KAM Tori Are No More Than Sticky

Rome, 5–8 February 2019

David Sauzin, CNRS

IMCCE UMR 8028 – Observatoire de Paris – PSL Research University

Theorem (B.Fayad-D.S. 2018)

Theorem (B.Fayad-D.S. 2018)

For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Giorgilli-Morbidelli 1995: "Superexponential stability of KAM tori" for analytic Hamiltonians

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Giorgilli-Morbidelli 1995: "Superexponential stability of KAM tori" for analytic Hamiltonians: perturb a quasi-convex integrable Hamiltonian,

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Giorgilli-Morbidelli 1995: "Superexponential stability of KAM tori" for analytic Hamiltonians: perturb a quasi-convex integrable Hamiltonian, consider a KAM torus with τ-Diophantine frequency,

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Giorgilli-Morbidelli 1995: "Superexponential stability of KAM tori" for analytic Hamiltonians: perturb a quasi-convex integrable Hamiltonian, consider a KAM torus with τ-Diophantine frequency, then nearby solutions stay close to the torus for an interval of time which is doubly exponentially large

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Giorgilli-Morbidelli 1995: "Superexponential stability of KAM tori" for analytic Hamiltonians: perturb a quasi-convex integrable Hamiltonian, consider a KAM torus with τ-Diophantine frequency, then ν-close solutions stay close to the torus for an interval of time which is doubly exponentially large

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Giorgilli-Morbidelli 1995: “Superexponential stability of KAM tori” for analytic Hamiltonians: perturb a quasi-convex integrable Hamiltonian, consider a KAM torus with τ-Diophantine frequency, then ν-close solutions stay close to the torus for an interval of time which is doubly exponentially large: $\exp\big(\exp\big(\nu^{-1/(\tau+1)}\big)\big)$.

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Giorgilli-Morbidelli 1995: "Superexponential stability of KAM tori" for analytic Hamiltonians: perturb a quasi-convex integrable Hamiltonian, consider a KAM torus with τ-Diophantine frequency, then ν-close solutions stay close to the torus for an interval of time which is doubly exponentially large: $\exp\big(\exp\big(\nu^{-1/(\tau+1)}\big)\big)$.
"STICKY"

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Giorgilli-Morbidelli 1995: "Superexponential stability of KAM tori" for analytic Hamiltonians: perturb a quasi-convex integrable Hamiltonian, consider a KAM torus with τ-Diophantine frequency, then ν-close solutions stay close to the torus for an interval of time which is doubly exponentially large: $\exp\big(\exp\left(\nu^{-1/(\tau+1)}\right)\big)$.
"STICKY"

Bounemoura-Fayad-Niederman 2017: extension to the Gevrey category.

Theorem (B.Fayad-D.S. 2018) *[more precise statement later]*
For any $N \geqslant 3$, the integrable quasi-convex Hamiltonian

$$H_0(\theta, r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N \qquad (\theta \in \mathbb{T}^N,\ r \in \mathbb{R}^N)$$

can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.

Giorgilli-Morbidelli 1995: "Superexponential stability of KAM tori" for analytic Hamiltonians: perturb a quasi-convex integrable Hamiltonian, consider a KAM torus with τ-Diophantine frequency, then ν-close solutions stay close to the torus for an interval of time which is doubly exponentially large: $\exp\left(\exp\left(\nu^{-1/(\tau+1)}\right)\right)$.
"STICKY"

Bounemoura-Fayad-Niederman 2017: extension to the Gevrey category. Also, for a residual and prevalent set of integrable Hamiltonians, for any small perturbation in Gevrey class, there is a set of almost full Lebesgue measure of KAM tori which are doubly exponentially stable.

Gevrey regularity: Given $\alpha \geqslant 1$, $L > 0$ and K a product of closed Euclidean balls and tori, we define uniformly Gevrey-(α, L) fcns:

Gevrey regularity: Given $\alpha \geqslant 1$, $L > 0$ and K a product of closed Euclidean balls and tori, we define uniformly Gevrey-(α, L) fcns:

$$G^{\alpha,L}(\mathbb{R}^M \times K) := \{f \in C^\infty(\mathbb{R}^M \times K) \mid \|f\|_{\alpha,L} < \infty\},$$

$$\|f\|_{\alpha,L} := \sum_{\ell\in\mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^M\times K)}$$

(with $\ell! = \ell_1! \ldots \ell_N!$).

Gevrey regularity: Given $\alpha \geqslant 1$, $L > 0$ and K a product of closed Euclidean balls and tori, we define uniformly Gevrey-(α, L) fcns:

$$G^{\alpha,L}(\mathbb{R}^M \times K) := \{f \in C^\infty(\mathbb{R}^M \times K) \mid \|f\|_{\alpha,L} < \infty\},$$

$$\|f\|_{\alpha,L} := \sum_{\ell\in\mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^M\times K)}$$

(with $\ell! = \ell_1! \dots \ell_N!$).

Nice properties: Banach algebra,

Gevrey regularity: Given $\alpha \geqslant 1$, $L > 0$ and K a product of closed Euclidean balls and tori, we define uniformly Gevrey-(α, L) fcns:

$$G^{\alpha,L}(\mathbb{R}^M \times K) := \{f \in C^\infty(\mathbb{R}^M \times K) \mid \|f\|_{\alpha,L} < \infty\},$$

$$\|f\|_{\alpha,L} := \sum_{\ell\in\mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^M\times K)}$$

(with $\ell! = \ell_1! \dots \ell_N!$).

Nice properties: Banach algebra, Cauchy-Gevrey inequalities,

Gevrey regularity: Given $\alpha \geqslant 1$, $L > 0$ and K a product of closed Euclidean balls and tori, we define uniformly Gevrey-(α, L) fcns:

$$G^{\alpha,L}(\mathbb{R}^M \times K) := \{f \in C^\infty(\mathbb{R}^M \times K) \mid \|f\|_{\alpha,L} < \infty\},$$

$$\|f\|_{\alpha,L} := \sum_{\ell \in \mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^M \times K)}$$

(with $\ell! = \ell_1! \dots \ell_N!$).

Nice properties: Banach algebra, Cauchy-Gevrey inequalities, the flow of a Gevrey vector field is a Gevrey map, etc.

Gevrey regularity: Given $\alpha \geqslant 1$, $L > 0$ and K a product of closed Euclidean balls and tori, we define uniformly Gevrey-(α, L) fcns:

$$G^{\alpha,L}(\mathbb{R}^M \times K) := \{f \in C^\infty(\mathbb{R}^M \times K) \mid \|f\|_{\alpha,L} < \infty\},$$

$$\|f\|_{\alpha,L} := \sum_{\ell \in \mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^M \times K)}$$

(with $\ell! = \ell_1! \dots \ell_N!$).

Nice properties: Banach algebra, Cauchy-Gevrey inequalities, the flow of a Gevrey vector field is a Gevrey map, etc.

Gevrey functions with compact support: if $\alpha > 1$

Gevrey regularity: Given $\alpha \geqslant 1$, $L > 0$ and K a product of closed Euclidean balls and tori, we define uniformly Gevrey-(α, L) fcns:

$$G^{\alpha,L}(\mathbb{R}^M \times K) := \{f \in C^\infty(\mathbb{R}^M \times K) \mid \|f\|_{\alpha,L} < \infty\},$$

$$\|f\|_{\alpha,L} := \sum_{\ell \in \mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^M \times K)}$$

(with $\ell! = \ell_1! \ldots \ell_N!$).

Nice properties: Banach algebra, Cauchy-Gevrey inequalities, the flow of a Gevrey vector field is a Gevrey map, etc.

Gevrey functions with compact support: if $\alpha > 1$, $z \in \mathbb{T} \times \mathbb{R}$ and $\nu > 0$, $G^{\alpha,L}(\mathbb{T} \times \mathbb{R})$ contains a function $0 \leqslant \eta_{z,\nu} \leqslant 1$ such that $\eta_{z,\nu} \equiv 1$ on $B(z, \nu/2)$, $\eta_{z,\nu} \equiv 0$ on $B(z,\nu)^c$

Gevrey regularity: Given $\alpha \geqslant 1$, $L > 0$ and K a product of closed Euclidean balls and tori, we define uniformly Gevrey-(α, L) fcns:

$$G^{\alpha,L}(\mathbb{R}^M \times K) := \{f \in C^\infty(\mathbb{R}^M \times K) \mid \|f\|_{\alpha,L} < \infty\},$$

$$\|f\|_{\alpha,L} := \sum_{\ell \in \mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^M \times K)}$$

(with $\ell! = \ell_1! \dots \ell_N!$).

Nice properties: Banach algebra, Cauchy-Gevrey inequalities, the flow of a Gevrey vector field is a Gevrey map, etc.

Gevrey functions with compact support: if $\alpha > 1$, $z \in \mathbb{T} \times \mathbb{R}$ and $\nu > 0$, $G^{\alpha,L}(\mathbb{T} \times \mathbb{R})$ contains a function $0 \leqslant \eta_{z,\nu} \leqslant 1$ such that $\eta_{z,\nu} \equiv 1$ on $B(z, \nu/2)$, $\eta_{z,\nu} \equiv 0$ on $B(z, \nu)^c$, $\|\eta_{z,\nu}\|_{\alpha,L} \leqslant \exp(c\nu^{-\frac{1}{\alpha-1}})$.

Gevrey regularity: Given $\alpha \geqslant 1$, $L > 0$ and K a product of closed Euclidean balls and tori, we define uniformly Gevrey-(α, L) fcns:

$$G^{\alpha,L}(\mathbb{R}^M \times K) := \{f \in C^\infty(\mathbb{R}^M \times K) \mid \|f\|_{\alpha,L} < \infty\},$$

$$\|f\|_{\alpha,L} := \sum_{\ell \in \mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^M \times K)}$$

(with $\ell! = \ell_1! \dots \ell_N!$).

Nice properties: Banach algebra, Cauchy-Gevrey inequalities, the flow of a Gevrey vector field is a Gevrey map, etc.

Gevrey functions with compact support: if $\alpha > 1$, $z \in \mathbb{T} \times \mathbb{R}$ and $\nu > 0$, $G^{\alpha,L}(\mathbb{T} \times \mathbb{R})$ contains a function $0 \leqslant \eta_{z,\nu} \leqslant 1$ such that $\eta_{z,\nu} \equiv 1$ on $B(z, \nu/2)$, $\eta_{z,\nu} \equiv 0$ on $B(z,\nu)^c$, $\|\eta_{z,\nu}\|_{\alpha,L} \leqslant \exp(c\nu^{-\frac{1}{\alpha-1}})$.

Fréchet space $\mathcal{G}^{\alpha,L}(\mathbb{R}^M \times K)$: cover the factor $\mathbb{R}^M$ by an increasing sequence of closed balls $\overline{B}_{R_j}$, choose $L_j = 2^{-j}L$, get a complete metric space with translation-invariant distance $d_{\alpha,L}$.

Theorem ... $\exists$ Gevrey perturbations of H_0 so that invariant tori are no more than doubly exponentially stable...

Theorem ... $\exists$ Gevrey perturbations of H_0 so that invariant tori are no more than doubly exponentially stable...

This is about the existence of "diffusive" invariant tori.

Theorem ... $\exists$ Gevrey perturbations of H_0 so that invariant tori are no more than doubly exponentially stable...

This is about the existence of "diffusive" invariant tori.

Definition Given a transformation T (or a flow) on a metric space (M, d) and $\nu > 0$, we say that:

Theorem ... $\exists$ Gevrey perturbations of H_0 so that invariant tori are no more than doubly exponentially stable...

This is about the existence of "diffusive" invariant tori.

Definition Given a transformation T (or a flow) on a metric space (M, d) and $\nu > 0$, we say that:

- A point z of M is *ν-diffusive* if there exist an initial condition $\hat{z} \in M$ and a positive integer (or real) t such that $d(\hat{z}, z) \leqslant \nu$, $t \leqslant E(\nu)$ and $d(T^t \hat{z}, z)$

$$E(\nu) = \mathrm{e}^{\mathrm{e}^{C\nu^{-\gamma}}} \quad \text{(with } C, \gamma > 0 \text{ to be chosen later)}$$

Theorem ... $\exists$ Gevrey perturbations of H_0 so that invariant tori are no more than doubly exponentially stable...

This is about the existence of "diffusive" invariant tori.

Definition Given a transformation T (or a flow) on a metric space (M, d) and $\nu > 0$, we say that:

- A point z of M is *ν-diffusive* if there exist an initial condition $\hat{z} \in M$ and a positive integer (or real) t such that $d(\hat{z}, z) \leqslant \nu$, $t \leqslant E(\nu)$ and $d(T^t \hat{z}, z)$

$$E(\nu) = \mathrm{e}^{\mathrm{e}^{C\nu^{-\gamma}}} \quad \text{(with } C, \gamma > 0 \text{ to be chosen later)}$$

Theorem ... $\exists$ Gevrey perturbations of H_0 so that invariant tori are no more than doubly exponentially stable...

This is about the existence of "diffusive" invariant tori.

Definition Given a transformation T (or a flow) on a metric space (M, d) and $\nu > 0$, we say that:

- A point z of M is *ν-diffusive* if there exist an initial condition $\hat{z} \in M$ and a positive integer (or real) t such that $d(\hat{z}, z) \leqslant \nu$, $t \leqslant E(\nu)$ and $d(T^t \hat{z}, z) \geqslant E(2\nu)$.

$$E(\nu) = \mathrm{e}^{\mathrm{e}^{C\nu^{-\gamma}}} \quad \text{(with } C, \gamma > 0 \text{ to be chosen later)}$$

Theorem ... $\exists$ Gevrey perturbations of H_0 so that invariant tori are no more than doubly exponentially stable...

This is about the existence of "diffusive" invariant tori.

Definition Given a transformation T (or a flow) on a metric space (M, d) and $\nu > 0$, we say that:

- A point z of M is *ν-diffusive* if there exist an initial condition $\hat{z} \in M$ and a positive integer (or real) t such that $d(\hat{z}, z) \leqslant \nu$, $t \leqslant E(\nu)$ and $d(T^t \hat{z}, z) \geqslant E(2\nu)$.
- A subset X of M is *ν-diffusive* if all points in X are ν-diffusive.

$$E(\nu) = \mathrm{e}^{\mathrm{e}^{C\nu^{-\gamma}}} \quad \text{(with } C, \gamma > 0 \text{ to be chosen later)}$$

Theorem ... $\exists$ Gevrey perturbations of H_0 so that invariant tori are no more than doubly exponentially stable...

This is about the existence of "diffusive" invariant tori.

Definition Given a transformation T (or a flow) on a metric space (M, d) and $\nu > 0$, we say that:

- A point z of M is *ν-diffusive* if there exist an initial condition $\hat{z} \in M$ and a positive integer (or real) t such that $d(\hat{z}, z) \leqslant \nu$, $t \leqslant E(\nu)$ and $d(T^t \hat{z}, z) \geqslant E(2\nu)$.
- A subset X of M is *ν-diffusive* if all points in X are ν-diffusive.
- A subset X of M is *diffusive* if there exists a sequence $\nu_n \to 0$ such that X is ν_n-diffusive for each n.

$$E(\nu) = \mathrm{e}^{\mathrm{e}^{C\nu^{-\gamma}}} \quad \text{(with } C, \gamma > 0 \text{ to be chosen later)}$$

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.
It is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $[\dot{\tau} = \frac{\partial H}{\partial s} = 1]$
$(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.
It is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or autonomous Hamiltonian $H(\theta, \tau, r, s) = s + h(\theta, r, \tau)$.

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $[\dot{\tau} = \frac{\partial H}{\partial s} = 1]$
$(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.
It is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or autonomous Hamiltonian $H(\theta, \tau, r, s) = s + h(\theta, r, \tau)$.
For arbitrary $\omega \in \mathbb{R}^n$, non-autonomous 1-periodic perturbations of

$$h_0(r) := (\omega, r) + \frac{1}{2}(r, r)$$

are equivalent to

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $[\dot{\tau} = \frac{\partial H}{\partial s} = 1]$
$(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.
It is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or autonomous Hamiltonian $H(\theta, \tau, r, s) = s + h(\theta, r, \tau)$.
For arbitrary $\omega \in \mathbb{R}^n$, non-autonomous 1-periodic perturbations of

$$h_0(r) := (\omega, r) + \tfrac{1}{2}(r, r)$$

are equivalent to certain autonomous perturbations of the integrable Hamiltonian $H_0(r, s) := s + h_0(r)$

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $[\dot{\tau} = \frac{\partial H}{\partial s} = 1]$
$(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.
It is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or autonomous Hamiltonian $H(\theta, \tau, r, s) = s + h(\theta, r, \tau)$.
For arbitrary $\omega \in \mathbb{R}^n$, non-autonomous 1-periodic perturbations of

$$h_0(r) := (\omega, r) + \frac{1}{2}(r, r)$$

are equivalent to certain autonomous perturbations of the integrable Hamiltonian $H_0(r, s) := s + h_0(r)$, for which $\mathcal{T}_{(r,s)} := \mathbb{T}^{n+1} \times \{(r, s)\}$ is an invariant quasi-periodic torus with frequencies $\dot{\theta} = \omega + r$, $\dot{\tau} = 1$ (for arbitrary r and s).

Coordinates $(\theta_1,\dots,\theta_n,r_1,\dots,r_n)$ in $\mathbb{T}^n\times\mathbb{R}^n$ or $(\theta_1,\dots,\theta_n,\tau,r_1,\dots,r_n,s)$ in $\mathbb{T}^{n+1}\times\mathbb{R}^{n+1}$, where $n:=N-1\geqslant 2$. $[\dot\tau=\frac{\partial H}{\partial s}=1]$

It is equivalent to consider a non-autonomous Hamiltonian $h(\theta,r,t)$ on $\mathbb{T}^n\times\mathbb{R}^n$ which depends 1-periodically on the time t or autonomous Hamiltonian $H(\theta,\tau,r,s)=s+h(\theta,r,\tau)$.

For arbitrary $\omega\in\mathbb{R}^n$, non-autonomous 1-periodic perturbations of

$$h_0(r):=(\omega,r)+\frac{1}{2}(r,r)$$

are equivalent to certain autonomous perturbations of the integrable Hamiltonian $H_0(r,s):=s+h_0(r)$, for which $\mathcal{T}_{(r,s)}:=\mathbb{T}^{n+1}\times\{(r,s)\}$ is an invariant quasi-periodic torus with frequencies $\dot\theta=\omega+r$, $\dot\tau=1$ (for arbitrary r and s).

THEOREM 1

Coordinates $(\theta_1, \dots, \theta_n, r_1, \dots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $[\dot{\tau} = \frac{\partial H}{\partial s} = 1]$
$(\theta_1, \dots, \theta_n, \tau, r_1, \dots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.
It is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or autonomous Hamiltonian $H(\theta, \tau, r, s) = s + h(\theta, r, \tau)$.
For arbitrary $\omega \in \mathbb{R}^n$, non-autonomous 1-periodic perturbations of

$$h_0(r) := (\omega, r) + \tfrac{1}{2}(r, r)$$

are equivalent to certain autonomous perturbations of the integrable Hamiltonian $H_0(r, s) := s + h_0(r)$, for which $\mathcal{T}_{(r,s)} := \mathbb{T}^{n+1} \times \{(r, s)\}$ is an invariant quasi-periodic torus with frequencies $\dot{\theta} = \omega + r$, $\dot{\tau} = 1$ (for arbitrary r and s).

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $[\dot{\tau} = \frac{\partial H}{\partial s} = 1]$
$(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.
It is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or autonomous Hamiltonian $H(\theta, \tau, r, s) = s + h(\theta, r, \tau)$.
For arbitrary $\omega \in \mathbb{R}^n$, non-autonomous 1-periodic perturbations of

$$h_0(r) := (\omega, r) + \tfrac{1}{2}(r, r)$$

are equivalent to certain autonomous perturbations of the integrable Hamiltonian $H_0(r, s) := s + h_0(r)$, for which $\mathcal{T}_{(r,s)} := \mathbb{T}^{n+1} \times \{(r, s)\}$ is an invariant quasi-periodic torus with frequencies $\dot{\theta} = \omega + r$, $\dot{\tau} = 1$ (for arbitrary r and s).

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$,

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $[\dot{\tau} = \frac{\partial H}{\partial s} = 1]$
$(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.
It is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or autonomous Hamiltonian $H(\theta, \tau, r, s) = s + h(\theta, r, \tau)$.
For arbitrary $\omega \in \mathbb{R}^n$, non-autonomous 1-periodic perturbations of

$$h_0(r) := (\omega, r) + \frac{1}{2}(r, r)$$

are equivalent to certain autonomous perturbations of the integrable Hamiltonian $H_0(r, s) := s + h_0(r)$, for which $\mathcal{T}_{(r,s)} := \mathbb{T}^{n+1} \times \{(r, s)\}$ is an invariant quasi-periodic torus with frequencies $\dot{\theta} = \omega + r$, $\dot{\tau} = 1$ (for arbitrary r and s).

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and

Coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$ in $\mathbb{T}^n \times \mathbb{R}^n$ or $[\dot{\tau} = \frac{\partial H}{\partial s} = 1]$
$(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$ in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, where $n := N - 1 \geqslant 2$.
It is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or autonomous Hamiltonian $H(\theta, \tau, r, s) = s + h(\theta, r, \tau)$.
For arbitrary $\omega \in \mathbb{R}^n$, non-autonomous 1-periodic perturbations of

$$h_0(r) := (\omega, r) + \frac{1}{2}(r, r)$$

are equivalent to certain autonomous perturbations of the integrable Hamiltonian $H_0(r, s) := s + h_0(r)$, for which $\mathcal{T}_{(r,s)} := \mathbb{T}^{n+1} \times \{(r, s)\}$ is an invariant quasi-periodic torus with frequencies $\dot{\theta} = \omega + r$, $\dot{\tau} = 1$ (for arbitrary r and s).

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable:

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\big(\exp\big(c\nu^{-\frac{1}{\alpha(\tau+1)}}\big)\big)$.

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\big(\exp\big(c\nu^{-\frac{1}{\alpha(\tau+1)}}\big)\big)$.

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\left(\exp\left(c\nu^{-\frac{1}{\alpha(\tau+1)}}\right)\right)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential.

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\big(\exp\big(c\nu^{-\frac{1}{\alpha(\tau+1)}}\big)\big)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential. Our "diffusiveness exponent" in $E(\nu) = \exp(\exp(C\nu^{-\gamma}))$ is $\gamma = \frac{1}{\alpha-1}$

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\left(\exp\left(c\nu^{-\frac{1}{\alpha(\tau+1)}}\right)\right)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential. Our "diffusiveness exponent" in $E(\nu) = \exp(\exp(C\nu^{-\gamma}))$ is $\gamma = \frac{1}{\alpha-1}$, exponents do not match

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\big(\exp\big(c\nu^{-\frac{1}{\alpha(\tau+1)}}\big)\big)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential. Our "diffusiveness exponent" in $E(\nu) = \exp(\exp(C\nu^{-\gamma}))$ is $\gamma = \frac{1}{\alpha-1}$, exponents do not match yet.

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\left(\exp\left(c\nu^{-\frac{1}{\alpha(\tau+1)}}\right)\right)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential. Our "diffusiveness exponent" in $E(\nu) = \exp(\exp(C\nu^{-\gamma}))$ is $\gamma = \frac{1}{\alpha-1}$, exponents do not match yet.

THEOREM 2

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\left(\exp\left(c\nu^{-\frac{1}{\alpha(\tau+1)}}\right)\right)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential. Our "diffusiveness exponent" in $E(\nu) = \exp(\exp(C\nu^{-\gamma}))$ is $\gamma = \frac{1}{\alpha-1}$, exponents do not match yet.

THEOREM 2 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there are $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\left(\exp\left(c\nu^{-\frac{1}{\alpha(\tau+1)}}\right)\right)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential. Our "diffusiveness exponent" in $E(\nu) = \exp(\exp(C\nu^{-\gamma}))$ is $\gamma = \frac{1}{\alpha-1}$, exponents do not match yet.

THEOREM 2 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there are $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ and $X_\varepsilon \subset [0,1]$ with $\mathrm{Leb}(X_\varepsilon) \geqslant 1 - \varepsilon$

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\left(\exp\left(c\nu^{-\frac{1}{\alpha(\tau+1)}}\right)\right)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential. Our "diffusiveness exponent" in $E(\nu) = \exp(\exp(C\nu^{-\gamma}))$ is $\gamma = \frac{1}{\alpha-1}$, exponents do not match yet.

THEOREM 2 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there are $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ and $X_\varepsilon \subset [0,1]$ with $\mathrm{Leb}(X_\varepsilon) \geqslant 1 - \varepsilon$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$,

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow$ $\mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\Big(\exp\big(c\nu^{-\frac{1}{\alpha(\tau+1)}}\big)\Big)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential. Our "diffusiveness exponent" in $E(\nu) = \exp(\exp(C\nu^{-\gamma}))$ is $\gamma = \frac{1}{\alpha-1}$, exponents do not match yet.

THEOREM 2 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there are $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ and $X_\varepsilon \subset [0,1]$ with $\mathrm{Leb}(X_\varepsilon) \geqslant 1 - \varepsilon$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and

THEOREM 1 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and the tori $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ are invariant and diffusive for H.

Note: ω τ-Diophantine $\Rightarrow \mathcal{T}_{(0,s)}$ is doubly exponentially stable: for any ν-close initial condition, the orbit stays within distance 2ν from $\mathcal{T}_{(0,s)}$ during time $\exp\left(\exp\left(c\nu^{-\frac{1}{\alpha(\tau+1)}}\right)\right)$. Theorem 1 shows that we cannot expect in general a stability better than doubly exponential. Our "diffusiveness exponent" in $E(\nu) = \exp(\exp(C\nu^{-\gamma}))$ is $\gamma = \frac{1}{\alpha-1}$, exponents do not match yet.

THEOREM 2 Given $\alpha > 1$, $L > 0$, $\varepsilon > 0$, there are $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ and $X_\varepsilon \subset [0,1]$ with $\mathrm{Leb}(X_\varepsilon) \geqslant 1 - \varepsilon$ such that $d_{\alpha,L}(h_0, h) < \varepsilon$, the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and, for each $r \in (X_\varepsilon + \mathbb{Z}) \times \mathbb{R}^{n-1}$ and $s \in \mathbb{R}$, the torus $\mathcal{T}_{(r,s)} = \mathbb{T}^{n+1} \times \{(r,s)\} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ is invariant and diffusive for H.

Method: Obtain first a discrete version of the results

Method: Obtain first a discrete version of the results by "Herman's synchronized diffusion" mechanism...

Method: Obtain first a discrete version of the results by "Herman's synchronized diffusion" mechanism...

Discrete version in the case $n = 2$:

Method: Obtain first a discrete version of the results by "Herman's synchronized diffusion" mechanism...

Discrete version in the case $n = 2$:

Phase space $M_1 \times M_2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$ with $M_1 := \mathbb{T} \times \mathbb{R}$, $M_2 := \mathbb{T} \times \mathbb{R}$.

Unperturbed integrable system: $T_0 := F_0 \times G_0 \colon M_1 \times M_2 \circlearrowleft$

with $F_0 \colon M_1 \circlearrowleft$ and $G_0 \colon M_2 \circlearrowleft$ defined by

$F_0(\theta_1, r_1) := (\theta_1 + \omega_1 + r_1, r_1), \quad G_0(\theta_2, r_2) := (\theta_2 + \omega_2 + r_2, r_2).$

Method: Obtain first a discrete version of the results by "Herman's synchronized diffusion" mechanism...

Discrete version in the case $n = 2$:

Phase space $M_1 \times M_2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$ with $M_1 := \mathbb{T} \times \mathbb{R}$, $M_2 := \mathbb{T} \times \mathbb{R}$.

Unperturbed integrable system: $T_0 := F_0 \times G_0 \colon M_1 \times M_2 \circlearrowleft$

with $F_0 \colon M_1 \circlearrowleft$ and $G_0 \colon M_2 \circlearrowleft$ defined by

$F_0(\theta_1, r_1) := (\theta_1 + \omega_1 + r_1, r_1)$, $\quad G_0(\theta_2, r_2) := (\theta_2 + \omega_2 + r_2, r_2)$.

$\mathcal{T}_0 := \mathbb{T}^2 \times \{(0,0)\}$ invariant torus with frequency $\omega = (\omega_1, \omega_2)$.

Method: Obtain first a discrete version of the results by "Herman's synchronized diffusion" mechanism...

Discrete version in the case $n = 2$:

Phase space $M_1 \times M_2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$ with $M_1 := \mathbb{T} \times \mathbb{R}$, $M_2 := \mathbb{T} \times \mathbb{R}$.

Unperturbed integrable system: $T_0 := F_0 \times G_0 \colon M_1 \times M_2 \circlearrowleft$

with $F_0 \colon M_1 \circlearrowleft$ and $G_0 \colon M_2 \circlearrowleft$ defined by

$F_0(\theta_1, r_1) := (\theta_1 + \omega_1 + r_1, r_1)$, $\quad G_0(\theta_2, r_2) := (\theta_2 + \omega_2 + r_2, r_2)$.

$\mathcal{T}_0 := \mathbb{T}^2 \times \{(0,0)\}$ invariant torus with frequency $\omega = (\omega_1, \omega_2)$.

Notation: $H \rightsquigarrow \Phi^H$ = time-1 map of the Hamiltonian flow

e.g. $T_0 = \Phi^{\omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}(r_1^2 + r_2^2)}$.

Method: Obtain first a discrete version of the results by "Herman's synchronized diffusion" mechanism...

Discrete version in the case $n = 2$:

Phase space $M_1 \times M_2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$ with $M_1 := \mathbb{T} \times \mathbb{R}$, $M_2 := \mathbb{T} \times \mathbb{R}$.

Unperturbed integrable system: $T_0 := F_0 \times G_0 \colon M_1 \times M_2 \circlearrowleft$

with $F_0 \colon M_1 \circlearrowleft$ and $G_0 \colon M_2 \circlearrowleft$ defined by

$F_0(\theta_1, r_1) := (\theta_1 + \omega_1 + r_1, r_1)$, $\quad G_0(\theta_2, r_2) := (\theta_2 + \omega_2 + r_2, r_2)$.

$\mathcal{T}_0 := \mathbb{T}^2 \times \{(0,0)\}$ invariant torus with frequency $\omega = (\omega_1, \omega_2)$.

Notation: $H \rightsquigarrow \Phi^H$ = time-1 map of the Hamiltonian flow e.g. $T_0 = \Phi^{\omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}(r_1^2 + r_2^2)}$. THEOREM 1 follows easily from

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

Method: Obtain first a discrete version of the results by "Herman's synchronized diffusion" mechanism...

Discrete version in the case $n = 2$:

Phase space $M_1 \times M_2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$ with $M_1 := \mathbb{T} \times \mathbb{R}$, $M_2 := \mathbb{T} \times \mathbb{R}$.

Unperturbed integrable system: $T_0 := F_0 \times G_0 \colon M_1 \times M_2 \circlearrowleft$

with $F_0 \colon M_1 \circlearrowleft$ and $G_0 \colon M_2 \circlearrowleft$ defined by

$F_0(\theta_1, r_1) := (\theta_1 + \omega_1 + r_1, r_1)$, $\quad G_0(\theta_2, r_2) := (\theta_2 + \omega_2 + r_2, r_2)$.

$\mathcal{T}_0 := \mathbb{T}^2 \times \{(0,0)\}$ invariant torus with frequency $\omega = (\omega_1, \omega_2)$.

Notation: $H \rightsquigarrow \Phi^H =$ time-1 map of the Hamiltonian flow e.g. $T_0 = \Phi^{\omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}(r_1^2 + r_2^2)}$. THEOREM 1 follows easily from

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1)$, $v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

Method: Obtain first a discrete version of the results by "Herman's synchronized diffusion" mechanism...

Discrete version in the case $n = 2$:

Phase space $M_1 \times M_2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$ with $M_1 := \mathbb{T} \times \mathbb{R}$, $M_2 := \mathbb{T} \times \mathbb{R}$.

Unperturbed integrable system: $T_0 := F_0 \times G_0 \colon M_1 \times M_2 \circlearrowleft$

with $F_0 \colon M_1 \circlearrowleft$ and $G_0 \colon M_2 \circlearrowleft$ defined by

$F_0(\theta_1, r_1) := (\theta_1 + \omega_1 + r_1, r_1), \quad G_0(\theta_2, r_2) := (\theta_2 + \omega_2 + r_2, r_2)$.

$\mathcal{T}_0 := \mathbb{T}^2 \times \{(0,0)\}$ invariant torus with frequency $\omega = (\omega_1, \omega_2)$.

Notation: $H \rightsquigarrow \Phi^H$ = time-1 map of the Hamiltonian flow e.g. $T_0 = \Phi^{\omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}(r_1^2 + r_2^2)}$. THEOREM 1 follows easily from

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

(2) $\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

(2) $\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

(2) $\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

There is also a THEOREM 2' which implies THEOREM 2...

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

(2) $\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

There is also a THEOREM 2' which implies THEOREM 2...

Key proposition: localized diffusive orbits:

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

(2) $\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

There is also a THEOREM 2' which implies THEOREM 2...

Key proposition: localized diffusive orbits:

PROPOSITION Let $\gamma = \frac{1}{\alpha - 1}$. For any $\nu > 0$ small enough and $\bar{r} \in \mathbb{R}$, there exist $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

(2) $\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

There is also a THEOREM 2' which implies THEOREM 2...

Key proposition: localized diffusive orbits:

PROPOSITION Let $\gamma = \frac{1}{\alpha-1}$. For any $\nu > 0$ small enough and $\bar{r} \in \mathbb{R}$, there exist $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) $u \equiv 0, v \equiv 0$ for $r_1 \notin (\bar{r} - \nu, \bar{r} + \nu)$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} \leqslant \mathrm{e}^{-c\nu^{-\gamma}}$,

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

(2) $\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

There is also a THEOREM 2' which implies THEOREM 2...

Key proposition: localized diffusive orbits:

PROPOSITION Let $\gamma = \frac{1}{\alpha-1}$. For any $\nu > 0$ small enough and $\bar{r} \in \mathbb{R}$, there exist $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) $u \equiv 0, v \equiv 0$ for $r_1 \notin (\bar{r} - \nu, \bar{r} + \nu)$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} \leqslant \mathrm{e}^{-c\nu^{-\gamma}}$,

(2) the set $\mathbb{T} \times (\bar{r} - \nu, \bar{r} + \nu) \times M_2$ is invariant and ν-diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

(2) $\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

There is also a THEOREM 2' which implies THEOREM 2...

Key proposition: localized diffusive orbits:

PROPOSITION Let $\gamma = \frac{1}{\alpha-1}$. For any $\nu > 0$ small enough and $\bar{r} \in \mathbb{R}$, there exist $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) $u \equiv 0, v \equiv 0$ for $r_1 \notin (\bar{r} - \nu, \bar{r} + \nu)$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} \leqslant \mathrm{e}^{-c\nu^{-\gamma}}$,

(2) the set $\mathbb{T} \times (\bar{r} - \nu, \bar{r} + \nu) \times M_2$ is invariant and ν-diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

PROP $\Rightarrow$ THEOREM 1': take $\nu = \nu_n = 10^{-n}\varepsilon$, $\bar{r} = \bar{r}_n = 2\nu_n$ and add up the corresponding u_n's and v_n's... (Disjoint supports!)

THEOREM 1' $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) u and v are flat for $r_1 = 0$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,

(2) $\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

There is also a THEOREM 2' which implies THEOREM 2...

Key proposition: localized diffusive orbits:

PROPOSITION Let $\gamma = \frac{1}{\alpha-1}$. For any $\nu > 0$ small enough and $\bar{r} \in \mathbb{R}$, there exist $\exists u \in G^{\alpha,L}(M_1), v \in G^{\alpha,L}(M_1 \times M_2)$ such that

(1) $u \equiv 0, v \equiv 0$ for $r_1 \notin (\bar{r} - \nu, \bar{r} + \nu)$, $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} \leqslant \mathrm{e}^{-c\nu^{-\gamma}}$,

(2) the set $\mathbb{T} \times (\bar{r} - \nu, \bar{r} + \nu) \times M_2$ is invariant and ν-diffusive for $T := \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big)$.

PROP $\Rightarrow$ THEOREM 1': take $\nu = \nu_n = 10^{-n}\varepsilon$, $\bar{r} = \bar{r}_n = 2\nu_n$ and add up the corresponding u_n's and v_n's... (Disjoint supports!)

PROP $\Rightarrow$ THEOREM 2': more elaborate...

Herman's synchronized diffusion mechanism relies on a "coupling lemma" which was used in the construction of examples

Herman's synchronized diffusion mechanism relies on a "coupling lemma" which was used in the construction of examples

- with drifting orbits, biasymptotic to infinity, with diffusion speed bounded from below (J.-P.Marco-D.S. 2003)

Herman's synchronized diffusion mechanism relies on a "coupling lemma" which was used in the construction of examples

- with drifting orbits, biasymptotic to infinity, with diffusion speed bounded from below (J.-P.Marco-D.S. 2003)
- with wandering polydiscs (J.-P.Marco-D.S. 2004)

Herman's synchronized diffusion mechanism relies on a "coupling lemma" which was used in the construction of examples

- with drifting orbits, biasymptotic to infinity, with diffusion speed bounded from below (J.-P.Marco-D.S. 2003)
- with wandering polydiscs (J.-P.Marco-D.S. 2004), with estimates for their size in L.Lazzarini-J.-P.Marco-D.S. 2018

Herman's synchronized diffusion mechanism relies on a "coupling lemma" which was used in the construction of examples

- with drifting orbits, biasymptotic to infinity, with diffusion speed bounded from below (J.-P.Marco-D.S. 2003)
- with wandering polydiscs (J.-P.Marco-D.S. 2004), with estimates for their size in L.Lazzarini-J.-P.Marco-D.S. 2018
- with qth iterate containing a subsystem isomorphic to a skew-product defined on $\frac{1}{q}\mathbb{Z} \times \{\omega_1, \omega_2\}^{\mathbb{Z}}$ giving rise to a random walk of step $\frac{1}{q}$ for r_1 (J.-P.Marco-D.S. 2004)

Herman's synchronized diffusion mechanism relies on a "coupling lemma" which was used in the construction of examples

- with drifting orbits, biasymptotic to infinity, with diffusion speed bounded from below (J.-P.Marco-D.S. 2003)
- with wandering polydiscs (J.-P.Marco-D.S. 2004), with estimates for their size in L.Lazzarini-J.-P.Marco-D.S. 2018
- with qth iterate containing a subsystem isomorphic to a skew-product defined on $\frac{1}{q}\mathbb{Z} \times \{\omega_1, \omega_2\}^{\mathbb{Z}}$ giving rise to a random walk of step $\frac{1}{q}$ for r_1 (J.-P.Marco-D.S. 2004)
- with a subsystem isomorphic to a transitive system on $(\mathbb{T} \times \mathbb{R})^{n-1} \times \{\omega_1, \ldots, \omega_r\}^{\mathbb{Z}}$,

Herman's synchronized diffusion mechanism relies on a "coupling lemma" which was used in the construction of examples

- with drifting orbits, biasymptotic to infinity, with diffusion speed bounded from below (J.-P.Marco-D.S. 2003)
- with wandering polydiscs (J.-P.Marco-D.S. 2004), with estimates for their size in L.Lazzarini-J.-P.Marco-D.S. 2018
- with qth iterate containing a subsystem isomorphic to a skew-product defined on $\frac{1}{q}\mathbb{Z} \times \{\omega_1, \omega_2\}^{\mathbb{Z}}$ giving rise to a random walk of step $\frac{1}{q}$ for r_1 (J.-P.Marco-D.S. 2004)
- with a subsystem isomorphic to a transitive system on $(\mathbb{T} \times \mathbb{R})^{n-1} \times \{\omega_1, \dots, \omega_r\}^{\mathbb{Z}}$, with convergence in law to a Brownian motion of the $n-1$ first action variables after rescaling,

Herman's synchronized diffusion mechanism relies on a "coupling lemma" which was used in the construction of examples

- with drifting orbits, biasymptotic to infinity, with diffusion speed bounded from below (J.-P.Marco-D.S. 2003)
- with wandering polydiscs (J.-P.Marco-D.S. 2004), with estimates for their size in L.Lazzarini-J.-P.Marco-D.S. 2018
- with qth iterate containing a subsystem isomorphic to a skew-product defined on $\frac{1}{q}\mathbb{Z} \times \{\omega_1, \omega_2\}^{\mathbb{Z}}$ giving rise to a random walk of step $\frac{1}{q}$ for r_1 (J.-P.Marco-D.S. 2004)
- with a subsystem isomorphic to a transitive system on $(\mathbb{T} \times \mathbb{R})^{n-1} \times \{\omega_1, \ldots, \omega_r\}^{\mathbb{Z}}$, with convergence in law to a Brownian motion of the $n-1$ first action variables after rescaling, ergodic if $n = 2$ or 3 (D.S. 2006, unpublished)

Herman's synchronized diffusion mechanism relies on a "coupling lemma" which was used in the construction of examples

- with drifting orbits, biasymptotic to infinity, with diffusion speed bounded from below (J.-P.Marco-D.S. 2003)
- with wandering polydiscs (J.-P.Marco-D.S. 2004), with estimates for their size in L.Lazzarini-J.-P.Marco-D.S. 2018
- with qth iterate containing a subsystem isomorphic to a skew-product defined on $\frac{1}{q}\mathbb{Z} \times \{\omega_1, \omega_2\}^{\mathbb{Z}}$ giving rise to a random walk of step $\frac{1}{q}$ for r_1 (J.-P.Marco-D.S. 2004)
- with a subsystem isomorphic to a transitive system on $(\mathbb{T} \times \mathbb{R})^{n-1} \times \{\omega_1, \ldots, \omega_r\}^{\mathbb{Z}}$, with convergence in law to a Brownian motion of the $n-1$ first action variables after rescaling, ergodic if $n = 2$ or 3 (D.S. 2006, unpublished)
- with a non-resonant elliptic fixed point attracting an orbit (B.Fayad-J.-P.Marco-D.S. 2018).

Herman's mechanism:

Herman's mechanism: Fine-tuned coupling of two twist maps:

Herman's mechanism: Fine-tuned coupling of two twist maps:

At exactly one point z_* of a well chosen periodic orbit of period q of the first twist map $F = \Phi^u \times F_0 : M_1 = \mathbb{T} \times \mathbb{R} \circlearrowleft$

Herman's mechanism: Fine-tuned coupling of two twist maps:

At exactly one point z_* of a well chosen periodic orbit of period q of the first twist map $F = \Phi^u \times F_0 : M_1 = \mathbb{T} \times \mathbb{R} \circlearrowleft$, the coupling will push the orbits in the second annulus $M_2 = \mathbb{T} \times \mathbb{R}$ upward, along a fixed vertical Δ

Herman's mechanism: Fine-tuned coupling of two twist maps:

At exactly one point z_* of a well chosen periodic orbit of period q of the first twist map $F = \Phi^u \times F_0 : M_1 = \mathbb{T} \times \mathbb{R} \circlearrowleft$, the coupling will push the orbits in the second annulus $M_2 = \mathbb{T} \times \mathbb{R}$ upward, along a fixed vertical Δ, by an amount $1/q$ that sends an invariant curve whose rotation number is a multiple of $1/q$ exactly to another one having the same property.

Herman's mechanism: Fine-tuned coupling of two twist maps:

At exactly one point z_* of a well chosen periodic orbit of period q of the first twist map $F = \Phi^u \times F_0 : M_1 = \mathbb{T} \times \mathbb{R} \circlearrowleft$, the coupling will push the orbits in the second annulus $M_2 = \mathbb{T} \times \mathbb{R}$ upward, along a fixed vertical Δ, by an amount $1/q$ that sends an invariant curve whose rotation number is a multiple of $1/q$ exactly to another one having the same property.

The dynamics of the q^{th} iterate of the coupled map on the line $\{z_*\} \times \Delta \subset M_1 \times M_2$ will thus drift at a linear speed

Herman's mechanism: Fine-tuned coupling of two twist maps:

At exactly one point z_* of a well chosen periodic orbit of period q of the first twist map $F = \Phi^u \times F_0 : M_1 = \mathbb{T} \times \mathbb{R} \circlearrowleft$, the coupling will push the orbits in the second annulus $M_2 = \mathbb{T} \times \mathbb{R}$ upward, along a fixed vertical Δ, by an amount $1/q$ that sends an invariant curve whose rotation number is a multiple of $1/q$ exactly to another one having the same property.

The dynamics of the q^{th} iterate of the coupled map on the line $\{z_*\} \times \Delta \subset M_1 \times M_2$ will thus drift at a linear speed: after q^2 iterates the point will have moved by 1 in the second action coordinate r_2

Herman's mechanism: Fine-tuned coupling of two twist maps:

At exactly one point z_* of a well chosen periodic orbit of period q of the first twist map $F = \Phi^u \times F_0 : M_1 = \mathbb{T} \times \mathbb{R} \circlearrowleft$, the coupling will push the orbits in the second annulus $M_2 = \mathbb{T} \times \mathbb{R}$ upward, along a fixed vertical Δ, by an amount $1/q$ that sends an invariant curve whose rotation number is a multiple of $1/q$ exactly to another one having the same property.

The dynamics of the q^{th} iterate of the coupled map on the line $\{z_*\} \times \Delta \subset M_1 \times M_2$ will thus drift at a linear speed: after q^2 iterates the point will have moved by 1 in the second action coordinate r_2, and after q^3 it will have moved by q.

Herman's mechanism: Fine-tuned coupling of two twist maps:

At exactly one point z_* of a well chosen periodic orbit of period q of the first twist map $F = \Phi^u \times F_0 : M_1 = \mathbb{T} \times \mathbb{R} \circlearrowleft$, the coupling will push the orbits in the second annulus $M_2 = \mathbb{T} \times \mathbb{R}$ upward, along a fixed vertical Δ, by an amount $1/q$ that sends an invariant curve whose rotation number is a multiple of $1/q$ exactly to another one having the same property.

The dynamics of the q^{th} iterate of the coupled map on the line $\{z_*\} \times \Delta \subset M_1 \times M_2$ will thus drift at a linear speed: after q^2 iterates the point will have moved by 1 in the second action coordinate r_2, and after q^3 it will have moved by q.

The diffusing orbits obtained this way are bi-asymptotic to infinity: their r_2-coordinates travel from $-\infty$ to $+\infty$ at average speed $1/q^2$.

Coupling lemma

Coupling lemma

$F\colon M_1 \circlearrowleft$ and $G_0\colon M_2 \circlearrowleft$ diffeomorphisms

$z_* \in M_1$ a q-periodic for F

$f\colon M_1 \to \mathbb{R}$ and $g\colon M_2 \to \mathbb{R}$ (Hamiltonian) functions.

Coupling lemma

$F\colon M_1 \circlearrowleft$ and $G_0\colon M_2 \circlearrowleft$ diffeomorphisms

$z_* \in M_1$ a q-periodic for F

$f\colon M_1 \to \mathbb{R}$ and $g\colon M_2 \to \mathbb{R}$ (Hamiltonian) functions.

Then $T := \Phi^{f\otimes g} \circ (F \times G_0)\colon M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \quad \text{for all } z_2 \in M_2.$$

We have denoted by $f \otimes g$ the function $(z_1, z_2) \mapsto f(z_1)g(z_2)$.

Coupling lemma

$F\colon M_1 \circlearrowleft$ and $G_0\colon M_2 \circlearrowleft$ diffeomorphisms

$z_* \in M_1$ a q-periodic for F

$f\colon M_1 \to \mathbb{R}$ and $g\colon M_2 \to \mathbb{R}$ (Hamiltonian) functions.

Synchronization Assumption

$$f(z_*) = 1, \quad \mathrm{d}f(z_*) = 0, \quad f(F^s(z_*)) = 0, \quad \mathrm{d}f(F^s(z_*)) = 0$$

for $1 \leqslant s \leqslant q-1$.

Then $T := \Phi^{f\otimes g} \circ (F \times G_0)\colon M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \quad \text{for all } z_2 \in M_2.$$

We have denoted by $f \otimes g$ the function $(z_1, z_2) \mapsto f(z_1)g(z_2)$.

Coupling lemma

$F\colon M_1 \circlearrowleft$ and $G_0\colon M_2 \circlearrowleft$ diffeomorphisms

$z_* \in M_1$ a q-periodic for F

$f\colon M_1 \to \mathbb{R}$ and $g\colon M_2 \to \mathbb{R}$ (Hamiltonian) functions.

Synchronization Assumption

$$f(z_*) = 1, \quad \mathrm{d}f(z_*) = 0, \quad f(F^s(z_*)) = 0, \quad \mathrm{d}f(F^s(z_*)) = 0$$

for $1 \leqslant s \leqslant q-1$.

Then $T := \Phi^{f\otimes g} \circ (F \times G_0)\colon M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \quad \text{for all } z_2 \in M_2.$$

We have denoted by $f \otimes g$ the function $(z_1, z_2) \mapsto f(z_1)g(z_2)$.

The point is that

$$\Phi^{f\otimes g}(z_1, z_2) = \big(\Phi^{g(z_2)\,f}(z_1), \Phi^{f(z_1)\,g}(z_2)\big) \quad \text{for all } (z_1, z_2).$$

$T := \Phi^{f \otimes g} \circ (F \times G_0) \colon\ M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \text{ for all } z_2 \in M_2.$$

$T := \Phi^{f \otimes g} \circ (F \times G_0) \colon \; M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \text{ for all } z_2 \in M_2.$$

$\psi := \Phi^g \circ G_0^q \colon \; M_2 \circlearrowleft$ appears as a subsystem of $T^q \colon \; M_1 \times M_2 \circlearrowleft$

$T := \Phi^{f\otimes g} \circ (F \times G_0)\colon\ M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \text{ for all } z_2 \in M_2.$$

$\psi := \Phi^g \circ G_0^q\colon\ M_2 \circlearrowleft$ appears as a subsystem of $T^q\colon\ M_1 \times M_2 \circlearrowleft$

To prove PROP:

Use $g(r_2, \theta_2) = -\frac{1}{q}\frac{\sin(2\pi\theta_2)}{2\pi}$, so $\psi =$ rescaled standard map

$$\psi(\theta_2, r_2) = (\theta_2 + q(\omega_2 + r_2), r_2 + \tfrac{1}{q}\cos(\theta_2 + q(\omega_2 + r_2)))$$

$T := \Phi^{f \otimes g} \circ (F \times G_0)\colon\ M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \left(z_*, \Phi^g \circ G_0^q(z_2)\right) \text{ for all } z_2 \in M_2.$$

$\psi := \Phi^g \circ G_0^q\colon\ M_2 \circlearrowleft$ appears as a subsystem of $T^q\colon\ M_1 \times M_2 \circlearrowleft$

To prove PROP:

Use $g(r_2, \theta_2) = -\frac{1}{q}\frac{\sin(2\pi\theta_2)}{2\pi}$, so $\psi =$ rescaled standard map

$$\psi(\theta_2, r_2) = (\theta_2 + q(\omega_2 + r_2), r_2 + \tfrac{1}{q}\cos(\theta_2 + q(\omega_2 + r_2)))$$

not close to integrable! Drift will take place

$T := \Phi^{f\otimes g} \circ (F \times G_0)\colon\ M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \text{ for all } z_2 \in M_2.$$

$\psi := \Phi^g \circ G_0^q\colon\ M_2 \circlearrowleft$ appears as a subsystem of $T^q\colon\ M_1 \times M_2 \circlearrowleft$

To prove PROP:

Use $g(r_2, \theta_2) = -\frac{1}{q}\frac{\sin(2\pi\theta_2)}{2\pi}$, so $\psi =$ rescaled standard map

$$\psi(\theta_2, r_2) = \left(\theta_2 + q(\omega_2 + r_2), r_2 + \tfrac{1}{q}\cos(\theta_2 + q(\omega_2 + r_2))\right)$$

not close to integrable! Drift will take place on $\{z_*\} \times \Delta$ with $\Delta := \{0\} \times \mathbb{R} \subset M_2$: $\psi^n(0, -\omega_2) = (0, -\omega_2 + \frac{n}{q})$ for all $n \in \mathbb{Z}$

$T := \Phi^{f\otimes g} \circ (F \times G_0) \colon M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \text{ for all } z_2 \in M_2.$$

$\psi := \Phi^g \circ G_0^q \colon M_2 \circlearrowleft$ appears as a subsystem of $T^q \colon M_1 \times M_2 \circlearrowleft$

To prove PROP:

Use $g(r_2, \theta_2) = -\frac{1}{q}\frac{\sin(2\pi\theta_2)}{2\pi}$, so $\psi =$ rescaled standard map

$$\psi(\theta_2, r_2) = (\theta_2 + q(\omega_2 + r_2), r_2 + \tfrac{1}{q}\cos(\theta_2 + q(\omega_2 + r_2)))$$

not close to integrable! Drift will take place on $\{z_*\} \times \Delta$ with $\Delta := \{0\} \times \mathbb{R} \subset M_2$: $\psi^n(0, -\omega_2) = (0, -\omega_2 + \frac{n}{q})$ for all $n \in \mathbb{Z}$

For the first factor, find a near-integrable system $F = \Phi^u \circ F_0$ with a q-periodic "σ-isolated" point, with σ not too small:

$T := \Phi^{f \otimes g} \circ (F \times G_0) \colon M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \text{ for all } z_2 \in M_2.$$

$\psi := \Phi^g \circ G_0^q \colon M_2 \circlearrowleft$ appears as a subsystem of $T^q \colon M_1 \times M_2 \circlearrowleft$

To prove PROP:

Use $g(r_2, \theta_2) = -\frac{1}{q}\frac{\sin(2\pi\theta_2)}{2\pi}$, so $\psi =$ rescaled standard map

$$\psi(\theta_2, r_2) = (\theta_2 + q(\omega_2 + r_2), r_2 + \tfrac{1}{q}\cos(\theta_2 + q(\omega_2 + r_2)))$$

not close to integrable! Drift will take place on $\{z_*\} \times \Delta$ with $\Delta := \{0\} \times \mathbb{R} \subset M_2$: $\psi^n(0, -\omega_2) = (0, -\omega_2 + \frac{n}{q})$ for all $n \in \mathbb{Z}$

For the first factor, find a near-integrable system $F = \Phi^u \circ F_0$ with a q-periodic "σ-isolated" point, with σ not too small: fulfilling Synchronization Assumption will make f exponentially large in σ.

$T := \Phi^{f\otimes g} \circ (F \times G_0) \colon M_1 \times M_2 \circlearrowleft$ satisfies

$$T^q(z_*, z_2) = \big(z_*, \Phi^g \circ G_0^q(z_2)\big) \text{ for all } z_2 \in M_2.$$

$\psi := \Phi^g \circ G_0^q \colon M_2 \circlearrowleft$ appears as a subsystem of $T^q \colon M_1 \times M_2 \circlearrowleft$

To prove PROP:

Use $g(r_2, \theta_2) = -\frac{1}{q}\frac{\sin(2\pi\theta_2)}{2\pi}$, so ψ = rescaled standard map

$$\psi(\theta_2, r_2) = (\theta_2 + q(\omega_2 + r_2), r_2 + \tfrac{1}{q}\cos(\theta_2 + q(\omega_2 + r_2)))$$

not close to integrable! Drift will take place on $\{z_*\} \times \Delta$ with $\Delta := \{0\} \times \mathbb{R} \subset M_2$: $\psi^n(0, -\omega_2) = (0, -\omega_2 + \frac{n}{q})$ for all $n \in \mathbb{Z}$

For the first factor, find a near-integrable system $F = \Phi^u \circ F_0$ with a q-periodic "σ-isolated" point, with σ not too small: fulfilling Synchronization Assumption will make f exponentially large in σ.

Then take q large enough to ensure that $v := f \otimes g$ is small...
(Indeed: want to achieve $\|u\| + \|v\| \leqslant \mathrm{e}^{-c\nu^{-\gamma}}$)

$$T = \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big), \quad v := f \otimes g = -\frac{1}{q} f(z_1) \frac{\sin(2\pi\theta_2)}{2\pi}$$

$$T = \Phi^{v} \circ \big((\Phi^{u} \circ F_0) \times G_0 \big), \quad v := f \otimes g = -\frac{1}{q} f(z_1) \frac{\sin(2\pi\theta_2)}{2\pi}$$

$\|f\|$ exponentially large w.r.t. σ

$$T = \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big), \quad v := f \otimes g = -\frac{1}{q} f(z_1) \frac{\sin(2\pi\theta_2)}{2\pi}$$

$\|f\|$ exponentially large w.r.t. σ

It so happens that σ must be taken exponentially small w.r.t. ν,

$$T = \Phi^{v} \circ \big((\Phi^{u} \circ F_0) \times G_0 \big), \quad v := f \otimes g = -\frac{1}{q} f(z_1) \frac{\sin(2\pi\theta_2)}{2\pi}$$

$\|f\|$ exponentially large w.r.t. σ

It so happens that σ must be taken exponentially small w.r.t. ν, i.e. $\|f\|$ is doubly exponentially large w.r.t. ν.

$$T = \Phi^v \circ \left((\Phi^u \circ F_0) \times G_0 \right), \quad v := f \otimes g = -\frac{1}{q} f(z_1) \frac{\sin(2\pi\theta_2)}{2\pi}$$

$\|f\|$ exponentially large w.r.t. σ

It so happens that σ must be taken exponentially small w.r.t. ν, i.e. $\|f\|$ is doubly exponentially large w.r.t. ν.

This is why we take q doubly exponentially large in ν

$$T = \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big), \quad v := f \otimes g = -\frac{1}{q} f(z_1) \frac{\sin(2\pi\theta_2)}{2\pi}$$

$\|f\|$ exponentially large w.r.t. σ

It so happens that σ must be taken exponentially small w.r.t. ν, i.e. $\|f\|$ is doubly exponentially large w.r.t. ν.

This is why we take q doubly exponentially large in ν and, in the end, the diffusion time q^3 is doubly exponentially large in ν !!

$$T = \Phi^v \circ \big((\Phi^u \circ F_0) \times G_0\big), \quad v := f \otimes g = -\frac{1}{q} f(z_1) \frac{\sin(2\pi\theta_2)}{2\pi}$$

$\|f\|$ exponentially large w.r.t. σ

It so happens that σ must be taken exponentially small w.r.t. ν, i.e. $\|f\|$ is doubly exponentially large w.r.t. ν.

This is why we take q doubly exponentially large in ν and, in the end, the diffusion time q^3 is doubly exponentially large in ν !!

A technical work is required to find $F = \Phi^u \circ F_0$ with the desired isolation property...

– Fine-tuning of rotation number of a certain circle diffeo, $\mathbb{T} \circlearrowleft$

– Another trick by Herman alllows us to embed it in a system of the form $F = \Phi^u \circ F_0 \colon M_1 \circlearrowleft$.