n-1}$. Then

$$\lim_{i\to\infty} \int_{\mathbf{T}^n} g_i(x_1)d\mu_F(x) = \lim_{i\to\infty} g_i(c) = I_c(c) = 1. \qquad (22)$$

On the other hand, since $\nu(E) = 0$ also $\mu_F(E) = 0$ (because μ_F and ν are equivalent) and (21) yields

$$\lim_{i\to\infty} \int_{\mathbf{T}^n} g_i(x_1)d\mu_F(x) = \int_{\mathbf{T}^n} I_c(x_1)d\mu_F(x) = \int_{\mathbf{T}^n} I_E(x)d\mu_F(x) = \mu_F(E) = 0$$

which contradicts (22). Hence F must have at least one fixed point in Δ^n.

An inductive argument will show that the existence of at least two fixed points in Δ^n leads to a contradiction. If $n = 1$, F is the identity map (see [22]) which is not expanding on $\mathbf{T}^1$. Now suppose that uniqueness has been established for $1 \le r < n$. Since Δ^n is homogeneous, we can assume that one of the fixed points is in 0. Let $a \in \Delta^n \setminus \{0\}$ be another fixed point. There exists a complex geodesic $\varphi \in \mathrm{Hol}(\Delta, \Delta^n)$ whose range contains 0 and a and

$$\varphi(\Delta) \subset \mathrm{Fix}_{\Delta^n}(F) \stackrel{\text{def}}{=} \{x \in \Delta^n \ : \ F(x) = x\}$$

(see [21]). At least one component of φ, say the first, is an automorphism of Δ, hence, the closure of the range of this complex geodesic intersects the boundary of the polydisc in at least one point $\tilde{z} \in \mathbf{T}^1 \times \overline{\Delta}^{n-1}$. Therefore, by permuting the last $n-1$ variables, we can assume that $\tilde{z}$ is of the form $(\tilde{u}, \tilde{v}) \in \mathbf{T}^r \times \Delta^s$ with $r \geq 1$, $s \geq 0$ and $r + s = n$.

Since F is continuous on $\overline{\Delta}^n$ we have that $F(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v})$. At each fixed point of F in Δ^n, the eigenvalues of the differential of F have absolute value less or equal to 1 (see [22]). Hence

$$|\det(D_{\varphi(\zeta)}F)| \leq 1 \ \ \forall \zeta \in \Delta \ \text{ implies that } \ |\det(D_{(\tilde{u},\tilde{v})}(F))| \leq 1.$$

On the other hand, since F is expanding, $|\det(D_{(u,v)}F)| > 1$ for each fixed point $(u, v) \in \mathbf{T}^n$; therefore the intersection $(\tilde{u}, \tilde{v})$ can not stay on the Šilov boundary $\mathbf{T}^n$. Then, since $f_i(\tilde{u}, \tilde{v}) = \tilde{u}_i$ for $i = 1, \dots, r$, by lemma 2.1 the first r component maps $f_1, \dots, f_r$ of F, do not depend on the last s variables. So we can define the map $F_1 = (f_1, \dots, f_r) \in \mathcal{I}(\Delta^r)$ that has 0 and $(\varphi_1(\frac{1}{2}), \dots, \varphi_r(\frac{1}{2})) \in \Delta^r$ as fixed points. Note that they are certainly different because φ_1 is an automorphism of Δ which fixes 0. Moreover,

$$D_{(u,v)}F^k = \begin{vmatrix} D_u F_1^k & 0 \\ * & * \end{vmatrix} \quad \forall (u, v) \in \mathbf{T}^r \times \mathbf{T}^s.$$

If $w \in T_u(\mathbf{T}^r)$, by the expanding hypothesis, the matrix $D_(u,v)F^k$ is invertible and there is a $w' \in \mathbb{R}^s$ such that

$$\|D_u F_1^k w\| = \|D_{(u,v)}F^k(w, w')\| \geq c\lambda^k \|(w, w')\| \geq c\lambda^k \|w\|.$$

Hence also F_1 is expanding on $\mathbf{T}^r$ and this contradicts the inductive assumption.

Q.E.D.

Now, in order to establish the theorem stated at the beginning of this paper, note that, by (13) the commutative property is easily extended from $\mathbf{T}^n$ to $\overline{\Delta}^n$, and that, by theorem 2.2, F has a unique fixed point $a \in \Delta^n$. Since $F(G(a)) = G(F(a)) = G(a)$ then $G(a) = a$. The second part of the theorem is proved in theorem 1.6.

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