BERNOULLI, RAMANUJAN, TOEPLITZ AND THE TRIANGULAR MATRICES

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Abstract. By using one of the definitions of the Bernoulli numbers, we prove that they solve particular *odd* and *even* lower triangular Toeplitz (l.t.T.) systems of linear equations. In a paper Ramanujan writes down a sparse lower triangular system solved by Bernoulli numbers; we observe that such system is equivalent to a sparse l.t.T. system. The attempt to obtain the sparse l.t.T. Ramanujan system from the l.t.T. odd and even systems, has led us to study efficient methods for solving generic l.t.T. systems. Such methods are here explained in detail in case n, the number of equations, is a power of a generic positive integer.

1. Introduction

The j-th Bernoulli number, $B_{2j}(0)$, is a rational number defined for any $j \in \mathbb{N}$, positive if j is odd and negative if j is even, whose denominator is known, in the sense that it is the product of all prime numbers p such that $p-1$ divides $2j$ [11], and, instead, only partial information are known about the numerator [44], [21], [38]. Shortly, $B_{2j}(0)$, $j \geq 1$, could be defined by the well known Euler formula $B_{2j}(0) = (-1)^{j+1}(2(2j)!/((2\pi)^{2j}))\sum_{k=1}^{+\infty} 1/k^{2j}$, involving the Zeta-Riemann function [3], [14]. May be the latter formula alone is sufficient to justify the past and present interest in investigating Bernoulli numbers (B.n.). Note that an immediate consequence of the Euler formula is the fact that the $B_{2j}(0)$ go to infinite as j diverges.

In literature one finds several identities involving B.n., and also several "explicit" formulas for them, which may appear more explicit than Euler formula since involve finite (instead of infinite) sums [16], [27], [20], [9], [42], [24], [8], [1]. Some of such identities/formulas have been used to define algorithms for the computation of the numerators of the B.n.. It is however interesting to note that there are efficient algorithms for such computations which exploit directly the expression of the B.n. in terms of the Zeta-Riemann function [10], [31], [38], [18]. See also [2], [20], [32]. As it is noted in [27], the B.n. appear in several fields of mathematics; in particular, the numerators of the B.n. and their factors play an important role in advanced number theory (see [28], [29], [44], [30], [21]). So, wider and wider lists of the "first" B.n. have been and are compiled, and also lists of the known factors of their numerators. The updating of these lists requires the implementation of efficient primality-test/integer-factorization algorithms on powerful parallel computers. For instance, by this way the numerator of $B_{200}(0)$ first has been proved not prime, and then has been factorized as the product of five prime integers. Two of such factors, respectively of 90 and 115 digits, have been found only very recently [45], [35].

A lower triangular Toeplitz (l.t.T.) matrix A is a square matrix such that $a_{ij} = 0$ if $i < j$, and $a_{i,j} = a_{i+1,j+1}$, for all i, j. The product of two l.t.T. matrices whatever order is used generates the same matrix, and such matrix is l.t.T.. If A l.t.T. is non

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singular then its inverse is l.t.T., and thus is uniquely defined by its first column. Such remarks simply follow from the fact that the set of all l.t.T. matrices is nothing else than the set $\{p(Z)\}$ of all polynomials in the *lower-shift* matrix $Z = (\delta_{i,j+1})$, and the fact that $\{p(X)\}$ is, for any square matrix X, a commutative matrix algebra closed under inversion.

Note that, given a $n \times n$ l.t.T. matrix A, multiplying A by a vector ($\mathcal{M}$), or solving a system whose coefficient matrix is A ($\mathcal{S}$), are both operations that can be performed in at most $O(n \log n)$ arithmetic operations, thus in an amount of operations significantly smaller than, for example, the $n(n+1)/2$ multiplicative operations required by the standard algorithm for lower triangular (non Toeplitz) matrices. Such performances are possible by introducing alternative algorithms which exploit, first, the strict relationship between the Toeplitz structure and the discrete Fourier transform – which can be resumed in circulant matrices [13] –, and second, the fast implementation, known as FFT, of the latter. However, for ($\mathcal{M}$) and ($\mathcal{S}$) it is not so clear what is the best possible alternative algorithm. In particular, two well known procedures performing the multiplication l.t.T. matrix $\times$ vector hold unchanged if the l.t.T. is replaced by a generic (full) Toeplitz matrix; so one guesses that better algorithms may be introduced, ad hoc for the l.t.T. case. Analogously, a widely known exact algorithm able to solve l.t.T. systems (or, more precisely, to compute the first column of the inverse of a l.t.T. matrix) in at most $O(n \log n)$ a.o., has essentially a recursive character, which is not so convenient from the point of view of the space complexity [33]. In order to avoid such drawback, one could use approximation inverse algorithm [6], [25]. See also [41], [40], [39], [4], [26], [37], [25]. Note that improving the constant c in the complexity $cn \log n$ of l.t.T. solvers is the subject of recent researches, see for example [17].

In this paper we emphasize the connection (may be also noted elsewhere, see f.i. [15]) between Bernoulli numbers and lower triangular Toeplitz matrices. This connection will finally result into new possible algorithms for computing simultaneously the first n Bernoulli numbers. More precisely, in Section 3 we prove that the vector $\mathbf{z} = (\,B_{2j}(0)x^j/(2j)!\,)_{j=0}^{+\infty}$, $x \in \mathbb{R}$ ($B_0(0) = 1$), solves three *type I* l.t.T. semi-infinite linear systems $A\mathbf{x} = \mathbf{f}$, named *even*, *odd* and *Ramanujan*, respectively. To such systems correspond other three systems, of *type II*, solved by the vector $Z^T\mathbf{z} = (\,B_{2j}(0)x^j/(2j)!\,)_{j=1}^{+\infty}$. Type I ad II l.t.T. systems have been obtained as follows:

- Introducing/considering three particular lower triangular systems solved by Bernoulli numbers. The first two, which we may call *almost-even* and *almost-odd*, are introduced by exploiting a well known power series expansion involving Bernoulli polynomials. It is interesting to note that the coefficient matrices of such systems are particular submatrices of the l.t. Tartaglia-Pascal matrix. The third one, the *almost-Ramanujan* system, is deduced from the 11 equations, solved by the absolute values of the first 11 B.n., listed by Ramanujan in the paper [36].

- Noting that the almost-even, almost-odd, and almost-Ramanujan systems are structured in such a way that their coefficient matrices can be forced to be Toeplitz. This result follows, for the first two systems, from the matrix series representation of the Tartaglia-Pascal matrix in terms of powers of a kind of regularly weighted lower shift matrix, and, for the third one, by a clever remark proved in the 11×11 case, and conjectured in the general case.

- Proving that each of the three l.t.T. systems so obtained – even, odd and Ramanujan –, which is solved by $\mathbf{z}$ (or $Z^T\mathbf{z}$), can be manipulated so to define a correspondent l.t.T. system whose solution is $Z^T\mathbf{z}$ (or $\mathbf{z}$).

The lower triangular system written by Ramanujan in [36] has the remarkable peculiarity to have two null diagonals alternating the nonnull ones. The same peculiarity is inherited by its (lower triangular) Toeplitz version, obtained in this paper (see (16), (18), (19)). The attempt to obtain the l.t.T. Ramanujan system, and, more in general, sparse and simple l.t.T. systems solved by Bernoulli numbers, directly from the odd and even l.t.T. systems, has led us to study, using a simple matrix formulation, fast direct (not recursive) solvers of generic l.t.T. systems. In fact, the process of making null two diagonals every one, can be repeated, so to finally transform the initial l.t.T. into the identity matrix. Each step of such sort of Gaussian elimination procedure – where diagonals, instead of columns, are nullified – is realized by a left multiplication by a suitable l.t.T. matrix whose dimension is in fact 2/3 smaller than in the previous step. This leads to a $O(n \log_3 n)$ exact solver of l.t.T. systems $A\mathbf{x} = \mathbf{f}$ where A is $n \times n$ with $n = 3^s$, the $\mathcal{G}_3$ algorithm, and then to a general exact low complexity $\mathcal{G}_b$ procedure, ad hoc for the case $n = b^s$ with b generic. The algorithm $\mathcal{G}_b$, with $b = 2, 3$ and b generic, is illustrated in Section 4. Note that the first step of $\mathcal{G}_3$ can be skipped when solving the Ramanujan l.t.T. system, as, we may say, it has been already performed explicitly by Ramanujan. The simplicity of the algorithm $\mathcal{G}_b$ and, in particular, the clearness of its matrix formulation could suggest its consideration in the wide literature on low complexity l.t.T. solvers [6], [5], [25], [12], [7], [33], [34], [41], [22], [37], [17], [43].

2. Preliminaries on lower triangular Toeplitz (l.t.T.) matrices

Let Z be the following $n \times n$ matrix

$$Z = \begin{bmatrix} 0 & & & & \\ 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & 1 & 0 \end{bmatrix}.$$

Z is usually called *lower-shift* due to the effect that its multiplication by a vector $\mathbf{v} = [v_0\, v_1\, \cdots\, v_{n-1}]^T \in \mathbb{C}^n$ produces: $Z\mathbf{v} = [0\, v_0\, v_1\, \cdots\, v_{n-2}]^T$. Let $\mathcal{L}$ be the subspace of $\mathbb{C}^{n\times n}$ of those matrices which commute with Z. It is simple to observe that $\mathcal{L}$ is a matrix algebra closed under inversion, that is if $A, B \in \mathcal{L}$ then $AB \in \mathcal{L}$ and if $A \in \mathcal{L}$ is nonsingular then $A^{-1} \in \mathcal{L}$. Let us investigate the structure of the matrices in $\mathcal{L}$. Let $A \in \mathbb{C}^{n\times n}$. Then

$$AZ = \begin{bmatrix} a_{12} & \cdot & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n2} & \cdot & a_{nn} & 0 \end{bmatrix}, \quad ZA = \begin{bmatrix} 0 & \cdots & 0 \\ a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ a_{n-11} & \cdots & a_{n-1n} \end{bmatrix}.$$

Forcing the equality between AZ and ZA we obtain the conditions $a_{12} = a_{13} = \ldots a_{1n} = a_{2n} = \ldots a_{n-1,n} = 0$ and $a_{i,j+1} = a_{i-1,j}$, $i = 2, \ldots, n$, $j = 1, \ldots, n-1$, from which one deduces the structure of $A \in \mathcal{L}$: A must be a lower triangular Toeplitz (l.t.T.) matrix, i.e. a matrix of the type

$$A = \begin{bmatrix} a_{11} & & & & \\ a_{21} & a_{11} & & & \\ a_{31} & a_{21} & a_{11} & & \\ \cdot & & \cdot & \cdot & \\ a_{n1} & \cdot & \cdot & a_{21} & a_{11} \end{bmatrix}. \tag{1}$$

It follows that $\dim \mathcal{L} = n$ and that, by a well known general result [23], $\mathcal{L}$ can be represented as the set of all polynomials in Z, i.e. $\mathcal{L} = \{p(Z) : p = \text{polynomials}\}$.

Actually, by investigating the powers of Z one realizes that the matrix A in (1) is exactly the polynomial $\sum_{k=1}^{n} a_{k1} Z^{k-1}$.

Note also that, as a consequence of the above arguments, the inverse of a l.t.T. matrix is still l.t.T., thus it is completely determined as soon as its first column is known. In Section 4 we will illustrate a low complexity exact algorithm $\mathcal{G}_b$ for the solution of a l.t.T. linear system $A\mathbf{x} = \mathbf{f}$, $A \in \mathcal{L}$, $A_{11} = 1$, where $n = b^s$ with $b = 2, 3$ and b generic. The implementation of $\mathcal{G}_b$ requires the availability of a procedure able to perform a matrix-vector product where the matrix is l.t.T. $m \times m$ with $m = b^j$ $(j = 2, \ldots, s)$. More efficient is such procedure, lower is the complexity of the algorithm. So, let us briefly discuss the complexity of the computation l.t.T. matrix×vector.

The product of a $m \times m$ l.t.T. matrix times a vector can be computed with much less than the $m(m+1)/2$ multiplications and $(m-1)m/2$ additions required by the obvious algorithm. In fact, two alternative known procedures reduce the required computation to a small number of FFT of order m or bm, i.e., if $m = b^j$, to no more than $c_b j b^j$ arithmetic operations, where c_b is a constant. This result is obtained by using a) two representations of a generic Toeplitz matrix involving circulant and (-1)-circulant matrix algebras and b) the fact that the matrices of such algebras are simultaneously diagonalized by the discrete Fourier transform (DFT) or scalings of the DFT [13]. For the sake of completeness, here we briefly recall the two representations, by noting however that both *do not improve at all when the Toeplitz matrix is lower triangular.* It follows that lower complexity procedures, for the l.t.T. matrix by vector product computation, could exist and would be welcome.

Let $\Pi_{\pm 1}$ be the $m \times m$ matrix $\Pi_{\pm 1} = Z^T \pm \mathbf{e}_m \mathbf{e}_1^T$, where Z is the $m \times m$ lower-shift matrix. Given any vector $\mathbf{a}^T = [a_1\, a_2 \cdots a_m]$, the *circulant* and *skew circulant* matrices whose first row is $\mathbf{a}^T$, are, respectively,

$$\mathcal{C}(\mathbf{a}) = \sum_{k=1}^{m} a_k \Pi_1^{k-1} \quad \text{and} \quad \mathcal{C}_{-1}(\mathbf{a}) := \sum_{k=1}^{m} a_k \Pi_{-1}^{k-1}$$

Let $T = (t_{i-j})_{i,j=1}^{m}$, $t_k \in \mathbb{C}$, be a Toeplitz matrix, and let $\mathbf{v}$ be any vector of $\mathbb{C}^m$. The matrix T is the upper left submatrix of a $bm \times bm$ circulant matrix $C(\mathbf{a})$, i.e.

$$T = \{\mathcal{C}(\mathbf{a})\}_m, \ \mathbf{a}^T = [t_0\ t_{-1}\ \cdot\ t_{-m+1}\ \mathbf{0}^T_{(b-2)m+1}\ t_{m-1}\ \cdot\ t_1]. \tag{2}$$

Thus

$$T\mathbf{v} = \{\, C(\mathbf{a}) \begin{bmatrix} \mathbf{v} \\ \mathbf{0}_{(b-1)m} \end{bmatrix} \}_m,$$

where the symbol $\{\mathbf{z}\}_m$ denotes the $m \times 1$ vector whose entries are the first m components of $\mathbf{z}$. If, for example, m is a power of b $(b = 2, 3, \ldots)$, such equality allows to perform very efficiently the product $T\mathbf{v}$, by the tool of FFT. The Toeplitz matrix T can be alternatively represented as the sum of a circulant and a skew circulant matrix. In fact,

$$\begin{aligned} T &= C(\mathbf{a}) + C_{-1}(\mathbf{a}'), \\ & a_i = \frac{1}{2}(t_{-i+1} + t_{m-i+1}),\ a'_i = \frac{1}{2}(t_{-i+1} - t_{m-i+1}),\ i = 1, \ldots, m \end{aligned} \tag{3}$$

$(t_m = 0)$. Again, if $m = b^j$ $(b = 2, 3, \ldots)$, from this formula one immediately deduces a procedure computing $T\mathbf{v}$ in no more than $c_b j b^j$ arithmetic operations.

3. Bernoulli numbers and triangular Toeplitz matrices

The following three conditions

$$B(x+1)-B(x)=nx^{n-1},\quad \int_0^1 B(x)\,dx=0,\quad B(x)\text{ polinomio}$$

uniquely define the function $B(x)$. It is a particular degree n monic polynomial called *n-th Bernoulli polynomial* and usually denoted by the symbol $B_n(x)$. It is simple to compute the first Bernoulli polynomials:

$$B_1(x)=x-\frac{1}{2},\ B_2(x)=x(x-1)+\frac{1}{6},\ B_3(x)=x(x-\frac{1}{2})(x-1),\ \dots\ .$$

$B_0(x)$ is assumed equal to 1.

It can be proved that Bernoulli polynomials define the coefficients of the power series representation of several functions, for instance to our aim it is useful to recall that the following power series expansion holds:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{+\infty}\frac{B_n(x)}{n!}t^n. \tag{4}$$

Moreover, Bernoulli polynomials satisfy many identities. Among all we recall the following ones, concerning the value of their derivatives and their property of symmetry with respect to the line $x=\frac{1}{2}$:

$$B_n'(x)=nB_{n-1}(x),\quad B_n(1-x)=(-1)^nB_n(x).$$

It is simple to observe as a consequence of their definition and of the last identity that all the Bernoulli polynomials with odd degree (except $B_1(x)$) vanish for $x=0$. On the contrary, the value that an even degree Bernoulli polynomial attains in the origin is different from zero and especially important. In particular, recall the following Euler formula

$$\zeta(2j)=\frac{|B_{2j}(0)|(2\pi)^{2j}}{2(2j)!},\quad \zeta(s)=\sum_{k=1}^{+\infty}\frac{1}{k^s},$$

which shows the strict relation between the numbers $B_{2j}(0)$ and the values that the Riemann Zeta function $\zeta(s)$ attains over all even positive integer numbers $2j$ [14], [3]. For instance, from such relation and from the fact that $\zeta(2j)\to 1$ if $j\to+\infty$, one deduces that $|B_{2j}(0)|$ tends to $+\infty$ almost the same way as $2(2j)!/(2\pi)^{2j}$ does. Another important formula involving the values $B_{2j}(0)$ is the Euler-Maclaurin formula [14], which is useful in particular for the computation of sums: if f is a smooth enough function over $[m,n]$, $m,n\in\mathbb{Z}$, then

$$\begin{aligned}\sum_{r=m}^{n}f(r) &= \frac{1}{2}[f(m)+f(n)]+\int_m^n f(x)\,dx \\ &\quad+\sum_{j=1}^{k}\frac{B_{2j}(0)}{(2j)!}[f^{(2j-1)}(n)-f^{(2j-1)}(m)]+u_{k+1},\end{aligned} \tag{5}$$

where

$$\begin{aligned}u_{k+1}&=\frac{1}{(2k+1)!}\int_m^n f^{(2k+1)}(x)\overline{B}_{2k+1}(x)\,dx\\ &=-\frac{1}{(2k)!}\int_m^n f^{(2k)}(x)\overline{B}_{2k}(x)\,dx\\ &=\frac{1}{(2k+2)!}\int_m^n f^{(2k+2)}(x)[B_{2k+2}(0)-\overline{B}_{2k+2}(x)]\,dx\end{aligned}$$

and $\overline{B}_n$ is $B_n|_{[0,1)}$ extended periodically over $\mathbb{R}$. Let us recall that the Euler-Maclaurin formula also leads to an important representation of the error of the trapezoidal rule $\mathcal{I}_h = h[\frac{1}{2}g(a) + \sum_{r=1}^{n-1} g(a+rh) + \frac{1}{2}g(b)]$, $h = \frac{b-a}{n}$, in the approximation of the definite integral $\mathcal{I} = \int_a^b g(x)\,dx$. Such representation, holding for functions g which are smooth enough in $[a,b]$, is obtained by setting $m = 0$ and $f(t) = g(a+th)$ in (5):

$$\mathcal{I}_h = \mathcal{I} + \sum_{j=1}^{k} \frac{h^{2j} B_{2j}(0)}{(2j)!}[g^{(2j-1)}(b) - g^{(2j-1)}(a)] + r_{k+1}, \tag{6}$$
$$r_{k+1} = \frac{g^{(2k+2)}(\xi) h^{2k+2}(b-a) B_{2k+2}(0)}{(2k+2)!},$$

$\xi \in (a,b)$. The representation in (6) of the error, in terms of even powers of h, shows the reason why the Romberg extrapolation method for estimating a definite integral is efficient, when combined with trapezoidal rule. From (6) it is indeed clear that $\tilde{\mathcal{I}}_{h/2} := (2^2\mathcal{I}_{h/2} - \mathcal{I}_h)/(2^2-1)$ approximates $\mathcal{I}$ with an error of order $O(h^4)$, whereas the error made by $\mathcal{I}_h$ and $\mathcal{I}_{h/2}$ is of order $O(h^2)$.

For these and other applications/theories where the *Bernoulli numbers* $B_{2j}(0)$ are involved, see for instance [30], [28], [27], [14], [3].

3.1. **Bernoulli numbers solve triangular Toeplitz systems.** From (4) it follows that Bernoulli numbers satisfy the following identity

$$\frac{t}{e^t - 1} = -\frac{1}{2}t + \sum_{k=0}^{+\infty} \frac{B_{2k}(0)}{(2k)!} t^{2k}.$$

Multiplying the latter by $e^t - 1$, expanding e^t in terms of powers of t, and setting to zero the coefficients of t^i of the right hand side, $i = 2, 3, 4, \ldots$, yields the following equations:

$$-\frac{1}{2}j + \sum_{k=0}^{[\frac{j-1}{2}]} \binom{j}{2k} B_{2k}(0) = 0, \quad j = 2, 3, 4, \ldots \ . \tag{7}$$

Now, putting together equations (7) for j even and for j odd, we obtain two lower triangular linear systems that uniquely define Bernoulli numbers:

$$\begin{bmatrix} \binom{2}{0} & & & & \\ \binom{4}{0} & \binom{4}{2} & & & \\ \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & & \\ \binom{8}{0} & \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \cdot \end{bmatrix},$$

$$\begin{bmatrix} \binom{1}{0} & & & & \\ \binom{3}{0} & \binom{3}{2} & & & \\ \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & & \\ \binom{7}{0} & \binom{7}{2} & \binom{7}{4} & \binom{7}{6} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 5/2 \\ 7/2 \\ \cdot \end{bmatrix}.$$

So, we can for instance easily compute the first Bernoulli numbers:

$$1,\ \frac{1}{6},\ -\frac{1}{30},\ \frac{1}{42},\ -\frac{1}{30},\ \frac{5}{66},\ -\frac{691}{2730},\ \frac{7}{6},\ -\frac{3617}{510}. \tag{8}$$

The coefficients matrices W_e and W_o of such linear systems turn out to have an analytic representation. In order to prove this fact, it is enough to observe that W_e and W_o are suitable submatrices of the Tartaglia-Pascal matrix X, which can be represented as a power series. More precisely, set

$$Y = Z \cdot \text{diag}\big(i : i = 1, 2, 3, \ldots\big), \quad \phi = Z \cdot \text{diag}\big((2i-1)2i : i = 1, 2, 3, \ldots\big)$$

where Z is the semi-infinite lower shift matrix $Z = (\delta_{i,j+1})_{i,j=1}^{+\infty}$, and note that for all i, j, $1 \le j \le i \le n$, we have

$$[\sum_{k=0}^{+\infty} \frac{1}{k!} Y^k]_{ij} = \frac{1}{(i-j)!}[Y^{i-j}]_{ij} = \frac{1}{(i-j)!} j \cdots (i-2)(i-1) = \binom{i-1}{j-1},$$

or, equivalently,

$$X := \begin{bmatrix} \binom{0}{0} \\ \binom{\mathit{1}}{\mathit{0}} & \binom{1}{1} \\ \binom{\mathbf{2}}{\mathbf{0}} & \binom{2}{1} & \binom{2}{2} \\ \binom{\mathit{3}}{\mathit{0}} & \binom{3}{1} & \binom{\mathit{3}}{\mathit{2}} & \binom{3}{3} \\ \binom{\mathbf{4}}{\mathbf{0}} & \binom{4}{1} & \binom{\mathbf{4}}{\mathbf{2}} & \binom{4}{3} & \binom{4}{4} \\ \binom{\mathit{5}}{\mathit{0}} & \binom{5}{1} & \binom{\mathit{5}}{\mathit{2}} & \binom{5}{3} & \binom{\mathit{5}}{\mathit{4}} & \binom{5}{5} \\ \binom{\mathbf{6}}{\mathbf{0}} & \binom{6}{1} & \binom{\mathbf{6}}{\mathbf{2}} & \binom{6}{3} & \binom{\mathbf{6}}{\mathbf{4}} & \binom{6}{5} & \binom{6}{6} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \sum_{k=0}^{+\infty} \frac{1}{k!} Y^k.$$

Then, simple investigations allows one to observe that W_e and W_o, which are the l.t. matrices obtained by putting together, respectively, the bold and italic binomial entries of X, can be represented in terms of power series in the matrix ϕ, i.e.

$$W_e = Z^T \phi \cdot \sum_{k=0}^{+\infty} \frac{1}{(2k+2)!} \phi^k, \quad W_o = \psi \cdot \sum_{k=0}^{+\infty} \frac{1}{(2k+1)!} \phi^k,$$

where $\psi = \text{diag}\big(2i-1 : i = 1, 2, 3, \ldots\big)$. We can therefore rewrite the two linear systems solved by Bernoulli numbers as follows:

$$\sum_{k=0}^{+\infty} \frac{2}{(2k+2)!} \phi^k \mathbf{b} = \mathbf{q}^e, \quad \mathbf{b} = \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ \cdot \end{bmatrix}, \; \mathbf{q}^e = \begin{bmatrix} 1 \\ 1/3 \\ 1/5 \\ 1/7 \\ \cdot \end{bmatrix}, \tag{9}$$

$$\sum_{k=0}^{+\infty} \frac{1}{(2k+1)!} \phi^k \mathbf{b} = \mathbf{q}^o, \quad \mathbf{q}^o = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ 1/2 \\ \cdot \end{bmatrix}. \tag{10}$$

Now, let us show that systems (9) and (10) are equivalent to two l.t.T. linear systems. Our aim is to replace ϕ, a matrix whose subdiagonal entries are all different, by a matrix whose subdiagonal entries are all equal.

Set $D = \text{diag}\,(d_1, d_2, d_3, \ldots)$, $d_i \neq 0$. By investigating the nonzero entries of the matrix $D\phi D^{-1}$, it is easy to observe that it can be forced to be equal to a matrix of the form xZ; just choose $d_k = x^{k-1}d_1/(2k-2)!$, $k = 1, 2, 3, \ldots$. So, if

$$D = \operatorname{diag}\big(1, \frac{x}{2!}, \frac{x^2}{4!}, \cdot\,, \frac{x^{n-1}}{(2n-2)!}, \cdot\,\big), \tag{11}$$

we have the equality $D\phi D^{-1} = xZ$.

Now, since $D\phi^k D^{-1} = (D\phi D^{-1})^k = x^k Z^k$, it is easy to show the equivalence of (9) and (10) with the following l.t.T. linear systems:

$$\sum_{k=0}^{+\infty} \frac{2x^k}{(2k+2)!} Z^k D\mathbf{b} = D\mathbf{q}^e, \tag{12}$$

$$\sum_{k=0}^{+\infty} \frac{x^k}{(2k+1)!} Z^k D\mathbf{b} = D\mathbf{q}^o. \tag{13}$$

Summarizing, let $\mathbf{z}$ be the vector $D\mathbf{b}$. Then the vector $\{\mathbf{b}\}_n$ whose entries are the first n Bernoulli numbers can be obtained by a two-phase procedure:

1. compute the first n components of the solution of the l.t.T. system (12) ((13)), i.e. compute $\{\mathbf{z}\}_n$ such that

$$\{\sum_{k=0}^{+\infty} \frac{2x^k}{(2k+2)!} Z^k\}_n \{\mathbf{z}\}_n = \{D\mathbf{q}^e\}_n \quad \Big(\{\sum_{k=0}^{+\infty} \frac{x^k}{(2k+1)!} Z^k\}_n \{\mathbf{z}\}_n = \{D\mathbf{q}^o\}_n\Big)$$

2. solve the linear system $\{D\}_n\{\mathbf{b}\}_n = \{\mathbf{z}\}_n$ over the rational field.

The computation in phase 1 can be done by means of the algorithm described in the next Section 4 at a computational cost of $O(n \log n)$ arithmetic operations. It is important to note that such algorithm can be made numerically stable by a suitable choice of the parameter x. For instance, the choice $x = (2\pi)^2$ would ensure the sequence $\{z_n\} = \{x^n B_{2n}(0)/(2n)!\}$ to be bounded; indeed in this case $|z_n| \to 2$ if $n \to +\infty$, due to Euler formula. So, in phase 1, one obtains n machine numbers which are good approximations in $\mathbb{R}$ of the quantities $x^i B_{2i}(0)/(2i)!$, $i = 0, 1, \ldots, n-1$. Then, in phase 2, one should find a way to deduce, from the machine numbers obtained, the rational Bernoulli numbers $B_{2i}(0)$, $i = 0, 1, \ldots, n-1$.

3.2. **The Ramanujan l.t.T. linear system solved by Bernoulli numbers.** In [36] Ramanujan writes explicitly 11 sparse equations solved by the absolute values of the 11 Bernoulli numbers $B_2(0)$, $B_4(0)$, …, $B_{22}(0)$. They are the first of an infinite set of sparse equations solved by the absolute values of all the Bernoulli numbers. The Ramanujan equations, written together and directly in terms of the $B_{2i} := B_{2i}(0)$, $i = 1, 2, \ldots, 11$, form the linear system displayed here below:

$$\begin{bmatrix} 1 &&&&&&&&&&& \\ 0 & 1 &&&&&&&&&& \\ 0 & 0 & 1 &&&&&&&&& \\ \frac{1}{3} & 0 & 0 & 1 &&&&&&&& \\ 0 & \frac{5}{2} & 0 & 0 & 1 &&&&&&& \\ 0 & 0 & 11 & 0 & 0 & 1 &&&&&& \\ \frac{1}{5} & 0 & 0 & \frac{143}{4} & 0 & 0 & 1 &&&&& \\ 0 & 4 & 0 & 0 & \frac{286}{3} & 0 & 0 & 1 &&&& \\ 0 & 0 & \frac{204}{5} & 0 & 0 & 221 & 0 & 0 & 1 &&& \\ \frac{1}{7} & 0 & 0 & \frac{1938}{7} & 0 & 0 & \frac{3230}{7} & 0 & 0 & 1 && \\ 0 & \frac{11}{2} & 0 & 0 & \frac{7106}{5} & 0 & 0 & \frac{3553}{4} & 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} B_2 \\ B_4 \\ B_6 \\ B_8 \\ B_{10} \\ B_{12} \\ B_{14} \\ B_{16} \\ B_{18} \\ B_{20} \\ B_{22} \\ \cdot \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{30} \\ \frac{1}{42} \\ \frac{1}{45} \\ -\frac{1}{132} \\ \frac{4}{455} \\ \frac{1}{120} \\ -\frac{1}{306} \\ \frac{3}{665} \\ \frac{1}{231} \\ -\frac{1}{552} \\ \cdot \end{bmatrix}.$$

For example, by using the last but three of Ramanujan equations, from the Bernoulli numbers $B_2(0), \dots, B_{16}(0)$ listed in (8), the following further Bernoulli numbers can be easily obtained:

$$B_{18}(0) = \frac{43867}{798}, \quad B_{20}(0) = -\frac{174611}{330}, \quad B_{22}(0) = \frac{854513}{138}.$$

Let R be the semi-infinite coefficient matrix of the above Ramanujan system. By recalling the definition of the semi-infinite lower shift matrix Z and of the semi-infinite vector $\mathbf{b} = [B_0(0)\, B_2(0)\, B_4(0)\, \cdot\,]^T$, the Ramanujan system can be shortly indicated as

$$R(Z^T\mathbf{b}) = Z^T\mathbf{f}\,, \tag{14}$$

where $Z^T\mathbf{f}$, $\mathbf{f} = [f_0\, f_1\, f_2\, f_3\, \cdot\,]^T$, denotes the right hand side vector, i.e. $f_1 = 1/6$, $f_2 = -1/30$, $f_3 = 1/42$, $f_4 = 1/45, \dots$.

Apparently the non-zero entries of R are not related with each other, and it seems so also for the entries of $\mathbf{f}$. That is, it seems to be not possible to guess, just by looking at the above 11 equations, the twelfth equation of the Ramanujan system. We can only guess that the non-zero entries of R are in the same positions as the non-zero entries of a l.t.T. matrix $\tilde{R}$ of the form $\sum_{k=0}^{+\infty} w_k Z^{3k}$, and, maybe, it is possible to guess the sign of the entries of $\mathbf{f}$.

Actually, by a clever remark we noted that the following identity *must hold*

$$R\Lambda^{-1} = \Lambda^{-1}\tilde{R}, \quad \Lambda = Z^T D Z = \operatorname{diag}\big(\frac{x^i}{(2i)!} : i = 1,2,3,\dots\big), \tag{15}$$

where D is defined in (11) and $\tilde{R}$ is the following l.t.T. matrix:

$$\tilde{R} = \sum_{k=0}^{+\infty} \frac{2x^{3k}}{(6k+2)!(2k+1)} Z^{3k}$$

$$= \begin{bmatrix}
1 \\
0 & 1 \\
0 & 0 & 1 \\
\frac{2x^3}{8!3} & 0 & 0 & 1 \\
0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\
0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\
\frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\
0 & \frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\
0 & 0 & \frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\
\frac{2x^9}{20!7} & 0 & 0 & \frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\
0 & \frac{2x^9}{20!7} & 0 & 0 & \frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}.$$

In fact, the 11×11 upper left submatrix of $R\Lambda^{-1}$ coincides with the 11×11 upper left submatrix of $\Lambda^{-1}\tilde{R}$.

Assuming that the conjecture (15) is true, we have the equalities $R(Z^T\mathbf{b}) = R\Lambda^{-1}(\Lambda Z^T\mathbf{b}) = \Lambda^{-1}\tilde{R}(Z^T D\mathbf{b})$, and thus, by (14),

$$\tilde{R}(Z^T D\mathbf{b}) = Z^T D\mathbf{f}. \tag{16}$$

Hence, the vector $Z^T D\mathbf{b}$ solves a l.t.T. system which, differently from the l.t.T. systems (12), (13), is sparse, since in its coefficient matrix two null diagonals alternate the nonnull ones. Such Ramanujan l.t.T. system will be defined more precisely in the following (see (18), (19)).

3.3. **A unifying theorem with 6 l.t.T. linear systems solved by Bernoulli numbers.** Now we collect in a Theorem three l.t.T. linear systems solved by the vector $D_x\mathbf{b}$, say of type I, and the corresponding l.t.T. linear systems solved by the vector $Z^T D_x\mathbf{b}$, say of type II (D_x is the matrix D in (11)). In fact, till now, we have only found two systems of type I, the even and odd systems (12) and (13), and, partially, one system of type II, the Ramanujan l.t.T. system (16) (note that for the latter system only the coefficient matrix has been written explicitly).

In the following, first we state a Proposition which allows one to state a system of type II from a system of type I, and viceversa. Then we state the Theorem, with the six l.t.T. linear systems solved by Bernoulli numbers, and we prove it by applying the Proposition to the even, odd, and Ramanujan l.t.T. systems found till now, and, in the same time, by completing the definition of the Ramanujan l.t.T. system.

Proposition 3.1 Let Z_{n-1} and Z_n be the upper-left $(n-1)\times(n-1)$ and $n\times n$ submatrices of the semi-infinite lower-shift matrix Z, respectively. Set $\beta = B_0(0)$. Assume that, for some α_j and f_j (or w_j), the following equality holds:

$$\sum_{j=0}^{n-1}\alpha_j Z_n^j\{D_x\mathbf{b}\}_n = \begin{bmatrix} \eta \\ \frac{x}{2!}f_1 \\ \frac{x^2}{4!}f_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}f_{n-1} \end{bmatrix} + \beta \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_{n-1} \end{bmatrix} \quad \Big(= \begin{bmatrix} w_0 \\ \frac{x}{2!}w_1 \\ \frac{x^2}{4!}w_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}w_{n-1} \end{bmatrix} \Big),$$

where η and μ are arbitrary parameters. Then we have

$$\sum_{j=0}^{n-2}\alpha_j Z_{n-1}^j\{Z^T D_x\mathbf{b}\}_{n-1} = \begin{bmatrix} \frac{x}{2!}f_1 \\ \frac{x^2}{4!}f_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}f_{n-1} \end{bmatrix} \quad \Big(= \begin{bmatrix} \frac{x}{2!}w_1 \\ \frac{x^2}{4!}w_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}w_{n-1} \end{bmatrix} - \beta \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_{n-1} \end{bmatrix} \Big).$$

Also the converse is true provided that $\alpha_0\beta = \eta + \beta\mu$ (or $\alpha_0\beta = w_0$).

Proof. Exploit the equality $Z_n = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{e}_1 & Z_{n-1} \end{bmatrix}$. The details are left to the reader. □

Theorem 3.2 Set $Z = (\delta_{i,j+1})_{i,j=1}^{+\infty}$, $\mathbf{a} = (a_i)_{i=0}^{+\infty}$, and $L(\mathbf{a}) = \sum_{i=0}^{+\infty} a_i Z^i$. Let $d(\mathbf{w})$ be the diagonal matrix with w_i as diagonal entries. Set

$$\mathbf{b} = [B_0(0)\ B_2(0)\ B_4(0)\ \cdot\ ]^T, \quad D_x = \operatorname{diag}\big(\frac{x^i}{(2i)!} : i = 0,1,2,\dots\big),\ x \in \mathbb{R},$$

where $B_{2i}(0)$, $i = 0,1,2,\dots$, are the Bernoulli numbers. Then *the vectors* $D_x\mathbf{b}$ *and* $Z^T D_x\mathbf{b}$ *solve the following l.t.T. linear systems*

$$L(\mathbf{a})\,(D_x\mathbf{b}) = D_x\mathbf{q}, \tag{17}$$

$$L(\mathbf{a})\,(Z^T D_x\mathbf{b}) = d(\mathbf{w}) Z^T D_x\mathbf{q}, \tag{18}$$

where the vectors $\mathbf{a} = (a_i)_{i=0}^{+\infty}$, $\mathbf{q} = (q_i)_{i=0}^{+\infty}$, and $\mathbf{w} = (w_i)_{i=1}^{+\infty}$, can assume respectively the values:

$$\begin{aligned} a_i^R &= \delta_{i=0 \bmod 3}\frac{2x^i}{(2i+2)!(\frac{2}{3}i+1)},\ i = 0,1,2,3,\dots \\ q_i^R &= \frac{1}{(2i+1)(i+1)}(1-\delta_{i=2 \bmod 3}\frac{3}{2}),\ i = 0,1,2,3,\dots \\ w_i^R &= 1-\delta_{i=0 \bmod 3}\frac{1}{\frac{2}{3}i+1},\ i = 1,2,3,\dots, \end{aligned} \tag{19}$$

(20)
$$a_i^e = \frac{2x^i}{(2i+2)!},\ i=0,1,2,3,\dots,\quad q_i^e = \frac{1}{2i+1},\ i=0,1,2,3,\dots$$
$$w_i^e = \frac{i}{i+1},\ i=1,2,3,\dots,$$

(21)
$$a_i^o = \frac{x^i}{(2i+1)!},\ i=0,1,2,3,\dots,\ q_0^o=1,\ q_i^o=\frac{1}{2},\ i=1,2,3,\dots$$
$$w_i^o = \frac{2i-1}{2i+1},\ i=1,2,3,\dots\ .$$

Proof. From the Ramanujan semi-infinite l.t.T. linear system (16), we obtain the following finite linear system

(22)
$$\sum_{j=0}^{n-2}\alpha_j Z_{n-1}^j\{Z^T D_x\mathbf{b}\}_{n-1} = \begin{bmatrix} \frac{x}{2!}f_1 \\ \frac{x^2}{4!}f_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}f_{n-1}\end{bmatrix},$$
$$\alpha_j = \delta_{j=0 \bmod 3}\frac{2x^j}{(2j+2)!(\frac{2}{3}j+1)},$$

$$f_1=\tfrac{1}{6},\ f_2=-\tfrac{1}{30},\ f_3=\tfrac{1}{42},\ f_4=\tfrac{1}{45},\ f_5=-\tfrac{1}{132},\ f_6=\tfrac{4}{455},$$
$$f_7=\tfrac{1}{120},\ f_8=,-\tfrac{1}{306}\ f_9=\tfrac{3}{665},\ f_{10}=\tfrac{1}{231},\ f_{11}=-\tfrac{1}{552},\dots\ .$$

Then, by Proposition 3.1, if $\eta + B_0(0)\mu = \alpha_0 B_0(0)$, we have that

$$\sum_{j=0}^{n-1}\alpha_j Z_n^j\{D_x\mathbf{b}\}_n = \begin{bmatrix}\eta \\ \frac{x}{2!}f_1 \\ \frac{x^2}{4!}f_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}f_{n-1}\end{bmatrix} + B_0(0)\begin{bmatrix}\mu \\ \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_{n-1}\end{bmatrix},$$

or, more precisely, that

$$(I+\frac{2}{8!3}x^3Z^3+\frac{2}{14!5}x^6Z^6+\frac{2}{20!7}x^9Z^9+\dots)\begin{bmatrix}\left(\frac{x^i}{(2i)!}B_{2i}(0)\right)_{i=0}^{11} \\ \cdot\end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ \frac{x}{2!}\frac{1}{6} \\ \frac{x^2}{4!}(-\frac{1}{30}) \\ \frac{x^3}{6!}\frac{1}{42}+\frac{2x^3}{8!3} \\ \frac{x^4}{8!}\frac{1}{45} \\ \frac{x^5}{10!}(-\frac{1}{132}) \\ \frac{x^6}{12!}\frac{4}{455}+\frac{2x^6}{14!5} \\ \frac{x^7}{14!}\frac{1}{120} \\ \frac{x^8}{16!}(-\frac{1}{306}) \\ \frac{x^9}{18!}\frac{3}{665}+\frac{2x^9}{20!7} \\ \frac{x^{10}}{20!}\frac{1}{231} \\ \frac{x^{11}}{22!}(-\frac{1}{552}) \\ \cdot \end{bmatrix} = \begin{bmatrix} \frac{1}{0!}(\frac{1}{1\cdot 1}) \\ \frac{x}{2!}(\frac{1}{3\cdot 2}) \\ \frac{x^2}{4!}(\frac{1}{5\cdot 3}-\frac{1}{5\cdot 2}) \\ \frac{x^3}{6!}(\frac{1}{7\cdot 4}) \\ \frac{x^4}{8!}(\frac{1}{9\cdot 5}) \\ \frac{x^5}{10!}(\frac{1}{11\cdot 6}-\frac{1}{11\cdot 4}) \\ \frac{x^6}{12!}(\frac{1}{13\cdot 7}) \\ \frac{x^7}{14!}(\frac{1}{15\cdot 8}) \\ \frac{x^8}{16!}(\frac{1}{17\cdot 9}-\frac{1}{17\cdot 6}) \\ \frac{x^9}{18!}(\frac{1}{19\cdot 10}) \\ \frac{x^{10}}{20!}(\frac{1}{21\cdot 11}) \\ \frac{x^{11}}{22!}(\frac{1}{23\cdot 12}-\frac{1}{23\cdot 8}) \\ \cdot \end{bmatrix}.$$

The latter equality is a clever remark that allows us to guess that $D_x\mathbf{b}$ must solve the following Ramanujan l.t.T. system of type I:

(23)
$$\sum_{i=0}^{+\infty}\alpha_i Z^i D_x\mathbf{b} = D_x\mathbf{q}^R,\ \alpha_i = a_i^R,$$

with a_i^R and q_i^R, $i = 0, 1, 2, 3, \ldots$, defined as in (19). Note that from the explicit expression of $\mathbf{q}^R$ just obtained, it follows an explicit expression for *all* the entries f_i of the original Ramanujan system (14), i.e. also for the f_i with $i \geq 12$

$$f_i = \frac{1}{(2i+1)(i+1)}(1 - \delta_{i=2 \bmod 3}\frac{3}{2} - \delta_{i=0 \bmod 3}\frac{1}{\frac{2}{3}i+1}), \quad i = 1, 2, 3, \ldots.$$

Note also that (22) can be rewritten as

$$\sum_{j=0}^{n-2} \alpha_j Z_{n-1}^j \{Z^T D_x \mathbf{b}\}_{n-1} = \operatorname{diag}(w_i,\ i = 1, 2, \ldots, n-1)\{Z^T D_x \mathbf{q}^R\}_{n-1}$$

for suitable w_i. Such w_i are easily obtained by imposing the equality

$$(1 - \delta_{i=2 \bmod 3}\frac{3}{2} - \delta_{i=0 \bmod 3}\frac{1}{\frac{2}{3}i+1}) = w_i(1 - \delta_{i=2 \bmod 3}\frac{3}{2}),$$

which leads to the formula:

$$w_i = 1 - \frac{\delta_{i=0 \bmod 3}\frac{1}{\frac{2}{3}i+1}}{1 - \delta_{i=2 \bmod 3}\frac{3}{2}} = 1 - \delta_{i=0 \bmod 3}\frac{1}{\frac{2}{3}i+1}.$$

So, the l.t.T. type I and type II systems (17), (18) and (19) hold.
Now let us consider the finite versions of the even and odd systems (12) and (13),

$$\sum_{j=0}^{n-1} \frac{2x^j}{(2j+2)!} Z_n^j \{D_x \mathbf{b}\}_n = \{D_x \mathbf{q}^e\}_n, \quad \sum_{j=0}^{n-1} \frac{x^j}{(2j+1)!} Z_n^j \{D_x \mathbf{b}\}_n = \{D_x \mathbf{q}^o\}_n,$$

and apply to them Proposition 3.1:

$$\sum_{j=0}^{n-2} \frac{2x^j}{(2j+2)!} Z_{n-1}^j \{Z^T D_x \mathbf{b}\}_{n-1}$$

$$= \begin{bmatrix} \frac{x}{2!}\frac{1}{3} \\ \frac{x^2}{4!}\frac{1}{5} \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}\frac{1}{2n-1} \end{bmatrix} - B_0(0) \begin{bmatrix} \frac{2x}{4!} \\ \frac{2x^2}{6!} \\ \cdot \\ \frac{2x^{n-1}}{(2n)!} \end{bmatrix} = \begin{bmatrix} x\frac{2}{4!} \\ x^2\frac{4}{6!} \\ x^3\frac{6}{8!} \\ \cdot \\ x^{n-1}\frac{2(n-1)}{(2n)!} \end{bmatrix},$$

$$\sum_{j=0}^{n-2} \frac{x^j}{(2j+1)!} Z_{n-1}^j \{Z^T D_x \mathbf{b}\}_{n-1}$$

$$= \begin{bmatrix} \frac{x}{2!}\frac{1}{2} \\ \frac{x^2}{4!}\frac{1}{2} \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}\frac{1}{2} \end{bmatrix} - B_0(0) \begin{bmatrix} \frac{x}{3!} \\ \frac{x^2}{5!} \\ \cdot \\ \frac{x^{n-1}}{(2n-1)!} \end{bmatrix} = \begin{bmatrix} x\frac{1}{3!2} \\ x^2\frac{3}{5!2} \\ x^3\frac{5}{7!2} \\ \cdot \\ x^{n-1}\frac{2n-3}{(2n-1)!2} \end{bmatrix}.$$

From the above identities it follows that

$$\sum_{j=0}^{n-2} \frac{2x^j}{(2j+2)!} Z_{n-1}^j \{Z^T D_x \mathbf{b}\}_{n-1} = 2 \begin{bmatrix} \frac{x}{2!}\frac{1}{4\cdot 3} \\ \frac{x^2}{4!}\frac{2}{6\cdot 5} \\ \frac{x^3}{6!}\frac{3}{8\cdot 7} \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}\frac{n-1}{2n(2n-1)} \end{bmatrix}$$

$$= \operatorname{diag}(\frac{i}{i+1}, i = 1 \ldots n-1)\{Z^T D_x \mathbf{q}^e\}_{n-1},$$

$$\sum_{j=0}^{n-2} \frac{x^j}{(2j+1)!} Z_{n-1}^j \{Z^T D_x \mathbf{b}\}_{n-1} = \begin{bmatrix} \frac{x}{2!}\frac{1}{2\cdot 3} \\ \frac{x^2}{4!}\frac{3}{2\cdot 5} \\ \frac{x^3}{6!}\frac{5}{2\cdot 7} \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}\frac{2n-3}{2(2n-1)} \end{bmatrix}$$
$$= \operatorname{diag}\left(\frac{2i-1}{2i+1}, i = 1 \dots n-1\right)\{Z^T D_x \mathbf{q}^o\}_{n-1}.$$

So, also even and odd type II linear systems (18), (20) and (21) hold. □

Remark. From (17) of Theorem 3.2 we have the following formula for $B_{2n}(0)$:

$$B_{2n}(0) = \frac{(2n)!}{x^n}(J\{L(\mathbf{a})\}_{n+1}^{-1}\mathbf{e}_1)^T\{D\mathbf{q}\}_{n+1}, \tag{24}$$

where $J = (\delta_{i,n+2-j})_{i,j=1}^{n+1}$ is the anti-identity. So, the computation of $\{L(\mathbf{a})\}_{n+1}^{-1}\mathbf{e}_1$ (for example via the $\mathcal{G}_b$ procedure in subsection 4.5) yields $B_{2n}(0)$.

Now it is clear that the first n Bernoulli numbers $B_{2i}(0)$, unless the factors $x^i/(2i)!$, solve l.t.T. systems $A\mathbf{x} = \mathbf{f}$, where A is the $n \times n$ upper left submatrix of the semi-infinite matrix $L(\mathbf{a})$ in (17) (or (18)). Throughout the next Section 4 we describe an algorithm that can be used to compute the products $(x^i/(2i)!)B_{2i}(0)$.

4. An algorithm for the solution of l.t.T. linear systems

In this section we first present an algorithm $\mathcal{G}_2$ of complexity $O(n \log_2 n)$ for the computation of $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{f}$, where A is a $n \times n$ l.t.T. matrix, with $[A]_{11} = 1$ (which of course is not restrictive) and $n = 2^s$ for some $s \in \mathbb{N}$. Then, in subsection 4.5, we illustrate the algorithm $\mathcal{G}_b$, which solves the more general case $n = b^s$, where b is a generic positive integer.

4.1. **Preliminary Lemmas.** Given a vector $\mathbf{v} = [v_0\ v_1\ v_2\ \cdots]^T$, $v_i \in \mathbb{C}$ (briefly $\mathbf{v} \in \mathbb{C}^{\mathbb{N}}$), let $L(\mathbf{v})$ be the semi-infinite l.t.T. matrix whose first column is $\mathbf{v}$, i.e.

$$L(\mathbf{v}) = \sum_{k=0}^{+\infty} v_k Z^k, \quad Z = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \cdot & \cdot \end{bmatrix}.$$

Lemma 4.1 Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{C}^{\mathbb{N}}$. Then $L(\mathbf{a})L(\mathbf{b}) = L(\mathbf{c})$ if and only if $L(\mathbf{a})\mathbf{b} = \mathbf{c}$.

Proof. If $L(\mathbf{a})L(\mathbf{b}) = L(\mathbf{c})$, then the first column of $L(\mathbf{a})L(\mathbf{b})$ must be equal to the first column of $L(\mathbf{c})$, and these are the vectors $L(\mathbf{a})\mathbf{b}$ and $\mathbf{c}$, respectively. Conversely, assume that $L(\mathbf{a})\mathbf{b} = \mathbf{c}$ and consider the matrix $L(\mathbf{a})L(\mathbf{b})$. It is l.t.T., being a product of l.t.T matrices, and, by hypothesis, its first column, $L(\mathbf{a})\mathbf{b}$, coincides with the vector $\mathbf{c}$, which in turn is the first column of the l.t.T. matrix $L(\mathbf{c})$. The thesis follows from the fact that l.t.T. matrices are uniquely defined by their first columns. □

Given a vector $\mathbf{v} = [v_0\, v_1\, v_2\, \cdots]^T \in \mathbb{C}^{\mathbb{N}}$, let E be the semi-infinite matrix with entries 0 or 1, which maps $\mathbf{v}$ into the vector $E\mathbf{v} = [v_0\, 0\, v_1\, 0\, v_2\, 0\, \cdots]^T$:

$$E = \begin{bmatrix} 1 & & & \\ 0 & & & \\ 0 & 1 & & \\ 0 & 0 & & \\ 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

In other words, the application of E to $\mathbf{v}$ has the effect of inserting a zero between two consecutive components of $\mathbf{v}$. It is easy to observe that

$$E^2 = \begin{bmatrix} 1 & & & \\ 0 & & & \\ 0 & & & \\ 0 & & & \\ 0 & 1 & & \\ 0 & 0 & & \\ 0 & 0 & & \\ 0 & 0 & & \\ 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad E^k = \begin{bmatrix} 1 & & & \\ \mathbf{0} & & & \\ 0 & 1 & & \\ \mathbf{0} & \mathbf{0} & & \\ 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \mathbf{0} = \mathbf{0}_{2^k-1},$$

that is the application of E^k to $\mathbf{v}$ has the effect of inserting $2^k - 1$ zeros between two consecutive components of $\mathbf{v}$.

Lemma 4.2 Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{\mathbb{N}}$ with $u_0 = v_0 = 1$. Then $L(E\mathbf{u})E\mathbf{v} = EL(\mathbf{u})\mathbf{v}$, and, more in general, for each $k \in \mathbb{N}$, $L(E^k\mathbf{u})E^k\mathbf{v} = E^kL(\mathbf{u})\mathbf{v}$.

Proof. By inspecting the vectors $L(E\mathbf{u})E\mathbf{v}$ and $EL(\mathbf{u})\mathbf{v}$ one observes that they are equal. By multiplying E on the left of the identity $L(E\mathbf{u})E\mathbf{v} = EL(\mathbf{u})\mathbf{v}$ and using the same identity also for the vectors $E\mathbf{u}$ and $E\mathbf{v}$, in place of $\mathbf{u}$ and $\mathbf{v}$ respectively, one observes that it also holds $L(E^2\mathbf{u})E^2\mathbf{v} = E^2L(\mathbf{u})\mathbf{v}$. And so on. $\square$

4.2. **The algorithm $\mathcal{G}_2$.** Let A be a $n \times n$ l.t.T. matrix, with $n = 2^s$ and $[A]_{11} = 1$. Assume we want to solve the system $A\mathbf{x} = \mathbf{f}$, $\mathbf{f} \in \mathbb{C}^n$. The algorithm presented below exploits the fact that A^{-1} is still a $n \times n$ l.t.T. matrix.

1. Compute the first column of the l.t.T. matrix A^{-1} by solving the particular linear system $A\mathbf{x} = \mathbf{e}_1$ via the $\mathcal{G}_2$ procedure of complexity $O(n \log_2 n)$ described in the next subsection 4.3, based upon Lemmas 4.1, 4.2 and their repeated application.
2. Use one of the representations (2), (3) (with $b = 2$) for the l.t.T. matrix A^{-1} to compute the matrix-vector product $A^{-1}\mathbf{f}$ with no more than $O(n \log_2 n)$ arithmetic operations.

4.3. **The computation of the first column of the inverse of a $n \times n$ l.t.T. matrix ($n = 2^s$).** The $\mathcal{G}_2$ procedure for the computation of $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{e}_1$ consists of two parts. In the first one, certain particular l.t.T. matrices, with the property that their successive left multiplication by the matrix A transforms A into the the identity matrix, are introduced and computed. In the second part such matrices are successively left multiplied by the vector $\mathbf{e}_1$. As it will be clear throughout what follows, the method is nothing more than a kind of Gaussian elimination, where diagonals are nullified instead of columns. The overall cost $O(n \log_2 n)$ comes from the fact that at each step of the first part a half of the remaining non null diagonals are nullified, and from the fact that in the second part the computations can be simplified by exploiting the fact that $\mathbf{e}_1$ has only one nonzero component.

The $\mathcal{G}_2$ procedure is shown in the particular case $n = 8$. When suitable it is briefly discussed what changes in the case $n = 2^s$, $s \in \mathbb{N}$; nevertheless such case can be easily deduced from the considered one, or by setting $b = 2$ in the general $\mathcal{G}_b$ procedure, reported in subsection 4.5.

First of all observe that the 8×8 matrix A can be thought as the upper-left submatrix of a semi-infinite l.t.T. matrix $L(\mathbf{a})$, whose first column is $[1\, a_1\, a_2 \, \cdot \, a_7\, a_8 \, \cdot \,]^T$. The matrix A is transformed into the identity by three steps:

Step 1. Look for $\hat{\mathbf{a}}$ such that

$$L(\mathbf{a})\hat{\mathbf{a}} = \begin{bmatrix} 1 & & & & & & & & \\ a_1 & 1 & & & & & & & \\ a_2 & a_1 & 1 & & & & & & \\ a_3 & a_2 & a_1 & 1 & & & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \hat{a}_6 \\ \hat{a}_7 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ a_1^{(1)} \\ 0 \\ a_2^{(1)} \\ 0 \\ a_3^{(1)} \\ 0 \\ \cdot \end{bmatrix} = E\mathbf{a}^{(1)}$$

for some $a_i^{(1)} \in \mathbb{C}$, and compute such $a_i^{(1)}$. The computation of $a_i^{(1)}$ requires, once $\hat{\mathbf{a}}$ is known, one matrix-vector product where the matrix is l.t.T. 8×8 ($2^s \times 2^s$), or, more precisely, two matrix-vector products where the matrices are l.t.T. 4×4 ($2^{s-1} \times 2^{s-1}$). We will see that $\hat{\mathbf{a}}$ is available with no computations.

Note that, due to Lemma 4.1, we have $L(\hat{\mathbf{a}})L(\mathbf{a}) = L(E\mathbf{a}^{(1)})$, that is the l.t.T. matrix $L(\mathbf{a})$ is transformed into a l.t.T. matrix which alternates a null diagonal to each nonnull diagonal.

Step 2. Look for $\hat{\mathbf{a}}^{(1)}$ such that

$$L(E\mathbf{a}^{(1)})E\hat{\mathbf{a}}^{(1)}$$

$$= \begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ a_1^{(1)} & 0 & 1 & & & & & & \\ 0 & a_1^{(1)} & 0 & 1 & & & & & \\ a_2^{(1)} & 0 & a_1^{(1)} & 0 & 1 & & & & \\ 0 & a_2^{(1)} & 0 & a_1^{(1)} & 0 & 1 & & & \\ a_3^{(1)} & 0 & a_2^{(1)} & 0 & a_1^{(1)} & 0 & 1 & & \\ 0 & a_3^{(1)} & 0 & a_2^{(1)} & 0 & a_1^{(1)} & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \hat{a}_1^{(1)} \\ 0 \\ \hat{a}_2^{(1)} \\ 0 \\ \hat{a}_3^{(1)} \\ 0 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ a_1^{(2)} \\ 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix} = E^2\mathbf{a}^{(2)}$$

for some $a_i^{(2)} \in \mathbb{C}$, and compute such $a_i^{(2)}$. Note that, due to Lemma 4.2, if $L(\mathbf{a}^{(1)})\hat{\mathbf{a}}^{(1)} = E\mathbf{a}^{(2)}$ then $L(E\mathbf{a}^{(1)})E\hat{\mathbf{a}}^{(1)} = E^2\mathbf{a}^{(2)}$. So, the computation of $a_i^{(2)}$ requires, once $\hat{\mathbf{a}}^{(1)}$ is known, one matrix-vector product where the matrix is l.t.T. 4×4 ($2^{s-1} \times 2^{s-1}$), or, more precisely, two matrix-vector products where the matrices are l.t.T. 2×2 ($2^{s-2} \times 2^{s-2}$). We will see that $\hat{\mathbf{a}}^{(1)}$ such that $L(\mathbf{a}^{(1)})\hat{\mathbf{a}}^{(1)} = E\mathbf{a}^{(2)}$ is available with no computations.

Also note that, due to Lemma 4.1, we have $L(E\hat{\mathbf{a}}^{(1)})L(E\mathbf{a}^{(1)}) = L(E^2\mathbf{a}^{(2)})$, that is the l.t.T. matrix $L(\mathbf{a})$ is transformed into a l.t.T. matrix which alternates three null diagonals to each nonnull diagonal.

Step 3. Look for $\hat{\mathbf{a}}^{(2)}$ such that

$$L(E^2\mathbf{a}^{(2)})E^2\hat{\mathbf{a}}^{(2)}$$

$$= \begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ 0 & 0 & 1 & & & & & & \\ 0 & 0 & 0 & 1 & & & & & \\ a_1^{(2)} & 0 & 0 & 0 & 1 & & & & \\ 0 & a_1^{(2)} & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & a_1^{(2)} & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & a_1^{(2)} & 0 & 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \hat{a}_1^{(2)} \\ 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix} = E^3\mathbf{a}^{(3)}$$

for some $a_i^{(3)} \in \mathbb{C}$, and compute such $a_i^{(3)}$. Note that, due to Lemma 4.2, if $L(\mathbf{a}^{(2)})\hat{\mathbf{a}}^{(2)} = E\mathbf{a}^{(3)}$ then $L(E^2\mathbf{a}^{(2)})E^2\hat{\mathbf{a}}^{(2)} = E^3\mathbf{a}^{(3)}$. So, the computation of $a_i^{(3)}$ requires, once $\hat{\mathbf{a}}^{(2)}$ is known, one matrix-vector product where the matrix is l.t.T. 2×2 ($2^{s-2} \times 2^{s-2}$), or, more precisely, two matrix-vector products where the matrices are l.t.T. 1×1 ($2^{s-3} \times 2^{s-3}$). That is, no operation in our case $n = 8$, where no entry $a_i^{(3)}$, $i \geq 1$, is needed. We will see that $\hat{\mathbf{a}}^{(2)}$ such that $L(\mathbf{a}^{(2)})\hat{\mathbf{a}}^{(2)} = E\mathbf{a}^{(3)}$ is available with no computations.

Also note that, due to Lemma 4.1, we have $L(E^2\hat{\mathbf{a}}^{(2)})L(E^2\mathbf{a}^{(2)}) = L(E^3\mathbf{a}^{(3)})$, that is the l.t.T. matrix $L(\mathbf{a})$ is transformed into a l.t.T. matrix which alternates seven null diagonals to each nonnull diagonal.

If $n = 2^s > 8$, then proceed this way, until the Step $s = \log_2 n$. If otherwise $n = 2^3 = 8$, then stop, since the first part of the $\mathcal{G}_2$ procedure is complete.

Summarizing, we have proved that

$$L(E^2\hat{\mathbf{a}}^{(2)})L(E\hat{\mathbf{a}}^{(1)})L(\hat{\mathbf{a}})L(\mathbf{a}) = L(E^3\mathbf{a}^{(3)}) \tag{25}$$

where the upper left 8×8 submatrices of $L(\mathbf{a})$ and of $L(E^3\mathbf{a}^{(3)})$ are the initial l.t.T. matrix A and the identity matrix, respectively:

$$L(\mathbf{a}) = \left[\begin{array}{cccc|cc} 1 & & & & & \\ a_1 & 1 & & & & \\ \cdot & \cdot & \cdot & & & \\ a_7 & \cdot & a_1 & 1 & & \\ \hline a_8 & a_7 & \cdot & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right], \quad L(E^3\mathbf{a}^{(3)}) = \left[\begin{array}{cccc|cc} 1 & & & & & \\ 0 & 1 & & & & \\ \cdot & \cdot & \cdot & & & \\ 0 & \cdot & 0 & 1 & & \\ \hline a_1^{(3)} & 0 & \cdot & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right].$$

The operations we did so far are: one product l.t.T. 8×8 $\cdot$ vector plus one product l.t.T. 4×4 $\cdot$ vector (if A were $n \times n$ with $n = 2^s$ the operations required would have been: one product l.t.T. $2^s \times 2^s$ $\cdot$ vector plus ... plus one product l.t.T. 4×4 $\cdot$ vector).

Now let us move to our main purpose, compute the first column of A^{-1}, and thus let us show the second part of the $\mathcal{G}_2$ procedure. Consider the following semi-infinite linear system:

$$L(\mathbf{a})\mathbf{z} = E^2\mathbf{v} \tag{26}$$

where $\mathbf{v}$ is a generic semi-infinite vector in $\mathbb{C}^{\mathbb{N}}$ (if A is $n \times n$ with $n = 2^s$, then the matrix E in (26) must be raised to the power $s - 1$ rather than 2). Such system can be rewritten as follows

$$\begin{bmatrix} A & \mathbf{0} & O \\ a_8 \cdots a_1 & 1 & \mathbf{0}^T \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \{\mathbf{z}\}_8 \\ z_8 \\ \cdot \end{bmatrix} = \begin{bmatrix} v_0 \\ 0 \\ 0 \\ 0 \\ v_1 \\ 0 \\ 0 \\ 0 \\ v_2 \\ \cdot \end{bmatrix}$$

that is, pointing out the upper part of the system, which consists of only 8 equations. Before proceeding further, let us note that $\{\mathbf{z}\}_8$ is such that $A\{\mathbf{z}\}_8 = [v_0\ 0\ 0\ 0\ v_1\ 0\ 0\ 0]^T$, $v_0, v_1 \in \mathbb{C}$. Therefore the choices $v_0 = 1$ and $v_1 = 0$, would make $\{\mathbf{z}\}_8$ equal to the vector we are looking for, $A^{-1}\mathbf{e}_1$.

By using the identity (25) and Lemma 4.2 one observes that the system $L(\mathbf{a})\mathbf{z} = E^2\mathbf{v}$ is equivalent to the following one

$$\begin{bmatrix} I_8 & O \\ \vdots & \ddots \end{bmatrix}\begin{bmatrix} \{\mathbf{z}\}_8 \\ \vdots \end{bmatrix} = L(E^3\mathbf{a}^{(3)})\mathbf{z} = L(\hat{\mathbf{a}})L(E\hat{\mathbf{a}}^{(1)})L(E^2\hat{\mathbf{a}}^{(2)})E^2\mathbf{v}$$
$$= L(\hat{\mathbf{a}})L(E\hat{\mathbf{a}}^{(1)})E^2L(\hat{\mathbf{a}}^{(2)})\mathbf{v} = L(\hat{\mathbf{a}})EL(\hat{\mathbf{a}}^{(1)})EL(\hat{\mathbf{a}}^{(2)})\mathbf{v}.$$

The matrices involved in the vector $L(\hat{\mathbf{a}})EL(\hat{\mathbf{a}}^{(1)})EL(\hat{\mathbf{a}}^{(2)})\mathbf{v}$ on the right hand side are lower triangular. Moreover, the upper left square submatrices of E of dimensions 8×8, 4×4 have the second half of their columns null, in fact

$$\{E\}_8 = \left[\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{array}\right], \quad \{E\}_4 = \left[\begin{array}{cc|cc} 1&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{array}\right].$$

These two observations let us obtain an effective representation of $\{\mathbf{z}\}_8$:

$$\begin{aligned} \{\mathbf{z}\}_8 &= \{L(\hat{\mathbf{a}})\}_8\{E\}_8\{L(\hat{\mathbf{a}}^{(1)})\}_8\{E\}_8\{L(\hat{\mathbf{a}}^{(2)})\}_8\{\mathbf{v}\}_8 \\ &= \{L(\hat{\mathbf{a}})\}_8\{E\}_{8,4}\{L(\hat{\mathbf{a}}^{(1)})\}_4\{E\}_{4,2}\{L(\hat{\mathbf{a}}^{(2)})\}_2\{\mathbf{v}\}_2. \end{aligned} \tag{27}$$

By using such formula, when $v_0 = 1$, $v_1 = 0$, the vector $\{\mathbf{z}\}_8$ can be computed by performing the operations: one product l.t.T. $4 \times 4 \cdot$ vector plus one product l.t.T. $8 \times 8 \cdot$ vector (if A is $n \times n$ with $n = 2^s$ the operations required would have been: one product l.t.T. $4 \times 4 \cdot$ vector plus $\dots$ plus one product l.t.T. $2^s \times 2^s \cdot$ vector), that is, as many operations as in the first part of the $\mathcal{G}_2$ procedure.

Theorem 4.3 If $cj2^j$ is an upper bound for the cost of the product l.t.T. $2^j \times 2^j$ $\cdot$ vector, then the overall cost of the $\mathcal{G}_2$ procedure for the computation of $A^{-1}\mathbf{e}_1$, where A is l.t.T. $n \times n$ with $n = 2^s$ and $A_{11} = 1$, is bounded by $\tilde{c}\sum_{j=2}^{s} j2^j = O(s2^s) = O(n \log_2 n)$.

We still have to prove that a vector $\hat{\mathbf{a}}$ such that $L(\mathbf{a})\hat{\mathbf{a}} = E\mathbf{a}^{(1)}$ is indeed available with no computations. To this aim it is sufficient to observe that

$$L(\mathbf{a})\big(\mathbf{e}_1 + \sum_{i=1}^{+\infty}(-1)^i a_i \mathbf{e}_{i+1}\big) = \mathbf{e}_1 + \sum_{i=1}^{+\infty} \delta_{i=0 \bmod 2}\big(2a_i + \sum_{j=1}^{i-1}(-1)^j a_j a_{i-j}\big)\mathbf{e}_{i+1}. \tag{28}$$

This can be verified by a direct calculation.

4.4. **Observations on the algorithm's core.** Given the vector $\mathbf{a} \in \mathbb{C}^{\mathbb{N}}$, the problem of the computation of $\hat{\mathbf{a}} \in \mathbb{C}^{\mathbb{N}}$ such that $L(\mathbf{a})\hat{\mathbf{a}} = E\mathbf{a}^{(1)}$, for some $\mathbf{a}^{(1)} \in \mathbb{C}^{\mathbb{N}}$ can also be seen as a polynomial arithmetic problem. In fact, due to Lemma 4.1, the identity $L(\mathbf{a})\hat{\mathbf{a}} = E\mathbf{a}^{(1)}$ is equivalent to the equality $L(\mathbf{a})L(\hat{\mathbf{a}}) = L(E\mathbf{a}^{(1)})$, i.e.

$$(\sum_{k=0}^{+\infty} a_k Z^k)(\sum_{k=0}^{+\infty} \hat{a}_k Z^k) = \sum_{k=0}^{+\infty} a_k^{(1)} Z^{2k}.$$

Therefore the polynomial arithmetic problem can be stated as follows:

given $a(z) = \sum_{k=0}^{+\infty} a_k z^k$, find a polynomial $\hat{a}(z) = \sum_{k=0}^{+\infty} \hat{a}_k z^k$ such that

$$\hat{a}(z)a(z) = a_0^{(1)} + a_1^{(1)}z^2 + a_2^{(1)}z^4 + \dots =: a^{(1)}(z)$$

for some coefficients $a_i^{(1)}$.

Such problem is a particular case of the more general problem: transform a *full* polynomial $a(z)$ into a *sparse* polynomial $a^{(1)}(z) = \sum_{k=0}^{+\infty} a_k^{(1)} z^{rk}$, for a fixed $r \in \mathbb{N}$. It is possible to describe explicitly a polynomial $\hat{a}(z)$ that realizes such transformation, in fact the following result holds

Proposition 4.4 Given $a(z) = \sum_{k=0}^{+\infty} a_k z^k$, set $\hat{a}(z) = a(zt)a(zt^2)\cdots a(zt^{r-1})$ where t is a r-th principal root of the unity ($t \in \mathbb{C}$, $t^r = 1$, $t^i \neq 1$ for $0 < i < r$). Then

$$\hat{a}(z)a(z) = a_0^{(1)} + a_1^{(1)} z^r + a_2^{(1)} z^{2r} + \ldots =: a^{(1)}(z) \tag{29}$$

for some $a_i^{(1)}$. Moreover, if the coefficients of a are real, then the coefficients of $\hat{a}$ are real.

Let us consider two corollaries of Proposition 4.4. For $r = 2$ we have $\hat{a}(z) = a(-z)$, that is we regain the result (28). It is clear that $a(-z)a(z) = a_0^{(1)} + a_1^{(1)} z^2 + a_2^{(1)} z^4 + \ldots$ (compare with the Graeffe root-squaring method [19]). In this case the coefficients of $\hat{a}$ are available with no computations, we only need to compute the new coefficients $a_i^{(1)}$.

For $r = 3$ we have $\hat{a}(z) = a(zt)a(zt^2)$, $t = e^{\mathbf{i}\frac{2\pi}{3}}$. By Proposition 4.4 the following equalities $a(z)a(zt)a(zt^2) = a_0^{(1)} + a_1^{(1)} z^3 + a_2^{(1)} z^6 + \ldots$ and

$$L(\hat{\mathbf{a}})L(\mathbf{a}) = L(E\mathbf{a}^{(1)}), \quad E = \begin{bmatrix} 1 & & & \\ 0 & & & \\ 0 & & & \\ 0 & 1 & & \\ 0 & 0 & & \\ 0 & 0 & & \\ 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \tag{30}$$

hold for some vector $\mathbf{a}^{(1)}$, and the coefficients of $\hat{a}(z) = a(zt)a(zt^2)$ are real, provided that the coefficients of a are. This time, the coefficients of $\hat{a}$ are not easily readable from the coefficients of a. In order to calculate them, observe that the polynomial equality $\hat{a}(z) = a(zt)a(zt^2)$ is equivalent to the matrix identity $L(\hat{\mathbf{a}}) = L(\mathbf{p})L(\mathbf{q})$, where $\mathbf{p}$ and $\mathbf{q}$ are the vectors of the coefficients of the polynomials in z $a(zt)$ and $a(zt^2)$, and therefore we get the following formula

$$\hat{\mathbf{a}} = L(\mathbf{p})\mathbf{q}, \quad p_i = a_i t^i, \; q_i = a_i t^{2i}, \tag{31}$$

which, taking into account that $t = e^{\mathbf{i}\frac{2\pi}{3}}$, becomes

$$\hat{\mathbf{a}} = L(\hat{\mathbf{p}})\hat{\mathbf{p}} + L(\hat{\mathbf{q}})\hat{\mathbf{q}}, \tag{32}$$

$$\hat{p}_i = \begin{cases} a_i & i = 3j \\ -\frac{1}{2}a_i & i \neq 3j \end{cases}, \quad \hat{q}_i = \begin{cases} 0 & i = 3j \\ -\sqrt{3}a_i/2 & i = 3j+1 \\ \sqrt{3}a_i/2 & i = 3j+2 \end{cases}, \; j = 0, 1, 2, \ldots.$$

Remark. There are infinite possible choices of $\hat{\mathbf{a}}$ for which $L(\mathbf{a})\hat{\mathbf{a}} = E\mathbf{a}^{(1)}$ for some vector $\mathbf{a}^{(1)}$. Looking for the simplest among them, i.e. for *the simplest vector* $\hat{\mathbf{a}}$ *such that* $(L(\mathbf{a})\hat{\mathbf{a}})_i = 0$, $i = 2, 3, 5, 6, 8, 9, \ldots$, we have guessed the following optimal choice for the $\hat{a}_i$:

$$\hat{a}_i^{opt} = -\sum_{r=0}^{\lfloor \frac{i-1}{2} \rfloor} a_r a_{i-r} + \delta_{i=0 \bmod 2} a_{\frac{i}{2}}^2 + 3 \begin{cases} \sum_{s \geq \frac{3-i}{6}}^{0} a_{\frac{i-3}{2}+3s} a_{\frac{i+3}{2}-3s} & i \text{ odd} \\ \sum_{s \geq \frac{6-i}{6}}^{0} a_{\frac{i-6}{2}+3s} a_{\frac{i+6}{2}-3s} & i \text{ even} \end{cases}. \tag{33}$$

After that, when we have known about the result in Proposition 4.4, we checked if our optimal $\hat{a}_i^{opt}$ were equal to the $\hat{a}_i$ obtained by setting $r = 3$ in the result, i.e. to the $\hat{a}_i$ defined by (31)-(32). As the reader can verify, the answer to our check was yes. May be the statement of Proposition 4.4 could be completed with the assertion that the polynomial $\hat{a}(z)$ proposed as solution of problem (29) is optimal. But is such assertion true in general?

So, a vector $\hat{\mathbf{a}}$ satisfying (30) is computable by one l.t.T. matrix-vector product, and therefore (choose $b = 3$ in (3),(2) of Section 2) the first 3^j entries of such vector can be obtained with $O(j3^j)$ arithmetic operations. This remark allows to state that $O(s3^s)$ arithmetic operations are sufficient to implement the algorithm $\mathcal{G}_3$ for the computation of $A^{-1}\mathbf{f}$, where A is l.t.T. $3^s \times 3^s$ (set $b = 3$ in the next section). Note that, in the $\mathcal{G}_3$ algorithm, the l.t.T. matrix A is transformed into the identity by s steps, each consisting in nullifying 2/3 of the remaining non null diagonals. Of course it is convenient to use $\mathcal{G}_3$ instead of $\mathcal{G}_2$ when solving the Ramanujan l.t.T. systems (17), (18), (19), where the coefficient matrix has two null diagonals which alternate the non null ones.

4.5. **The case $n = b^s$, b generic.** Let A be a $n \times n$ l.t.T. matrix with $A_{11} = 1$ and $n = b^s$, where b is a fixed integer greater than 1. We know that if the first column of the l.t.T. A^{-1} is known then any system $A\mathbf{z} = \mathbf{f}$, $\mathbf{f} \in \mathbb{C}^n$, can be solved in $O(sb^s)$ arithmetic operations (see Section 2). In this section it is described the $\mathcal{G}_b$ procedure for the computation of $A^{-1}\mathbf{e}_1$.

The $\mathcal{G}_b$ procedure is structured in two parts and is very similar to the $\mathcal{G}_2$ procedure, described in detail in subsection 4.3. The main difference is in the fact that at each of the s steps of the first part, where A is transformed in the identity matrix, $(b-1)/b$ (instead of 1/2) of the remaining non null diagonals are nullified. Such operation is done by applying Lemma 4.1 and an extension of Lemma 4.2, which is now stated (its proof is left to the reader).

Let E be the matrix with entries equal to zero or one which, applied to $\mathbf{u} \in \mathbb{C}^{\mathbb{N}}$, has the effect of inserting $b-1$ zeros between two consecutive components of $\mathbf{u}$. Thus

$$E = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b-1}, \quad E^k = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b^k-1},$$

$$\mathbf{u} = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ \cdot \end{bmatrix}, \quad E\mathbf{u} = \begin{bmatrix} 1 \\ \mathbf{0} \\ u_1 \\ \mathbf{0} \\ u_2 \\ \cdot \end{bmatrix}, \quad L(E\mathbf{u}) = \begin{bmatrix} 1 & & & & & \\ \mathbf{0} & I & & & & \\ u_1 & \mathbf{0}^T & 1 & & & \\ \mathbf{0} & u_1 I & \mathbf{0} & I & & \\ u_2 & \mathbf{0}^T & u_1 & \mathbf{0}^T & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b-1}.$$

Lemma 4.5 If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{\mathbb{N}}$ and $u_0 = v_0 = 1$, then $L(E\mathbf{u})E\mathbf{v} = EL(\mathbf{u})\mathbf{v}$. Moreover, $L(E^k\mathbf{u})E^k\mathbf{v} = E^kL(\mathbf{u})\mathbf{v}$, $\forall\, k \in \mathbb{N}$.

The given $n \times n$ matrix A can be thought as the upper-left submatrix of a semi-infinite l.t.T. matrix $L(\mathbf{a})$, whose first column is $[1\, a_1\, a_2\ \cdot\ a_{b^s-1}\, a_{b^s}\ \cdot\ ]^T$. The two parts of the $\mathcal{G}_b$ procedure are described here below.

Part I. Set $\mathbf{a}^{(0)} := \mathbf{a}$, and find $\hat{\mathbf{a}}^{(0)}$, $\mathbf{a}^{(1)}$ such that

$$L(\mathbf{a}^{(0)})\hat{\mathbf{a}}^{(0)} = E\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ a_1^{(1)} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b-1} \ \Big(\text{thus} \ \ L(\hat{\mathbf{a}}^{(0)})L(\mathbf{a}^{(0)}) = L(E\mathbf{a}^{(1)}) \ \Big).$$

Then, for $k = 2, \dots, s$ find $\hat{\mathbf{a}}^{(k-1)}$, $\mathbf{a}^{(k)}$ such that

$$L(\mathbf{a}^{(k-1)})\hat{\mathbf{a}}^{(k-1)} = E\mathbf{a}^{(k)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ a_1^{(k)} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b-1}$$

$$\Big(\text{thus} \ \ L(E^{k-1}\mathbf{a}^{(k-1)})E^{k-1}\hat{\mathbf{a}}^{(k-1)} = E^k\mathbf{a}^{(k)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ a_1^{(k)} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b^k-1},$$

$$\text{and} \quad L(E^{k-1}\hat{\mathbf{a}}^{(k-1)})L(E^{k-1}\mathbf{a}^{(k-1)}) = L(E^k\mathbf{a}^{(k)}) \ \Big).$$

After the above s steps, we obtain the following identity

$$L(E^{s-1}\hat{\mathbf{a}}^{(s-1)})L(E^{s-2}\hat{\mathbf{a}}^{(s-2)}) \cdots L(E\hat{\mathbf{a}}^{(1)})L(\hat{\mathbf{a}}^{(0)})L(\mathbf{a}^{(0)}) = L(E^s\mathbf{a}^{(s)}) \tag{34}$$

where the upper left $b^s \times b^s$ submatrices of $L(\mathbf{a}^{(0)})$ and of $L(E^s\mathbf{a}^{(s)})$ are, respectively, the initial l.t.T. matrix A and the identity matrix,

$$L(\mathbf{a}^{(0)}) = \left[\begin{array}{c|cc} A & \mathbf{0} & O \\ \hline a_{b^s} \ \cdots \ a_1 & 1 & \mathbf{0}^T \\ \cdot & \cdot & \cdots \end{array}\right], \quad L(E^s\mathbf{a}^{(s)}) = \left[\begin{array}{c|cc} I_{b^s} & \mathbf{0} & O \\ \hline a_1^{(s)}\mathbf{e}_1^T & 1 & \mathbf{0}^T \\ \cdot & \cdot & \cdots \end{array}\right].$$

Part II. By (34), any system of type $L(\mathbf{a}^{(0)})\mathbf{z} = E^{s-1}\mathbf{v}$, $\mathbf{v} \in \mathbb{C}^{\mathbb{N}}$, is equivalent to the following linear system

$$\begin{aligned} L(E^s\mathbf{a}^{(s)})\mathbf{z} &= L(\hat{\mathbf{a}}^{(0)})L(E\hat{\mathbf{a}}^{(1)}) \cdots L(E^{s-2}\hat{\mathbf{a}}^{(s-2)})L(E^{s-1}\hat{\mathbf{a}}^{(s-1)})E^{s-1}\mathbf{v} \\ &= L(\hat{\mathbf{a}}^{(0)})EL(\hat{\mathbf{a}}^{(1)})E \cdots EL(\hat{\mathbf{a}}^{(s-2)})EL(\hat{\mathbf{a}}^{(s-1)})\mathbf{v}. \end{aligned}$$

Thus any vector $\{\mathbf{z}\}_n$, $n = b^s$, such that

$$A\{\mathbf{z}\}_n = \{L(\mathbf{a})\}_n\{\mathbf{z}\}_n = \begin{bmatrix} v_0 \\ \mathbf{0} \\ v_1 \\ \mathbf{0} \\ \cdot \\ v_{b-1} \\ \mathbf{0} \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b^{s-1}-1}$$

(for example, the vector $A^{-1}\mathbf{e}_1$ we are looking for), can be represented as follows

$$\begin{aligned} \{\mathbf{z}\}_n &= \{L(\hat{\mathbf{a}}^{(0)})\}_n\{E\}_n\{L(\hat{\mathbf{a}}^{(1)})\}_n\{E\}_n \\ &\quad \cdots \{E\}_n\{L(\hat{\mathbf{a}}^{(s-2)})\}_n\{E\}_n\{L(\hat{\mathbf{a}}^{(s-1)})\}_n\{\mathbf{v}\}_n \\ &= \{L(\hat{\mathbf{a}}^{(0)})\}_n\{E\}_{n,\frac{n}{b}}\{L(\hat{\mathbf{a}}^{(1)})\}_{\frac{n}{b}}\{E\}_{\frac{n}{b},\frac{n}{b^2}} \\ &\quad \cdots \{E\}_{b^3,b^2}\{L(\hat{\mathbf{a}}^{(s-2)})\}_{b^2}\{E\}_{b^2,b}\{L(\hat{\mathbf{a}}^{(s-1)})\}_b\{\mathbf{v}\}_b, \end{aligned} \tag{35}$$

where $\{M\}_{j,k}$ denotes the $j \times k$ upper left submatrix of M. The latter formula allows to compute $\{\mathbf{z}\}_n$ efficiently.

Let us resume and count the operations required to implement the $\mathcal{G}_b$ procedure. In the following, n is equal to b^s and $\mathbf{0}$ denotes $\mathbf{0}_{b-1}$. In the first part, for $k = 1, \dots, s$ one has to compute, by performing $\varphi_{n/b^{k-1}}$ arithmetic operations, the vectors $\{\hat{\mathbf{a}}^{(k-1)}\}_{n/b^{k-1}}$ and $\{\mathbf{a}^{(k)}\}_{n/b^k}$, i.e. scalars $\hat{a}_i^{(k-1)}$ and $a_i^{(k)}$ such that

$$\begin{bmatrix} 1 & & & & \\ a_1^{(k-1)} & 1 & & & \\ a_2^{(k-1)} & a_1^{(k-1)} & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \\ a_{\frac{n}{b^{k-1}}-1}^{(k-1)} & \cdot & a_2^{(k-1)} & a_1^{(k-1)} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1^{(k-1)} \\ \hat{a}_2^{(k-1)} \\ \cdot \\ \hat{a}_{\frac{n}{b^{k-1}}-1}^{(k-1)} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \\ a_1^{(k)} \\ \mathbf{0} \\ \cdot \\ a_{\frac{n}{b^k}-1}^{(k)} \\ \mathbf{0} \end{bmatrix},$$

$k = 1, \dots, s$ (note that there is no $a_i^{(s)}$ to be computed).

Remark. The $n/b^{k-1} \times n/b^{k-1}$ l.t.T. by vector products, $k = 1, \dots, s-1$, that one has to perform in order to compute the vector $\{\mathbf{a}^{(k)}\}_{n/b^k}$ (once $\{\hat{\mathbf{a}}^{(k-1)}\}_{n/b^{k-1}}$ is available), can be in fact replaced with a number b of $n/b^k \times n/b^k$ l.t.T. by vector products, $k = 1, \dots, s-1$. The proof of this fact is left to the reader.

In the second part, when applying (35), one has to compute the $b \times b$ l.t.T. by vector product $\{L(\hat{\mathbf{a}}^{(s-1)})\}_b\{\mathbf{v}\}_b$ (which requires no operation if $v_0 = 1$, $v_i = 0$, $i \geq 1$), and $b^j \times b^j$ l.t.T. by vector products of type

$$\{L(\hat{\mathbf{a}}^{(s-j)})\}_{b^j} \begin{bmatrix} 1 \\ \mathbf{0} \\ \bullet_1 \\ \mathbf{0} \\ \cdot \\ \bullet_{b^{j-1}-1} \\ \mathbf{0} \end{bmatrix}, \quad j = 2, \dots, s-1, s.$$

Now, assume the number φ_{b^j} and the number of arithmetic operations required by a $b^j \times b^j$ l.t.T. by vector product both bounded by $c_b j b^j$ for some constant c_b. (As we have seen, such assumption is satisfied if $b = 2, 3$; however, by Proposition 4.4, one easily realize that it is satisfied for any larger value of b, it is enough to choose a larger constant c_b). Then the total amount of the above operations is smaller than $O(sb^s) = O(n \log_b n)$. In particular, if $v_0 = 1$, $v_i = 0$, $i \geq 1$, by such amount of operations one obtains the first column of A^{-1}, and therefore the $\mathcal{G}_b$ algorithm defines a l.t.T. linear system solver of complexity $O(n \log_b n)$.

5. Concluding remarks

We have introduced three semi-infinite l.t.T. linear systems, named odd, even and Ramanujan, whose solution involves Bernoulli numbers. Each of them is presented in two versions, see Theorem 3.2 in Section 3. Moreover, by using a simple matrix formulation, we have described in detail a low complexity solver of generic l.t.T. systems of linear equations, see Section 4. Such solver has been conceived when studying how to transform the "full" l.t.T. odd or even systems (12), (13) into the sparse lower triangular (Toeplitz) Ramanujan system (14) ((17)-(18)-(19)), or, possibly, into more sparse l.t.T. systems, solved by Bernoulli numbers and with coefficients as simple as those of the Ramanujan one. In fact, the required operation, i.e. nullifying the second, third, fifth, sixth, eighth, ninth, and so on, diagonals of the even and odd systems, could be applied to *generic* full l.t.T. systems $A\mathbf{x} = \mathbf{f}$, and, moreover, could be repeated $\log_3 n$ times, so to finally transform the l.t.T.

coefficient matrix A into the identity. This remark has naturally led to the l.t.T. system solver $\mathcal{G}_3$, particularly suitable to solve the l.t.T. Ramanujan system (17)-(18)-(19), and then to the general $\mathcal{G}_b$ algorithm, where at each step are nullified $(b-1)/b$ of the remaining non null diagonals of the coefficient matrix. Note that our original aim, i.e. find vectors $\mathbf{z}^{eR}, \mathbf{z}^{oR} \in \mathbb{C}^{\mathbb{N}}$ such that $L(\mathbf{a}^e)\mathbf{z}^{eR}$ and $L(\mathbf{a}^o)\mathbf{z}^{oR}$ are equal to or more sparse than and as simple as the vector $\mathbf{a}^R$ – with $\mathbf{a}^R, \mathbf{a}^e, \mathbf{a}^o$ defined in (19), (20), (21)) –, has not been reached in this work. Moreover, since the solutions of the three (six) l.t.T. linear systems listed in Theorem 3.2 are not exactly the Bernoulli numbers, it also remains to study an efficient way to extract them from such solutions.

Note. Some of the contents of this work have been the subject of a communication held at the 2012–edition of the annual Italian meeting "Due Giorni di Algebra Lineare Numerica" (Genova, 16–17 Febbraio 2012; speaker: Carmine Di Fiore). See www.dima.unige.it/∼dibenede/2gg/home.html . There the authors have known of the result in Proposition 4.4.

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