# A general mechanism of diffusion in Hamiltonian Systems

DYNAMICAL SYSTEMS: FROM GEOMETRY TO MECHANICS

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U. Roma Tor Vergata, February 5–8, 2019

# Goals of the talk

• The problem of Arnold diffusion consists in studying in which Hamiltonian systems the effects of perturbations can accumulate over time to produce effects much larger than the size of the perturbations. Specially in integrable systems.

# Goals of the talk

• We will describe a recent mechanism based on the presence of Normally Hyperbolic Invariant Manifolds with stable and unstable manifolds which intersect.

• The mechanism is rather robust.

• It does not need that the perturbations are Hamiltonian (applies to small dissipation problems or for space craft maneuvers that involve burns).

• Can be applied to concrete problems

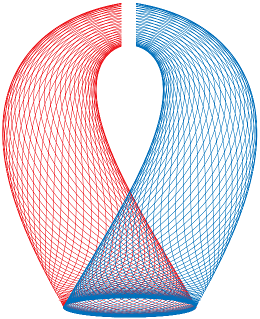
• Enjoys remarkable genericity properties since it does not require non-generic assumptions (for instance convexity).

# Outline

1. Background
2. Shadowing lemmas for NHIM's
3. Perturbative results
4. A general diffusion result
5. Application: Diffusion in a priori unstable systems

# Normal hyperbolicity

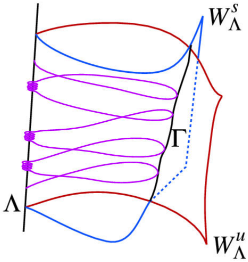
- **Normally hyperbolic invariant manifold (NHIM)**:
  - $f : M \to M$, $C^r$-smooth, $r \geq r_0$, $m = \dim M$.
  - $f(\Lambda) \subset \Lambda$, $n_c = \dim \Lambda$.
  - $TM = T\Lambda \oplus E^u \oplus E^s$
  - $n_s = \dim E^s$, $n_u = \dim E^u$.
  - $m = n_c + n_s + n_u$
  - $\exists\ C > 0,\ 0 < \lambda < \mu^{-1} < 1$, s.t. $\forall\ x \in \Lambda$
    $v \in E^s_x \Leftrightarrow \|Df^k_x(v)\| \leq C\lambda^k \|v\|, \forall k \geq 0$
    $v \in E^u_x \Leftrightarrow \|Df^k_x(v)\| \leq C\lambda^{-k} \|v\|, \forall k \leq 0$
    $v \in T_x\Lambda \Leftrightarrow \|Df^k_x(v)\| \leq C\mu^{|k|} \|v\|, \forall k \in \mathbb{Z}$



In this case $W^{u,s}(\Lambda) = \bigcup_{x \in \Lambda} W^{u,s}(x)$

# Scattering map: homoclinic channel

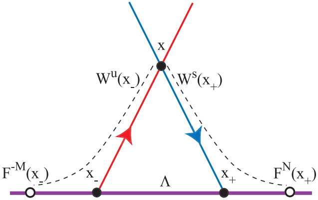
- Assume that $f$ has a Normally Hyperbolic Invariant Manifold (NHIM) $\Lambda$
- Assume $W^u(\Lambda)$ intersects transversally $W^s(\Lambda)$ along a homoclinic manifold $\Gamma$ satisfying certain extra transversality conditions ($\Gamma$ is transverse to the foliation).
- We call $\Gamma$ an homoclinic channel.



# Scattering map

## Definition

- Wave maps: $\Omega_\pm : \Gamma \to \Lambda$,
  $\Omega_\pm(x) = x_\pm \Leftrightarrow x \in W^{s,u}(x_\pm) \cap \Gamma$
- Restrict $\Gamma$ so that $\Omega_\pm$ diffeomorphisms
- Scattering map: $s : \Omega_-(\Gamma) \to \Omega_+(\Gamma)$
  given by $s = \Omega_+ \circ (\Omega_-)^{-1}$




## Properties

- $s$ is symplectic, if $M, \Lambda, f$ are symplectic
  [Delshams,de la Llave,Seara,2008]

$s(x_-) = x_+ \leftrightsquigarrow d(f^{-m}(x), f^{-m}(x_-)) \to 0$, $d(f^n(x), f^n(x_+)) \to 0$, as $m, n \to \infty$

# A general Shadowing Lemma for NHIM's

**Theorem 1** [Gidea, de la Llave, S.]

Given $f : M \to M$, is a $C^r$-map, $r \geq r_0$, $\Lambda \subseteq M$ NHIM, $\Gamma \subseteq M$ homoclinic channel. $s = s^\Gamma : \Omega_-(\Gamma) \to \Omega_+(\Gamma)$ is the scattering map associated to $\Gamma$.
Assume that $\Lambda$ and $\Gamma$ are compact.
Then, for every $\delta > 0$ there exists $m^* \in \mathbb{N}$ and a family of functions $n_i^* : \mathbb{N}^{2i+1} \to \mathbb{N}$, $i \geq 0$, such that, for every pseudo-orbit $\{y_i\}_{i\geq 0}$ in $\Lambda$ of the form

$$y_{i+1} = f^{m_i} \circ s \circ f^{n_i}(y_i),$$

for all $i \geq 0$, with $m_i \geq m^*$ and $n_i \geq n_i^*(n_0, \ldots, n_{i-1}, n_i, m_0, \ldots, m_{i-1})$, there exists an orbit $\{z_i\}_{i\geq 0}$ of $f$ in $M$ such that, for all $i \geq 0$,

$$z_{i+1} = f^{m_i+n_i}(z_i), \quad \text{and} \quad d(z_i, y_i) < \delta.$$

$n^*$ and $m_i^*$ also depend on the angle between $(W^u, W^s)$ along $\Gamma$

Related result: Gelfreich, Turaev Arnold Diffusion in a priori chaotic symplectic maps, Commun. Math. Phys., 2017, talk of A. Clarke

# A general Shadowing Lemma for NHIM's: Proof

The result is true if we use several scattering maps to build the pseudo-orbit: $y_{i+1} = f^{m_i} \circ s_{\alpha_i} \circ f^{n_i}(y_i)$
We have two proofs, one uses the topological method of correctly aligned windows.
The one we present here uses the obstruction argument.
We build a nested sequence of closed balls $B_{i+1} \subset B_i \subset B_\delta(y_0)$ ($y_0$ is the first point of the pseudo-orbit), such that:
if $z_0 \in B_k = \bigcap_{0 \le i \le k} B_i$,

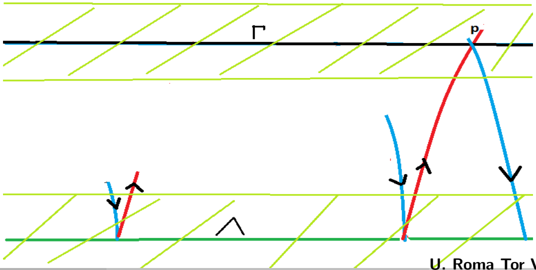
- $z_0 \in B_\delta(y_0)$
- $z_{i+1} = f^{m_i+n_i}(z_i) \in B_\delta(y_{i+1})$ for $i = 0, 1 \ldots, k$, for any $k \in \mathbb{N}$.

Moreover, taking $z_0 \in B_\infty = \bigcap_{i \ge 0} B_i \neq \emptyset$, one has that:
$z_{i+1} \in B_\delta(y_{i+1})$ for any $i \in \mathbb{N}$.

- The argument will be done by induction.
- At every step of the process we will have several choices which give us different orbits

# Choice of $m^*$

- We will take $\delta > 0$ and consider $V_\Lambda$ and $V_\Gamma$ contained in neighborhoods of size $\delta$ of the compact manifolds $\Lambda$ and $\Gamma$.
- We define $m^* = m^*(\delta)$ such that: given any point $p \in \Gamma$, for any $m \geq m^*$, one has that $f^{\pm m}(p) \in V_\Lambda$.
- Moreover, this property also holds for points in $W^{u,s}(\Lambda) \cap V_\Gamma$ when iterating them backwards or forward respectively.
- We will give an extra condition to $m^*$.



# General step

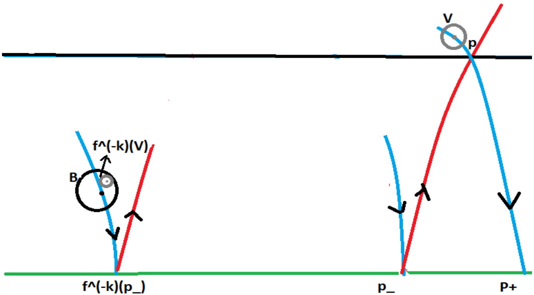
- Assume we have $p \in \Gamma$ and let $p^-, p^+ \in \Lambda$, such that $s(p^-) = p^+$ • $x \in W^s(f^{-k}(p^-))$, $B = B_\rho(x)$, $\rho > 0$ small enough

$$B \subset B_\delta(f^{-k}(p^-)) \subset V_\Lambda,$$

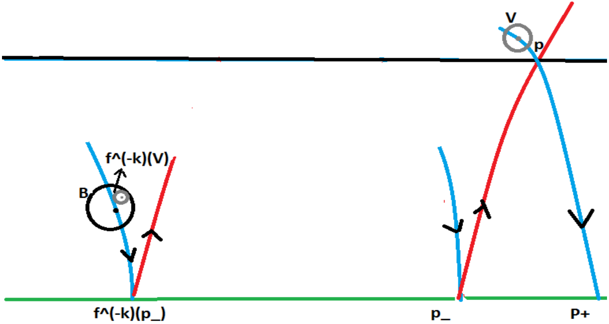
- $W^s(p^+)$ intersects transversally $W^u(\Lambda)$ at the homoclinic point $p$
- Lambda Lemma: there exists $k^* > 0$ such that:

if $k > k^*$, there exists a point $\bar{x} \in W^s(p^+) \cap V_\Gamma$ such that $f^{-k}(\bar{x}) \in B$.

- By continuity, $\exists\ V \subset V_\Gamma$ centered at $\bar{x}$ such that $f^{-k}(\bar{x}) \in f^{-k}(V) \subset B$.



- The value of $k^*$ depends on $\rho$ (and $\delta$) and also on the angle of intersection of the stable and unstable manifolds of $\Lambda$ along $\Gamma$.
- The point $\bar{x}$ and its neighborhood $V$ depend on the $k > k^*$ we choose.



# Inductive construction

• We construct the shadowing orbit $\{z_i\}$ once the pseudo-orbit $\{y_i\}$ is given.

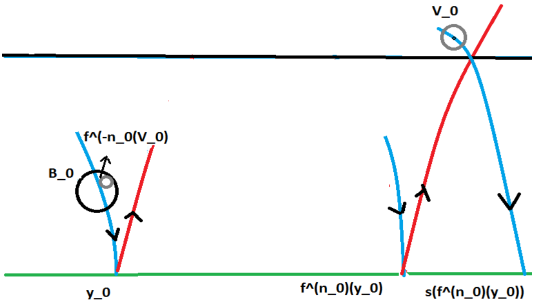
• Remember $y_{i+1} = f^{m_i}(s(f^{n_i}(y_i)))$, then $z_{i+1} = f^{m_i}(f^{n_i}(z_i))$.

• The required values of $n_i^*$, and $m^*$ do not depend of the given pseudo-orbit, but only on the numbers $n_i, m_j$.

# Inductive construction. First step

Fisrt step: $p^- = f^{n_0}(y_0)$, $p^+ = s(f^{n_0}(y_0))$

- Choose $x_0 \in W^s(y_0)$ and $B_0 = B_{\rho_0}(x_0)$ of radius $\rho_0 > 0$:
$$B_0 \subset B_\delta(y_0) \subset V_\Lambda, \quad x_0 \in B_0 \cap W^s(y_0) \neq \emptyset.$$
- There exists $m^* = k^*(\rho_0, \delta)$ such that, taking $k = n_0 > n_0^* = m^*$, $\exists \bar{x}_0 \in W^s(s(f^{n_0}(y_0))) \cap V_\Gamma$ and a a ball $V_0 \subset V_\Gamma$: such that $f^{-n_0}(\bar{x}_0) \in f^{-n_0}(V_0) \subset B_0 \subset B_\delta(y_0) \subset V_\Lambda$.



# Inductive construction. Intermediate step

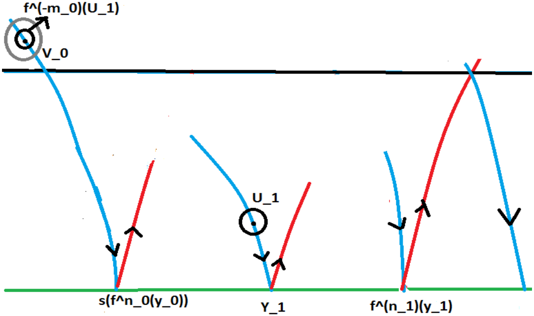
The value of $\rho_0$ and therefore the value of $m^*$ will be fixed from now on.
Remember:

- $y_1 = f^{m_0}(s(f^{n_0}(y_0))$.
- $\bar{x}_0 \in W^s(s(f^{n_0}(y_0)))$

Therefore $f^{m_0}(\bar{x}_0) \in W^s(f^{m_0}(s(f^{n_0}(y_0))) = W^s(y_1)$.

# Inductive construction. Intermediate step

1. We know that, if $m_0 > m^*$,
$$f^{m_0}(\bar{x}_0) \in W^s(f^{m_0}(s(f^{n_0}(y_0))) = W^s(y_1) \in V_\Lambda.$$
2. By continuity there exists a ball $U_1$ centered at $f^{m_0}(\bar{x}_0)$ such that:
$$U_1 \subset B_\delta(y_1) \subset V_\Lambda,\ f^{m_0}(\bar{x}_0) \in U_1\ f^{-m_0}(U_1) \subset V_0 \subset V_\Gamma.$$

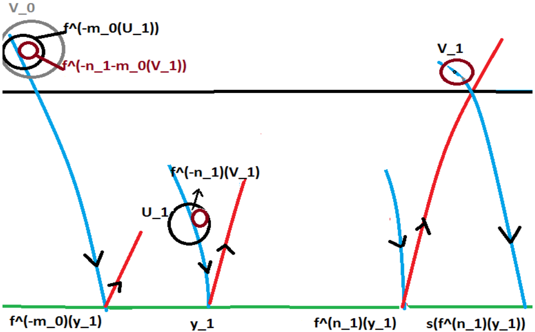


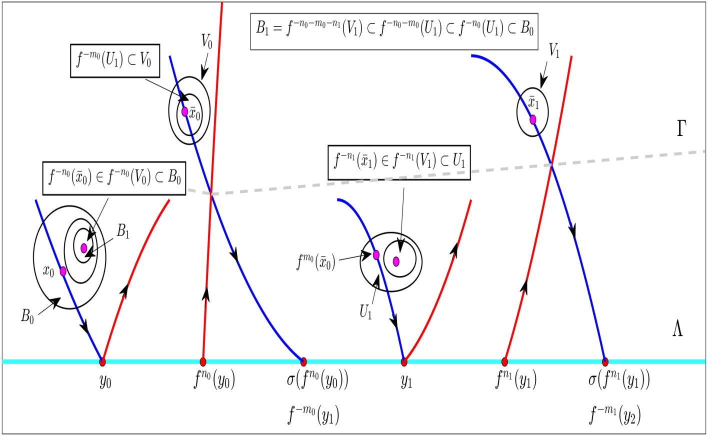
# Inductive construction. Second step

1. We use the general step with $p^- = f^{n_1}(y_1)$, $p^+ = s(f^{n_1}(y_1))$ and $k = n_1$.
2. We have $x_1 = f^{m_0}(\bar{x}_0) \in U_1 \subset B_\delta(y_1) \subset V_\Lambda$, $\quad x_1 \in U_1 \cap W^s(y_1) \neq \emptyset$ .

Taking $n_1 > n_1^* = k^*$ which depends on the size of $U_1$ (and $\delta$)
There is point $\bar{x}_1 \in W^s(s(f^{n_1}(y_1)))$ and a ball $V_1$ centered at $\bar{x}_1$ such that:

$$f^{-n_1}(\bar{x}_1) \in f^{-n_1}(V_1) \quad \subset U_1 \subset \quad B_\delta(y_1),$$






$$B_1 = f^{-(n_0+m_0+n_1)}(V_1).$$

# Conclusions of the first two steps of the induction process

1. If we now take $B_1 = f^{-(n_0+m_0+n_1)}(V_1)$, we have:

$$\begin{aligned} B_1 = f^{-(n_0+m_0+n_1)}(V_1) &= f^{-(n_0+m_0)} \circ f^{-(n_1)}(V_1) \\ \subset f^{-(n_0)} \circ f^{-m_0}(U_1) &\subset f^{-(n_0)}(V_0) \subset B_0. \end{aligned} \tag{1}$$

Moreover, if we take $z_0 \in B_1$ it satisfies:

$$\begin{aligned} z_0 &\in B_0 \subset B_\delta(y_0), \\ f^{n_0+m_0}(z_0) &\in f^{-n_1}(V_1) \subset U_1 \subset B_\delta(y_1). \end{aligned}$$

And we proceed by induction

# Remarks on the shadowing

- The results just needs the existence of a NHIM with stable and unstable manifolds which intersect transversally
- The system does not need to be Hamiltonian
- The more homoclinic channels, the better.
- No assumptions on the dynamics on the manifold $\Lambda$.

# Shadowing Lemma for pseudo-orbits of the scattering map

If we deal with a concrete system we can build the pseudo-orbit, if not we can use the following:

**Theorem 2 [Gidea, de la Llave, S.]**
$f : M \to M$ smooth map, $\Lambda \subseteq M$ NHIM, $\Gamma \subseteq M$ homoclinic channel, $s$ scattering map.

- $f$ preserves a measure $\mu$ absolutely continuous with respect to the Lebesgue measure on $\Lambda$,
- $s$ sends positive measure sets to positive measure sets.
- $\{x_i\}_{i=0,\dots,N}$ be a finite pseudo-orbit of the scattering map: $x_{i+1} = s(x_i)$, $i = 0, \dots, N-1$, $N \geq 1$,
- $\{x_i\}_{i=0,\dots,n} \subset \mathcal{U} \subseteq \Lambda$, $\mathcal{U}$ open set, almost every point of $\mathcal{U}$ recurrent for $f_{|\Lambda}$.

Then, for every $\delta > 0$ there exists an orbit $\{z_i\}_{i=0,\dots,N}$ of $f$ in $M$, with $z_{i+1} = f^{k_i}(z_i)$ for some $k_i > 0$, such that $d(z_i, x_i) < \delta$ for all $i = 0, \dots, N$.

# Shadowing Lemma for pseudo-orbits of the scattering map

The result is true if we use several scattering maps to build the pseudo-orbit: $x_{i+1} = s_{\alpha_i}(x_i)$

**Proof:**

- Choose a small ball $B_0 \subseteq \mathcal{U} \subset \Lambda$ centered at $x_0$ such that $B_i := s^i(B_0) \subseteq \mathcal{U}$, and $\mathrm{diam}(B_i) \leq \delta/2$, for all $i = 0, \ldots, N$.
- As $x_{i+1} = s(x_i)$, one has $x_i \in B_i$ for all $i$.
- We will use the recurrence hypothesis to produce a new pseudo-orbit $\{y_i\}$, with $y_{i+1} = f^{m_i} \circ s \circ f^{n_i}(y_i)$, where $m_i, n_i$ are as in **Theorem 1**, such that $y_i \in B_i$ for all $i$, and hence $d(y_i, x_i) \leq \delta/2$.
- The shadowing theorem (**Theorem 1**) will provide us with a true orbit $\{z_i\}$ with $z_{i+1} = f^{m_i+n_i}(z_i)$, such that $d(z_i, y_i) \leq \delta/2$, hence $d(z_i, x_i) < \delta$.

# Inductive construction of pseudo-orbits.

Starting with $B_0$, we construct inductively a nested sequence of subsets $\Sigma_i \subset B_0$ of positive measure of $B_0$, such that each set is carried onto a positive measure subset of $B_i$, $i = 1, \ldots, N$, via successive applications of some large powers of $f$ interspersed with applications of $s$.
Consider the value $n_0^*$ provided by the previous theorem for $\delta/2$.

- Let $A_0 := B_0$, let $n_0 > n_0^*$ and $U_0 \subset A_0$ of positive measure, such that

$$\Sigma_0 := U_0 \subseteq A_0 \subset B_0$$

  Has positive measure and its points return to $B_0$ after $n_0$ iterates.
  Consider the set $V_0 = f^{n_0}(U_0) \subseteq B_0$, which has positive measure.

- Then consider the set $A_1' := s(V_0) \subseteq B_1$, which has positive measure in $B_1$.

- Consider the value $m^*$ given by previous Theorem for $\delta/2$.
  There exists a set of positive measure $U_1' \subset A_1'$ such that its points return to $A_1' \subseteq B_1$ after $m_0 > m^*$ iterates.

- Then the set $A_1 = f^{m_0}(U_1') \subseteq B_1$ also has positive measure in $B_1$.

# Inductive construction of pseudo-orbits.

- Each point $y_1 \in A_1 = f^{m_0}(U_1')$ is of the form $y_1 = f^{m_0}(x')$, for some $x' \in U_1'$
- Such $x'$ is of the form $x' = s(x)$ for some $x \in V_0$; and each such $x$ is of the form $x = f^{n_0}(y_0)$ for some $y_0 \in U_0 = \Sigma_0$.
- Each $y_1 \in A_1$ can be written as

$$y_1 = f^{m_0} \circ s \circ f^{n_0}(y_0)$$

  for some $y_0 \in \Sigma_0$, $n_0 \geq n_0^*$ and $m_0 \geq m^*$.
- Denote by $\Sigma_1$ the set of points $y_0 \in \Sigma_0$ which correspond, to some point $y_1 \in A_1$.
- We obviously have $\Sigma_1 \subseteq \Sigma_0$ and is a positive measure subset of $B_0$.

• Proceeding by induction we will find subsets $A_j \subseteq B_j$, which have positive measure in $B_j$, such that each point $y_j \in A_j$ is of the form

$$y_j = f^{m_{j-1}} \circ s \circ f^{n_{j-1}} \circ \ldots \circ f^{m_0} \circ s \circ f^{n_0}(y_0), \qquad (2)$$

some $y_0 \in A_0 \subset B_0$,

• $\Sigma_j$ is the set of points $y_0$ for which the corresponding $y_j$ given by (2) is in $A_j$.

• Then we have that $\Sigma_j \subseteq \Sigma_{j-1} \subseteq \ldots \subseteq \Sigma_0$, and that $\Sigma_j$ is a positive measure subset of $B_0$.

• Starting with any $y_0 \in \Sigma_N$, and taking $y_{i+1} = f^{m_i}(s(f^{n_i}(y_i)))$, $i = 1, \ldots, N$, **Theorem 1** gives a true orbit $\{z_i\}$ with $z_{i+1} = f^{m_i+n_i}(z_i)$, such that $d(z_i, y_i) \leq \delta/2$, hence $d(z_i, x_i) < \delta$.

- **Theorem 2** tell us that, if the system has recurrence, we can follow any heteroclinic connexion between points in $\Lambda$
- It is not necessary to know the dynamics of the base points
- No need of invariant tori, periodic orbits. Aubry-Mather sets etc
- The only thing to verify is that the system has a NHIM with stable and unstable manifolds which intersect transversaly.
- Now we will give conditions (easy to verify and generic) to ensure that, in the perturbative setting, a System satifies the Hypotheses of **Theorem 2**.
- The conditions are verifiable for concrete systems and are satisfyied by generic perturbations for "lots" of systems.
- In particular, in the Hamiltonian case, the Hamiltonian does not need to be convex.

# A Perturbative result

**Theorem 3 [Gidea, de la Llave, S.]**
Given $H_\varepsilon$, and $f_\varepsilon$ the time 1 map. Assume for all $0 < \varepsilon < \varepsilon_0$ there exist

- NHIM $\Lambda_\varepsilon$.
- Homoclinic channel $\Gamma_\varepsilon$.
- Scattering map $s_\varepsilon = \mathrm{Id} + \varepsilon J\nabla S + O(\varepsilon^2)$
- Consider the vector field $\dot{x} = J\nabla S(x)$.
- Suppose that $J\nabla S(x_0) \neq 0$ at some point $x_0 \in \Lambda_\varepsilon$.
  take $\gamma_\varepsilon : [0,1] \to \Lambda_\varepsilon$ be an integral curve through $x_0$.
- Suppose that:$\gamma_\varepsilon([0,1]) \subset \mathcal{U} \subset \Lambda_\varepsilon$, and a.e. point in $\mathcal{U}$ is recurrent for $f_{\varepsilon|\Lambda_\varepsilon}$.

Then for every $\delta > 0$, there exists an orbit $\{z_i\}_{i=0,\dots,n}$ of $f_\varepsilon$ in $M$, with $n = O(\varepsilon^{-1})$, such that for all $i = 0, \dots, n-1$,

$$z_{i+1} = f_\varepsilon^{k_i}(z_i), \quad \text{for some } k_i > 0, \text{ and}$$

$$d(z_i, \gamma_\varepsilon(t_i)) < \delta + K\varepsilon, \text{ for } t_i = i \cdot \varepsilon,$$

where $0 = t_0 < t_1 < \dots < t_n \leq 1$.

# A perturbative result

**Proof:**

• The scattering map is given by $s_\varepsilon = \mathrm{Id} + \varepsilon J \nabla S + O(\varepsilon^2)$

• Its orbits are close to the orbits obtained by applying the Euler method of step $\varepsilon$ to the vector field

$$\dot{x} = J\nabla S(x)$$

• If we take:

$$x_0 = \gamma_\varepsilon(0), \quad x_{i+1} = s_\varepsilon(x_i) \in \mathcal{U} \subset \Lambda,$$

one has

$$d(\gamma_\varepsilon(t_i), x_i) < K\varepsilon, \quad i = 0, \dots, n, \ \ n = O(1/\varepsilon)$$

• Apply **Theorem 2** and obtain an orbit $z_{i+1} = F_\varepsilon^{k_i}(z_i)$ in $M$, for some $k_i > 0$, s.t. $d(z_i, x_i) < \delta$ for all $i = 0, \dots, n$

• Clearly $d(z_i, \gamma_\varepsilon(t_i)) < \delta + K\varepsilon$ for all $i = 0, \dots, n$

# A Perturbative result

Analogously, if

- Scattering map $s_\varepsilon = \mathrm{Id} + \mu(\varepsilon) J\nabla S + g(\mu(\varepsilon)),\ g(\mu(\varepsilon)) = o(\mu(\varepsilon)),\ \mu(0) = 0$ $(\mu(\varepsilon) = \varepsilon,\ g(\mu(\varepsilon)) = \varepsilon^2$ previous case)

Then for every $\delta > 0$, there exists an orbit $\{z_i\}_{i=0,\ldots,n}$ of $f_\varepsilon$ in $M$, with $n = O(\varepsilon^{-1})$, such that for all $i = 0, \ldots, n-1$,

$$z_{i+1} = f_\varepsilon^{k_i}(z_i), \quad \text{for some } k_i > 0, \text{ and}$$

$$d(z_i, \gamma_\varepsilon(t_i)) < \delta + K(\mu(\varepsilon) + |g(\mu(\varepsilon))/\mu(\varepsilon)|), \text{ for } t_i = i \cdot \mu(\varepsilon),$$

where $0 = t_0 < t_1 < \ldots < t_n \leq 1$.
This can be useful when the size of the transversality is not the "standard" one (a priori stable)

# A general diffusion result

**Corollary [Gidea, de la Llave, S.]** $H_\varepsilon = H_0 + \varepsilon H_1$. Assume for all $0 < \varepsilon < \varepsilon_0$ there exist

- NHIM $\Lambda_\varepsilon$
- Homoclinic channel $\Gamma_\varepsilon$.
- Scattering map $s_\varepsilon$:
  $s_\varepsilon = \mathrm{Id} + \varepsilon J\nabla S + O(\varepsilon^2)$,
- In $\Lambda_\varepsilon$ we have some coordinates $(I, \phi) \in \mathbb{R}^d \times \mathbb{T}^d$

If $J\nabla S(I, \phi)$ is transverse to some level set $\{I = I_*\}$ of $I$, then $\exists \varepsilon_1 < \varepsilon_0$, $\exists C > 0$, s.t. $\forall \varepsilon < \varepsilon_1$ $\exists x(t)$ with

$$\|I(x(T)) - I(x(0))\| > C, \text{ for some } T > 0.$$

- **Remark:**
  - There are no requirements on the inner dynamics, except of being conservative

# A general diffusion result

**Proof:**

- $J\nabla S(I,\phi)$ transverse to $\{I = I_0\} \Rightarrow J\nabla S(I,\phi)$ transverse to $\{I = I_*\}$ with $\|I_* - I_0\| < C$, for some $C > 0$ independent of $\varepsilon$
$\Rightarrow$ there is a strip $\mathcal{S}$ of $\phi$-size $O(1)$ consisting of trajectories of the Hamiltonian system $\dot{x} = J\nabla S(x)$ along which $I$ changes $O(1)$ $\Rightarrow$ there are orbits of the map $s_\varepsilon$ along which $I$ changes $O(1)$.
- We have two possibilities
  - There is a bounded domain through the inner dynamics, then we have Poincaré recurrence and Theorem 3 applies and we have orbits of $f_\varepsilon$ whose action $I$ changes $O(1)$
  - There are orbits of $f_{\varepsilon 1 \Lambda_\varepsilon}$ whose action $I$ changes $O(1)$.
- In both cases we have diffusion: combining outer and inner dynamics or only by the inner dynamics

# Application

**Diffusion in an a priori unstable system**

$$H_\varepsilon(p,q,I,\phi,t) \;=\; \underbrace{h_0(I)+\sum_{i=1}^{n}\pm\left(\frac{1}{2}p_i^2+V_i(q_i)\right)}_{H_0}+\varepsilon H_1(p,q,I,\phi,t;\varepsilon),$$

$(p,q,I,\phi,t)\in\mathbb{R}^n\times\mathbb{T}^n\times\mathbb{R}^d\times\mathbb{T}^d\times\mathbb{T}^1$

**Theorem 4 [Gidea, de la Llave, S.]**
Under the earlier assumptions,
there exists $\varepsilon_0>0$, and $C>0$ such that, for each $\varepsilon\in(0,\varepsilon_0)$, there exists a trajectory $x(t)$ such that

$$\|I(x(T))-I(x(0))\|>C \text{ for some } T>0.$$

- We make no asumptions on the dynamics of $h_0$. No need of KAM tori, Aubry Mather sets etc, do not require any property on $\partial^2 h_0/\partial I^2 \neq 0$
- No convexity of the unperturbed Hamiltonian; the argument works even if $\partial^2 h_0/\partial I^2$ degenerate or non-positive definite (e.g., non-twist maps)
- We allow strong resonances etc.
- Any dimension.
- Works for perturbations in an open and dense set satisfying explicit non-degeneracy conditions

# Proof of Theorem 4:

- Penduli $\rightsquigarrow$ homoclinic orbit $(p_i^0(\sigma), q_i^0(\sigma))$ to $(0,0)$
- Consider the Poincaré function:
  $$L(\tau, I, \phi, s) = -\int_{-\infty}^{\infty} \Big[H_1(p^0(\tau+\sigma), q^0(\tau+\sigma), I, \phi+\omega(I)\sigma, s+\sigma; 0) \\ -H_1(0,0,I,\phi+\omega(I)\sigma, s+\sigma; 0)\Big]\, dt$$
- For generic $H_1$, the equation $\frac{\partial}{\partial\tau}L(\tau, I, \phi, s) = 0$ has a non degenerate solution $\tau = \tau^*(I, \phi, s)$
- Define $\mathcal{L}(I, \phi, s) = L(\tau^*(I, \phi, s), I, \phi, s)$ and $\mathcal{L}^*(I, \theta) = \mathcal{L}(I, \theta, 0)$
- Then: $s_\varepsilon(I, \phi) = \mathrm{Id}(I, \phi) + \varepsilon J\nabla\mathcal{L}^*(I, \phi - \omega(I)s) + O(\varepsilon^2)$
- For generic $H_1$, $\nabla\mathcal{L}^*$ is transverse to some level set $\{I = I_0\}$
- Apply **Theorem 3** and **Corollary**.