Explicit Characterization of Consensus in a Distributed Estimation Problem on Chain Graphs

Cristiano Maria Verrelli*, Patrizio Tomei*, Carmine Di Fiore** (University of Rome ``Tor Vergata" - *Electronic Engineering Department, **Mathematics Department)

Due Giorni di Algebra Lineare Numerica e Applicazioni, January 2025

Multi-agent estimation problem with consensus

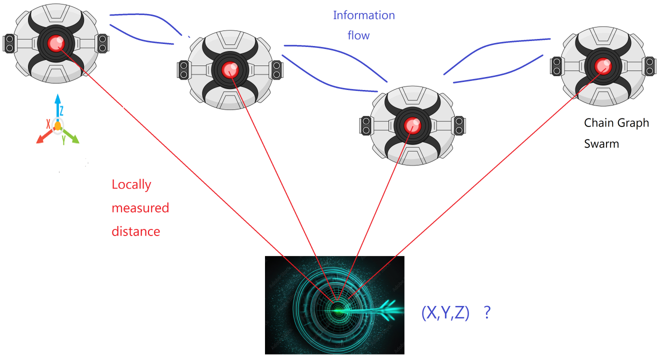
Designing differential equations for time-varying vectors $\hat{\Theta}^{[i]}(t), i = 1, \ldots, p$ – named Θ-estimates at each node $i = 1, \ldots, p$ – such that all of them exponentially converge to the unknown constant parameter vector $\Theta \in \mathbb{R}^m$ (consensus) defined by the set of linear time-varying equations:

$$
\begin{aligned}
y_1(t) &= \phi_1^{\mathrm{T}}(t)\Theta & (1)\\
y_i(t) &= \phi_i^{\mathrm{T}}(t)\Theta, \quad i = 2, \ldots, p-1 & (2)\\
y_p(t) &= \phi_p^{\mathrm{T}}(t)\Theta, & (3)
\end{aligned}
$$

where y_i are the locally measured outputs and $\phi_i(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}^m$ are the local regressor vectors, $i = 1, \ldots, p$, each of them assumed to be available at the running time at each node of an undirected chain graph and under the condition that each estimation scheme at the node (agent) can share information – namely, its own Θ-estimate – with the neighbours only.

Illustrative example

Consider the case in which a set of agents – a swarm of drones – at nodes face the local identification problem in which they cannot consistently estimate the parameter vector (position of a target object in the space) in isolation while having to engage in communication with their neighbours.



Illustrative example (details)

In particular, at least two drones with positions (x_i, y_i, z_i) in the space (at least one of them constant in time, $i = 1, \ldots, p$) have to identify the position (x_o, y_o, z_o) of the target object in the space by just sharing information with the neighbours; each drone knows its own position in the space and measures its (Euclidean) distance $D_i(t) = \sqrt{(x_i - x_o)^2 + (y_i - y_o)^2 + (z_i - z_o)^2}$ from the target object, which leads to the output:

$$\begin{aligned} y_i &= D_i^2 - x_i^2 - y_i^2 - z_i^2 \\ &= \overbrace{[-2x_i, -2y_i, -2z_i, 1]}^{\phi_i^{\mathrm{T}}} \overbrace{\begin{bmatrix} x_o \\ y_o \\ z_o \\ x_o^2 + y_o^2 + z_o^2 \end{bmatrix}}^{\Theta} \end{aligned}$$

in form (1)-(3).

Θ-identifiability conditions (even when $p < m$)

Assumption A1

The elements of the regressor vectors are assumed to be continuous and uniformly bounded over $[0, +\infty)$ as functions of time.

Assumption A2

The corresponding regressor matrix $\Phi^{\mathrm{T}}(\cdot) \in \mathbb{R}^{p \times m}$

$$\Phi^{\mathrm{T}}(\cdot) \quad = \quad \begin{bmatrix} \phi_1^{\mathrm{T}}(\cdot) \\ \cdots \\ \phi_p^{\mathrm{T}}(\cdot) \end{bmatrix} \tag{4}$$

is assumed to be persistently exciting (PE), i.e., there exist (known) positive reals c_p and T_p such that the following condition $[\mathbb{I} \in \mathbb{R}^{m \times m}]$ holds:

$$\int_t^{t+T_p} \Phi(\tau)\Phi^{\mathrm{T}}(\tau)\mathrm{d}\tau \quad \geqq \quad c_p\mathbb{I}, \quad \forall\, t \geq 0. \tag{5}$$

Original aim and contribution

Aim

To show how the positive definite nature of a quadratic form associated with the tridiagonal block structure

$$T = \begin{bmatrix} \mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \dots & \dots & \mathbb{O} \\ -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \dots & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \mathbb{O} \\ \dots & \dots & \dots & \dots & \dots & \dots & -\mathbb{I} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \dots & -\mathbb{I} & \mathbb{I} \end{bmatrix}. \tag{6}$$

is crucial to innovatively prove the exponential achievement – with an explicit characterization of the exponential convergence – of the distributed parameter estimation task on an undirected chain graph, in which an original neighbourhood-based decentralized parallel architecture (reducing the computational burden of a centralized estimation scheme on the graph) is adopted.

Contribution

In place of weaker contradiction arguments to prove that the *Cooperative PE Condition* guarantees exponential consensus, here original proofs of convergence are able to provide an explicit characterization of the exponentially achieved consensus.

Extension

Nevertheless, the problem of asymptotically identifying time-varying parameters that are periodic with known periods can be innovatively solved as well when $p \geq m$ and Φ is constant. Adaptive tools have just to be replaced by repetitive learning tools within the same theoretical framework.

Design of Estimators

Penalization of the mismatch between the parameter estimates

Design

$$\dot{\hat{\Theta}}^{[i]}(t) = \phi_i(t)\left(y_i(t) - \phi_i^{\mathrm{T}}(t)\hat{\Theta}^{[i]}(t)\right) - \eta_i(t). \quad (7)$$

Taking into account that any internal i-th estimation scheme ($i = 2, \ldots, p-1$) can use only the information coming from the $(i-1)$-th and the $(i+1)$-th estimation schemes, whereas the 1-st and the p-th estimation schemes can use only the information provided by the 2-nd and the $(p-1)$-th, respectively, we determine

$$\begin{aligned}
\eta_1 &= \left(\hat{\Theta}^{[1]} - \hat{\Theta}^{[2]}\right) \\
\eta_i &= \left(\hat{\Theta}^{[i]} - \hat{\Theta}^{[i-1]}\right) + \left(\hat{\Theta}^{[i]} - \hat{\Theta}^{[i+1]}\right), i = 2, \ldots, p-1 \\
\eta_p &= \left(\hat{\Theta}^{[p]} - \hat{\Theta}^{[p-1]}\right). \quad (8)
\end{aligned}$$

Error system

A crucial tridiagonal block matrix

Defining the estimation errors $\tilde{\Theta}^{[i]} = \Theta - \hat{\Theta}^{[i]}$, the error system reads

$$\begin{bmatrix} \dot{\tilde{\Theta}}^{[1]}(t) \\ \dot{\tilde{\Theta}}^{[2]}(t) \\ \dot{\tilde{\Theta}}^{[3]}(t) \\ \cdots \\ \dot{\tilde{\Theta}}^{[p]}(t) \end{bmatrix} = -(\Lambda(t) + T) \begin{bmatrix} \tilde{\Theta}^{[1]}(t) \\ \tilde{\Theta}^{[2]}(t) \\ \tilde{\Theta}^{[3]}(t) \\ \cdots \\ \tilde{\Theta}^{[p]}(t) \end{bmatrix} \tag{9}$$

where $\Lambda(t) + T$ is a tridiagonal block matrix in $\mathbb{R}^{pm \times pm}$ with

$$\Lambda(t) = \mathrm{diag}[\phi_1(t)\phi_1^{\mathrm{T}}(t), \ldots, \phi_p(t)\phi_p^{\mathrm{T}}(t)]. \tag{10}$$

Theoretical results

Lemma

Assume that $\Phi(t)$, besides Assumption A1, satisfies the following hypothesis:
A3. the entries of $\Phi(t)$ are analytical functions of time t.
Consider system (9) and assume that there exist (known) positive reals c_{pG} and T_{pG} such that the condition:

$$\int_t^{t+T_{pG}} G(\tau)\mathrm{d}\tau \geqq c_{pG}\mathbb{I}, \quad \forall\, t \geq 0 \tag{11}$$

holds. Then the n-dimensional extended error vector $\tilde{\Theta}^{[e]}(t) = [\tilde{\Theta}^{[1]\mathrm{T}}(t), \dots, \tilde{\Theta}^{[p]\mathrm{T}}(t)]^{\mathrm{T}}$ ($n = pm$) globally exponentially converges to zero.

Meaningfully, the theorem below shows how the weakest and least restrictive condition (5) actually implies condition (11) and thus, used in conjunction with the previous Lemma, provides the proof that the solution to the error system, under Assumptions A1-A3, globally exponentially converges to zero.

Theorem

Under Assumption A2 [namely, condition (5)], there exist explicitly computable positive reals c_{pG} and T_{pG} such that the condition (11) holds true.

Application to electrical motors

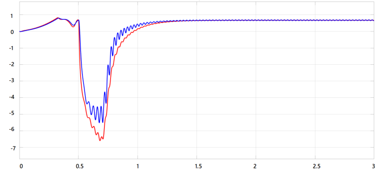


Figure: First components of the estimate vectors $\hat{\Theta}^{[1]}$ and $\hat{\Theta}^{[2]}$, both of them converging to the first component of Θ, namely Ψ_e (SI units).

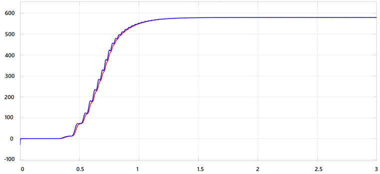


Figure: Second components of the estimate vectors $\hat{\Theta}^{[1]}$ and $\hat{\Theta}^{[2]}$, both of them converging to the second component of Θ, namely ωT_L (SI units).

Extension to periodic parameters

Learning-based design of estimators

Design

$$\hat{\Theta}^{[i]}(t) \quad = \quad \hat{\Theta}^{[i]}(t,\alpha) + \phi_i \tilde{y}_i(t) - \eta_i(t) \tag{12}$$

where: α is the vector $[\alpha_1, \dots, \alpha_m]^{\mathrm{T}}$; $\eta_i(t)$ is the same vector in (8);

$$\hat{\Theta}^{[i]}(t,\alpha) \quad = \quad [\hat{\theta}_1^{[i]}(t-\alpha_1), \dots, \hat{\theta}_m^{[i]}(t-\alpha_m)]^{\mathrm{T}},$$

leading to

$$\begin{bmatrix} \tilde{\Theta}^{[1]}(t) \\ \dots \\ \tilde{\Theta}^{[p]}(t) \end{bmatrix} = \begin{bmatrix} \tilde{\Theta}^{[1]}(t,\alpha) \\ \dots \\ \tilde{\Theta}^{[p]}(t,\alpha) \end{bmatrix} - (\Lambda + T) \begin{bmatrix} \tilde{\Theta}^{[1]}(t) \\ \dots \\ \tilde{\Theta}^{[p]}(t) \end{bmatrix}$$

where

$$\tilde{\Theta}^{[i]}(t,\alpha) \quad = \quad [\tilde{\theta}_1^{[i]}(t-\alpha_1), \dots, \tilde{\theta}_m^{[i]}(t-\alpha_m)]^{\mathrm{T}}.$$

Illustrative example (continued)

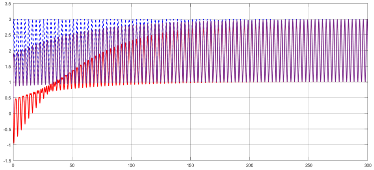


Figure: Two drones with shared information: $\hat{\theta}_1^{[i]}(t)$, $i = 1, 2$, converging to $\theta_1(t)$ (time in seconds).

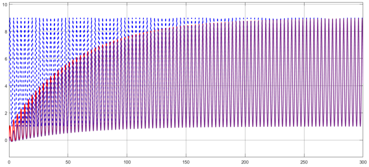


Figure: Two drones with shared information: $\hat{\theta}_2^{[i]}(t)$, $i = 1, 2$, converging to $\theta_2(t)$ (time in seconds).

Questions?