DYNAMICAL SYSTEMS: FROM GEOMETRY TO MECHANICS
Rome, Italy

GEOMETRY AND MECHANICS

One of the famous difficult problems in Mechanics is the three body problem. And we have seen many interesting lectures at this conference about this.

Problem

What are the possible phenomena that can occur in the mechanics of many bodies.

For this one can study geometric models. Poincare investigated the problem of three bodies by iterating geometrically simple mappings on a plane. In fact he used complex polynomials on $\mathbb{C}$. One can also use Henon maps, combining reflections and foldings on the plane.

Problem

What phenomena can one find when one investigates iterations of maps $F : M \to M$?

Abstract: I will lecture about ongoing joint work with Arosio, Benini and Peters. This mixes the theories of iteration of entire functions in one complex variable and polynomial Henon maps in two complex variables.
There have been many lectures already.
Eric Bedford has already lectured about polynomial Henon maps in two complex variables.
And **Jasmin Raissy** has discussed entire maps in two variables in her lecture. She focused on Fatou components
and **Nuria Fagella** did the same for entire functions in the complex plane. Nuria focused on wandering Fatou components and the same topic was also discussed in the talk by **Pierre Berger.**
Wandering Fatou components will also be a topic in this lecture.

Dynamics of transcendental Hénon maps

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Plan of talk

- Polynomials on $\mathbb{C}$
- Transcendental functions on $\mathbb{C}$
- Henon maps on $\mathbb{C}^2$
- Transcendental Henon maps on $\mathbb{C}^2$

This is a joint work with L. Arosio, A. M. Benini and H. Peters.

What is holomorphic dynamics?

Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. Holomorphic dynamics studies the behaviour of the orbits $(z_0, f(z_0), f^2(z_0), \dots)$, where $z_0 \in X$.

Example

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial in one complex variable. Its Fatou set is the open set where the family (f^n) is equicontinuous. Its complement is called the Julia set.

Polynomial dynamics

There exists a radius $R > 0$ such that $D(0, R)^{\complement}$ is mapped into itself and every orbit starting in $D(0, R)^{\complement}$ goes to infinity. Hence the escaping set $I_\infty := \{z \colon f^n(z) \to \infty\}$ is a Fatou component.

Classification of invariant components [Fatou-Julia]

An invariant Fatou component Ω different from I_∞ is either

- the basin of attraction of an attracting fixed point $|f'(p)| < 1$ in Ω,
- the basin of attraction of a parabolic fixed point $f'(p) = 1$ in $\partial\Omega$,
- a Siegel disk, biholomorphically equivalent to an irrational rotation on the unit disk $\mathbb{D}$.

There is no wandering Fatou component, that is $\Omega \colon f^n(\Omega) \neq f^m(\Omega)$ for all $n \neq m$. [Sullivan '85]

Transcendental dynamics

If $f\colon \mathbb{C} \to \mathbb{C}$ is transcendental (entire with essential singularity at ∞), there can be

- escaping wandering domain [Baker '76]:

$$f(z) = z + \sin z + 2\pi,$$

- oscillating wandering domain [Eremenko-Lyubich '87]
- it is an open question whether there can be orbitally bounded wandering domains.

Theorem

(Benini-Fornæss-Peters (2018)) All entire transcendental functions have infinite entropy.

What about $\mathbb{C}^2$?

A polynomial Hénon map is $F(z, w) = (p(z) - \delta w, z)$, where $p \in \mathbb{C}[z]$ and $\delta \neq 0$ is a constant [Hénon '76]. It is an automorphism of $\mathbb{C}^2$ with constant jacobian δ.

Oscillating and escaping wandering domains cannot exist. Bounded wandering domains?

Theorem (Astorg-Buff-Dujardin-Peters-Raissy)

There is a polynomial map on $\mathbb{C}^2$ with a wandering domain with bounded orbits. (This map is not invertible)

Theorem (Han Peters-David Hahn (2018))

There is an invertible polynomial map on $\mathbb{C}^4$ with a wandering domain with bounded orbits.

Definition

We introduce the family of *transcendental Hénon maps* of the type $F(z, w) = (f(z) - \delta w, z)$, where f is a transcendental function and $\delta \neq 0$ is a constant.

Every such F is an automorphism with constant jacobian δ and has nontrivial dynamics:

Theorem (Arosio-Benini-Fornæss-Peters (2018), Huu Tai Terje Nguyen (2018))

Every transcendental Henon map F has a periodic point p, $F^{\circ n}(p) = p$.

We have the existence of an escaping orbit for any transcendental Henon map. This is known already for entire functions on $\mathbb{C}$.

Theorem

Let $F(z,w) = (f(z) - \delta w, z)$ where f is an entire transcendental function. Then there exists an orbit (z_n, w_n) so that $z_n \to \infty$ and $w_n/z_n \to 0$.

Theorem

The Julia set of a Henon map is always nonempty.

Proof.

If the Julia set is empty, then there is a subsequence $F^{\circ n_k}$ which converges uniformly on compact sets to a holomorphic map $G : \mathbb{C}^2 \to \mathbb{P}^2$. Since there is an escaping orbit, G must map at least one point to the line at infinity. The line at infinity is the zero set of a holomorphic function locally. By the Hurwitz theorem it follows that G maps all of $\mathbb{C}^2$ to the line at infinity. However, since F has a periodic point, this is a contradiction. □

We explain the main ingredient in the construction of an escaping orbit. It is similar to the proof in one variable. The key ingredient is Wiman Valiron theory.

Let $f(z) = \sum_n a_n z^n$ be an entire transcendental function. For any radius r, let $M(r)$ be the maximum value of $|f(z)|, |z| = r$. Note that $a_n r^n \to 0$. Hence there is a power $n = N(r)$ which maximizes $|a_n| r^n$. For a given r, pick a point $w_r, |w_r| = r$ for which $|f(w_r)| = M(r)$. Then in a small disc around w_r, f is very close to a monomial, $(z/w_r)^{N(r)} f(w_r)$. This shows that the image of this disc maps much closer to infinity and the image will cover a very thich annulus. This makes it possible to repeat and thereby construct an escaping orbit.
More precisely, the main result in Wiman Valiron Theory is the following, but I wont say anything more about it.

Theorem (Wiman-Valiron estimates)

Let f be entire transcendental, $\frac{1}{2} < \alpha < 1$. Let q be a positive integer. Let $r > 0$ and let w_r be a point of maximum modulus for r, that is, such that $|w_r| = r$ and $|f(w_r)| = M(r)$. Let z be such that

$$|z - w_r| < \frac{r}{(N(r))^{\alpha}}, \tag{1}$$

then

$$f(z) = \left(\frac{z}{w_r}\right)^{N(r)} f(w_r)(1 + \epsilon_0), \tag{2}$$

$$f^{(j)}(z) = \frac{N(r)^j}{w_r^j} f(z)(1 + \epsilon_j), \tag{3}$$

for all $1 \leq j \leq q$, where ϵ_i are functions converging uniformly to 0 in z as $r \to \infty$ provided r stays outside an exceptional set E of finite logarithmic measure.

The disk $\left\{|z - w_r| < \frac{r}{(N(r))^\alpha}\right\}$ is called a *Wiman-Valiron disk*.

We next discuss the theorem mentioned earlier.

Theorem

(Benini-Fornæss-Peters (2018)) All entire transcendental functions have infinite entropy.

This is a first step towards proving that entire Henon maps have infinite entropy. This is still open.

Example

- The map $f = e^{i\theta} \to e^{2i\theta}$ doubles distance. The iterate $f^{\circ n}(e^{i\theta}) \to e^{2^n i\theta}$ multiply distances by 2^n. The entropy normalizes this to $\frac{\log(2^n)}{n} = \log 2$.
- The map $z \to z^2$ on $\mathbb{C}$ has entropy $\log 2$. This comes from the unit circle. The inside of the circle converges to zero and gives no entropy. The same goes for the outside.
- The map $z \to z^k$ has entropy $\log k$.
- A polynomial P of degree d has entropy $\log d$. A key property is that if R is large enough, then the image $P(\Delta(0, R)) \supset \Delta(0, R)$ and moreover for each $w \in \Delta(0, R)$, there are d preimages $z_1, \ldots, z_d \in \Delta(0, R)$ (counted with multiplicy)

Topological Entropy

Definition (Topological Entropy)

Let $f : X \to X$ be a self-map of a compact metric space (X, d). A set $A \subset X$ is called (n, δ)*-separated*, for $n \in \mathbb{N}$ and $\delta > 0$, if for any $z \neq w \in A$ there exists $k \leq n - 1$ such that $d(f^k(z), f^k(w)) > \delta$. Let $K(n, \delta)$ be the maximal cardinality of an (n, δ)-separated set. Then the *topological entropy* is defined as

$$top(f) = \sup_{\delta>0}\{\limsup_{n\to\infty} \frac{1}{n} \log K(n, \delta)\}.$$

Example

Let $f = \sum_k \epsilon_k z^{n_k}$ for a rapidly increasing sequence n_k and rapidly decreasing sequence ϵ_k. Then f has infinite entropy on $\mathbb{C}$. There will be a sequence R_k so that $f(\Delta(0, R_k)) \supset \Delta(0, R_k)$ and moreover for each $w \in \Delta(0, R_k)$, there are n_k preimages $z_1, \dots, z_d \in \Delta(0, R_k)$ (counted with multiplicy)

Topological Entropy

in the case when the space X is not compact, it is not clear how to define entropy. One possibility is to restrict to compact subsets.

Definition (Topological Entropy in the noncompact case)

Let $f : X \to X$ be a self-map of a metric space (X, d). Let $Y \subset X$ be a compact subset. A set $A \subset Y$ is called (n, δ)*-separated*, for $n \in \mathbb{N}$ and $\delta > 0$, if for any $z \neq w \in A$ for which $f^k(z), f^k(w) \in Y$ for all $k \leq n - 1$, there exists $k \leq n - 1$ such that $d(f^k(z), f^k(w)) > \delta$. Let $K(n, \delta, Y)$ be the maximal cardinality of an (n, δ)-separated set. Then the *topological entropy* is defined as

$$top(Y, f) = \sup_{\delta>0}\{\limsup_{n\to\infty} \frac{1}{n} \log K(n, \delta, Y)\}.$$

$$top(f) = \sup_{Y \subset X} top(Y, f).$$

We show that a similar result as for polynomials (see an above example, point 4) also holds for all entire functions:

Theorem

Let f be a transcendental entire function, and let $n \in \mathbb{N}$. There exists a non-empty bounded open set $V \subset \mathbb{C}$ so that $V \subset f(V)$ and such that any point in V has at least n preimages for f in V counted with multiplicity.

The Kobayashi metric

The Kobayashi metric appeared in Jasmins talk where it was an essential tool to describe a Fatou component equivalent to $\mathbb{C} \times \mathbb{C}^*$ It was also an essential tool. in Nurias talk about internal dynamics of wandering Fatou components for entire transcendental maps on $\mathbb{C}$. A key property of the Kobayashi metric is that it is distance decreasing under holomorphic maps.

Lemma

The Kobayashi metric on $\mathbb{C} \setminus \{0, 1\}$ is larger than $\frac{1}{2|z| \log |z|}$ for all large enough $|z|$.

This implies that if $f : \Delta(0, 1) \to \mathbb{C} \setminus \{0, 1\}$, then if $|f(0)|$ is very large, then $|f(z)|$ is very large for all $|z| < 1/2$. The reason is that the Kobayashi metric is distance decreasing.
More generally, if $C \subset\subset D \subset \mathbb{C}$ and $f : D \to \mathbb{C} \setminus \{0, 1\}$ and $|f(p)|$ is very large for some $p \in C$, then $|f(p)|$ is very large for all $p \in C$.

Note on entire transcendental functions f: The max value $M(R)$ for f on the circle of radius R goes to infinity faster then any power R^j of R. Another important fact: The Picard theorem says that all values in $\mathbb{C}$ except at most 1 are taken infinitely many times. This has an important consequence:

Lemma

There exist for any j arbitrarily large R so that $M(R) > R^j$ and the minimum $m(R)$ on the circle is less than 1.

Corollary

Let f be entire, transcendental. Then there exist arbitrarily large R so that the image of the annulus $A_R = \{R/2 < |z| < 2R\}$ cannot avoid both 0 and 1.

In fact, we can prove a stronger result: The point 1 can be replaced by any value $\alpha \in A_R$.

Corollary

Let f be entire, transcendental. Then there exist arbitrarily large R so that if $f \neq 0$ on A_R, then $f(A_R) \supset A_R$.

This suffices to prove that nonvanishing entire transcendental functions have infinite entropy.

Note that if we replace A_R by two halves, D_R, midpoints $\theta = \theta_R$, then f will have roots because D_R is simply connected.

Corollary

Let f be entire, transcendental. Let n be an integer. Then there exist arbitrarily large R so that if $f \neq 0$ on A_R, then $f(A_R) \supset A_R$ and covers A_R at least n times.

We can finally do the same argument, replacing 0 by any point in A_R.

Theorem

Let f be a transcendental function. Let $n \in \mathbb{N}$. Then there exist arbitrarily large R and j large and $\theta \in [0, 2\pi]$ so that either $A_R \subset f(D_R)$ or else there exists $\alpha \in A_R \setminus f(D_R)$ so that $\left(A_R \setminus \Delta(\alpha, \frac{1}{R^{j/2}})\right) \subset f(D_R)$. In the latter case, each $\beta \in \left(A_R \setminus \Delta(\alpha, \frac{1}{R^{j/2}})\right)$ has at least n distinct and uniformly separated preimages in D_R.

Using this, we prove:

Theorem

(Benini, Fornæss, Peters, 2018) All entire transcendental $f : \mathbb{C} \to \mathbb{C}$ *(not a polynomial) have infinite topological entropy.*

Theorem (Arosio-Benini-Fornæss-Peters, 2017)

There are examples of transcendental Hénon maps with

- *an escaping wandering domain biholomorphic to* $\mathbb{C}^2$,
- *an oscillating wandering domain biholomorphic to* $\mathbb{C}^2$.

The oscillating wandering domain

Let $0 < a < 1$. We construct a sequence of maps

$$F_k(z,w) = (f_k(z) + aw, az) \to F$$

with oscillating orbit (P_n) and

$$\operatorname{diam} F^n(B(P_0,1)) \to 0. \tag{4}$$

We ensure that every F_k has a saddle fixed point at the origin.
Assume that we defined F_k with an orbit $P_1, \ldots, P_{n_k}$.
First step: use the Lambda Lemma to construct a new oscillation $Q_0, \ldots, Q_N$ coming in along the stable manifold of F_k and going out along the unstable manifold of F_k.

Second step: use Runge approximation to obtain F_{k+1} connecting the old orbit $P_0, \ldots, P_{n_k}$ with the new oscillation (Q_j) via a contracting detour $T_0, \ldots, T_M$, long enough to neutralize (possible) expansion on (Q_j). We modify only the 1-dimensional function f_k.
Finally we send Q_N far away and obtain the point $P_{n_{k+1}}$.

Why are the P_j's in different Fatou components?

Let Ω_j be the Fatou component containing P_j. Assume by contradiction that $\Omega_0 = \Omega_m$.
All limit functions on Ω_0 are constant.
Let K be a compact neighborhood of 0 which does not contain any nonzero point of period m of F. Then there exists $P_{n_j} \to P \neq 0, P \in K$.
By normality, $F^{n_j} \to P$ on Ω_0, but

$$F^{n_j}(P_m) = F^m(F^{n_j}(P_0)) \to F^m(P) \neq P.$$

Now I go into work in progress, so things have not been written up.

Theorem

(Arosio, Benini, F, Peters) Let F be a transcendental Henon map. Then there can be no fixed point which is an isolated point in the Julia set.

We assume that 0 is an isolated fixed point in the Julia set.
(1) First we prove that 0 must be repelling. (2) Secondly we show that this is impossible.
(1) Choose two real numbers $0 < \delta << \epsilon < 1$. Let $A = \{\delta < \|z\| < \epsilon\}$. Let U be the connected component of the Fatou set which is punctured at the origin. If ϵ is small enough, A will divide U into three connected components, A, B, C where $B = \{0 < \|z\| \leq \delta\}$ and $C = U \setminus (A \cup B)$. If there exists R so that $F^n(A) \subset \mathbb{B}(0, R)$ for all n, then by the maximum principle $F^n(B) \subset \mathbb{B}(0, R)$ for all n and then 0 is in the Fatou set, a contradiction. Hence there must exist a sequence n_k so that F^{n_k} converges uniformly on A to the line at infinity. In particular there is an n so that $f^n(A) \cap \{\|z\| < \epsilon\} = \emptyset$. We also have that $U = F^n(A) \cup F^n(B) \cup F^n(C)$ which again divides U into three disjoint connected sets. Clearly $F^n(B)$ contains a punctured neighborhood of the origin. It follows that $\{0 < \|z\| < \epsilon\} \subset F^n(B)$. This implies that $F^{-n}(\{\|z\| < \epsilon\} \subset \{\|z\| < \delta\}$. Hence both eigenvalues of $(F^{-n})'(0)$ are strictly less than one. Hence the same is true for $(F^{-1})'(0)$ so indeed 0 is a repelling fixed point for F.

(2) Suppose that 0 is an isolated repelling fixed point in the Julia set and let U be the Fatou component with a puncture at 0. Since the Jacobian is larger than one, all limits of F^n must be in the line at infinity. Let V be the subset of $\mathbb{C}^2$ consisting of those points for which $F^{-n}(z) \to 0$. This is a Fatou Bieberbach domain. Since F^{-1} has an escaping point, V is not the whole space. So V has a boundary point p. Let $A = \{\delta < \|z\| < \epsilon\}$ for $0 < \delta << \epsilon << 1$. Then the sequence $F^n(A)$ converges uniformly to infinity, and hence cannot cluster at p. But there are points q arbitrarily close to p so that $F^{-n}(q) \to 0$. Hence for some n, $F^{-n}(q) \in A$. Contradiction.

A stronger goal is the following:

Theorem

(to be verified) There is no isolated point in the Julia set

and finally:

Theorem

(to be verified) The Fatou set is pseudoconvex.

Thank you for listening!