Preconditioned finite element method: the Poisson problem on the square

Consider the Poisson problem on $\Omega = [0,1]\times[0,1]$. Let τ_0 be the triangulation of Ω obtained by tracing the line through the points $(1,0)$ and $(0,1)$. Note that $h_0 = \sqrt{2}$. Let τ_j, $\mathcal{V}_j$, Π_j, $j = 1,2,\ldots$, be the triangulations of Ω, the subspaces of $\mathcal{V} = H^1_0(\Omega)$ and the interpolating operators defined starting from τ_0 via the rule indicated in (toe_1f). Note that $\mathrm{diam}\,(\tau_j) = h_j = 2^{-j}\sqrt{2}$. Note also that the number of the inner nodes of τ_j is n^2 where $n = 2^j - 1$. We call them $x_{j,1} = \frac{1}{2^j}(1,1)$, …, $x_{j,2^j-1} = \frac{1}{2^j}(2^j-1,1)$, $x_{j,2^j} = \frac{1}{2^j}(1,2)$, …, $x_{j,(2^j-1)^2} = \frac{1}{2^j}(2^j-1,2^j-1)$, and consider the corresponding elements of the Lagrange basis $\varphi_{j,k}$, $k = 1,\ldots,(2^j-1)^2$, of $\mathcal{V}_j$.

Exploiting the Lagrange basis. If u_φ is fixed in $H^1(\Omega)$ such that $u_\varphi|_{\partial\Omega} = \varphi$, then the scalars $(w_j)_k$ solving the linear system

$$\sum_{k=1}^{(2^j-1)^2}(w_j)_k\int_\Omega \nabla\varphi_{j,k}\nabla\varphi_{j,i} = F(\varphi_{j,i}) = \int_\Omega f\varphi_{j,i} - \int_\Omega \nabla u_\varphi\nabla\varphi_{j,i}, \quad i = 1,\ldots,(2^j-1)^2,$$

define a function $w_j = \sum (w_j)_k\varphi_{j,k} \in \mathcal{V}_j$ which approximates $w \in \mathcal{V} = H^1_0(\Omega)$, $a(w,v) = F(v)$, $\forall\, v\in\mathcal{V}$, and thus a function $u_j = u_\varphi + w_j$ which approximates the solution $u = u_\varphi + w$ of the variational version of the Poisson differential problem $-\Delta u = f$, $x\in\Omega$, $u = \varphi$, $x \in \partial\Omega$.

Let A be the coefficient matrix of such system. We know that it is positive definite (because $\int_\Omega \nabla u\nabla v$ is coercive on $H^1_0(\Omega)$) and that, for each row, there are at most seven nonzero entries (just look at τ_j and at the support of the $\varphi_{j,i}$, i generic). Let us compute its entries a_{ik},

$$a_{ik} = \int_\Omega \nabla\varphi_{j,k}\nabla\varphi_{j,i} = \int_{s(\varphi_{j,k})\cap s(\varphi_{j,i})}\nabla\varphi_{j,k}\nabla\varphi_{j,i}.$$

First, draw a zoom of τ_j around the inner node i, so to see only the nodes linked to i, i.e. $i-n, i-n+1, i-1, i+1, i+n-1, i+n$. Call T_1 the triangle of τ_j whose verteces are $i-1, i, i+n-1$, and T_2, T_3, T_4, T_5, T_6 the other triangles of τ_j having i as a vertex, in a clockwise order. Here below are reported the values of $\nabla\varphi_{j,i}(x)$ on these triangles, unless the factor $1/\delta_j$, $\delta_j := 1/2^j$:

$$\begin{bmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 \\ (1,0) & (0,-1) & (-1,-1) & (-1,0) & (0,1) & (1,1) \end{bmatrix}.$$

Observe that the value of $\nabla\varphi_{j,i+n-1}$ on T_1 is equal to the value of $\nabla\varphi_{j,i}$ on T_5; analogously, the value of $\nabla\varphi_{j,i-1}$ on T_6 is equal to the value of $\nabla\varphi_{j,i}$ on T_4, and so on. Finally, observe that all the triangles T_k, $k = 1,\ldots,6$, have area equal to $\delta_j^2/2$.

It follows that, for $i = 1,\ldots,n^2$:

$$\begin{aligned} a_{ii} &= \textstyle\int_{s(\varphi_{j,i})}\nabla\varphi_{j,i}\nabla\varphi_{j,i} = \int_{\cup_{k=1}^6 T_k}\|\nabla\varphi_{j,i}\|^2 \\ &= \textstyle\int_{T_1}\|\nabla\varphi_{j,i}\|^2 + \int_{T_2}\|\nabla\varphi_{j,i}\|^2 + \int_{T_3}\|\nabla\varphi_{j,i}\|^2 + \int_{T_4}\|\nabla\varphi_{j,i}\|^2 + \int_{T_5}\|\nabla\varphi_{j,i}\|^2 + \int_{T_6}\|\nabla\varphi_{j,i}\|^2 \\ &= \tfrac{1}{\delta_j^2}\tfrac{\delta_j^2}{2} + \tfrac{1}{\delta_j^2}\tfrac{\delta_j^2}{2} + \tfrac{2}{\delta_j^2}\tfrac{\delta_j^2}{2} + \tfrac{1}{\delta_j^2}\tfrac{\delta_j^2}{2} + \tfrac{1}{\delta_j^2}\tfrac{\delta_j^2}{2} + \tfrac{2}{\delta_j^2}\tfrac{\delta_j^2}{2} = 4; \end{aligned}$$

for $i = 2, \ldots, n^2$, $i \neq n+1, \ldots, (n-1)n+1$:

$$\begin{aligned} a_{i,i-1} &= \int_{s(\varphi_{j,i-1})\cap s(\varphi_{j,i})} \nabla\varphi_{j,i-1}\nabla\varphi_{j,i} = \int_{T_1\cup T_6} \nabla\varphi_{j,i-1}\nabla\varphi_{j,i} \\ &= \int_{T_1} \nabla\varphi_{j,i-1}\nabla\varphi_{j,i} + \int_{T_6} \nabla\varphi_{j,i-1}\nabla\varphi_{j,i} \\ &= \frac{\delta_j^2}{2}\frac{1}{\delta_j}(-1,-1)\frac{1}{\delta_j}(1,0) + \frac{\delta_j^2}{2}\frac{1}{\delta_j}(-1,0)\frac{1}{\delta_j}(1,1) = -1; \end{aligned}$$

for $i = n, \ldots, n^2$, $i \neq n, 2n, \ldots, n^2$:

$$\begin{aligned} a_{i,i-n+1} &= \int_{s(\varphi_{j,i-n+1})\cap s(\varphi_{j,i})} \nabla\varphi_{j,i-n+1}\nabla\varphi_{j,i} = \int_{T_4\cup T_5} \nabla\varphi_{j,i-n+1}\nabla\varphi_{j,i} \\ &= \int_{T_4} \nabla\varphi_{j,i-n+1}\nabla\varphi_{j,i} + \int_{T_5} \nabla\varphi_{j,i-n+1}\nabla\varphi_{j,i} \\ &= \frac{\delta_j^2}{2}\frac{1}{\delta_j}(0,-1)\frac{1}{\delta_j}(-1,0) + \frac{\delta_j^2}{2}\frac{1}{\delta_j}(1,0)\frac{1}{\delta_j}(0,1) = 0; \end{aligned}$$

for $i = n+1, \ldots, n^2$:

$$\begin{aligned} a_{i,i-n} &= \int_{s(\varphi_{j,i-n})\cap s(\varphi_{j,i})} \nabla\varphi_{j,i-n}\nabla\varphi_{j,i} = \int_{T_5\cup T_6} \nabla\varphi_{j,i-n}\nabla\varphi_{j,i} \\ &= \int_{T_5} \nabla\varphi_{j,i-n}\nabla\varphi_{j,i} + \int_{T_6} \nabla\varphi_{j,i-n}\nabla\varphi_{j,i} \\ &= \frac{\delta_j^2}{2}\frac{1}{\delta_j}(-1,-1)\frac{1}{\delta_j}(0,1) + \frac{\delta_j^2}{2}\frac{1}{\delta_j}(0,-1)\frac{1}{\delta_j}(1,1) = -1; \end{aligned}$$

and finally, for $i = k+1, \ldots, n^2$: $a_{i,i-k} = 0$, $\forall k \neq 0, 1, n-1, n$, because for such values of k the measure of the set $s(\varphi_{j,i-k}) \cap s(\varphi_{j,i})$ is zero.

Then, since $a_{ik} = a_{ki}$, $\forall\, i, k$, we can conclude that A is a $n \times n$ block matrix of the form

$$A = \begin{bmatrix} B & -I & & & \\ -I & B & -I & & \\ & -I & \ddots & \ddots & \\ & & \ddots & B & -I \\ & & & -I & B \end{bmatrix}$$

whose diagonal blocks are all equal to the following $n \times n$ matrix B,

$$B = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 4 & -1 \\ & & & -1 & 4 \end{bmatrix},$$

where $n = 2^j - 1$.

Eigenvalues of A. Now let us compute the eigenvalues of A (we already know that they are real and positive). Let $\mathbf{v}_k$ be the $n \times 1$ vector whose i-th entry is $\sin(ik\pi/(n+1))$, $1 \leq i, k \leq n$. Then

$$B\mathbf{v}_k = [4 - 2\cos(k\pi/(n+1))]\mathbf{v}_k, \quad k = 1, \ldots, n.$$

Moreover, if we set $[\cdot] = [4 - 2\cos(k\pi/(n+1))]$, then

$$\begin{bmatrix} B & -I & & \\ -I & B & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & B \end{bmatrix} \begin{bmatrix} \alpha_1\mathbf{v}_k \\ \alpha_2\mathbf{v}_k \\ \vdots \\ \alpha_n\mathbf{v}_k \end{bmatrix} = \lambda \begin{bmatrix} \alpha_1\mathbf{v}_k \\ \alpha_2\mathbf{v}_k \\ \vdots \\ \alpha_n\mathbf{v}_k \end{bmatrix}$$

iff

$$\begin{aligned}
&\alpha_1 B\mathbf{v}_k - \alpha_2\mathbf{v}_k = \lambda\alpha_1\mathbf{v}_k\\
&-\alpha_1\mathbf{v}_k + \alpha_2 B\mathbf{v}_k - \alpha_3\mathbf{v}_k = \lambda\alpha_2\mathbf{v}_k\\
&\dots\\
&-\alpha_{n-1}\mathbf{v}_k + \alpha_n B\mathbf{v}_k = \lambda\alpha_n\mathbf{v}_k
\end{aligned}$$

iff

$$\begin{aligned}
&\alpha_1[\cdot]\mathbf{v}_k - \alpha_2\mathbf{v}_k = \lambda\alpha_1\mathbf{v}_k\\
&-\alpha_1\mathbf{v}_k + \alpha_2[\cdot]\mathbf{v}_k - \alpha_3\mathbf{v}_k = \lambda\alpha_2\mathbf{v}_k\\
&\dots\\
&-\alpha_{n-1}\mathbf{v}_k + \alpha_n[\cdot]\mathbf{v}_k = \lambda\alpha_n\mathbf{v}_k
\end{aligned}$$

iff

$$\begin{bmatrix} [\cdot] & -1 & & \\ -1 & [\cdot] & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & [\cdot] \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \lambda \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix},$$

and the latter equation is verified for $\lambda = [\cdot] - 2\cos(s\pi/(n+1))$ and $\alpha_r = \sin(rs\pi/(n+1))$, $r = 1,\dots,n$ ($s = 1,\dots,n$). So, the eigenvalues of A are:

$$4 - 2(\cos\frac{k\pi}{n+1} - \cos\frac{s\pi}{n+1}) = 4 - 2(\cos\frac{k\pi}{2^j} + \cos\frac{s\pi}{2^j}), \quad 1 \le k, s \le n = 2^j - 1.$$

Thus

$$\mu_2(A) = \frac{1+\cos(\pi/(n+1))}{1-\cos(\pi/(n+1))} = \frac{1+\cos(\pi/(2^j))}{1-\cos(\pi/(2^j))} = O(n^2) = O((2^j)^2),$$

where $\mu_2(A)$ denotes the condition number of A in the 2-norm. (Recall that, since A is positive definite, $\mu_2(A)$ is simply the ratio of the greatest and the smallest eigenvalues of A).

For example, if $j = 2$, then A is a 9×9 matrix and

$$\mu_2(A) = \frac{1+\cos(\pi/4)}{1-\cos(\pi/4)} = \frac{2+\sqrt{2}}{2-\sqrt{2}} = 3 + 2\sqrt{2}.$$

If $j = 3$, A is a 49×49 matrix, and $\mu_2(A) \approx 25$. If $j = 4$, A is a 225×225 matrix, and $\mu_2(A) \approx 103$. And so on.

Exploiting the hierarchical basis. Consider now the hierarchical basis $\tilde{\varphi}_{j,k}$, $k = 1,\dots,(2^j-1)^2$, of $\mathcal{V}_j$. If u_φ is fixed in $H^1(\Omega)$ such that $u_\varphi|_{\partial\Omega} = \varphi$, then the scalars $(\tilde{w_j})_k$ solving the linear system

$$\sum_{k=1}^{(2^j-1)^2} (\tilde{w_j})_k \int_\Omega \nabla\tilde{\varphi}_{j,k}\nabla\tilde{\varphi}_{j,i} = F(\tilde{\varphi}_{j,i}) = \int_\Omega f\tilde{\varphi}_{j,i} - \int_\Omega \nabla u_\varphi \nabla\tilde{\varphi}_{j,i}, \quad i = 1,\dots,(2^j-1)^2,$$

define a function $w_j = \sum(\tilde{w_j})_k\tilde{\varphi}_{j,k} \in \mathcal{V}_j$ which approximates $w \in \mathcal{V} = H_0^1(\Omega)$, $a(w,v) = F(v)$, $\forall\, v \in \mathcal{V}$, and thus a function $u_j = u_\varphi + w_j$ which approximates the solution $u = u_\varphi + w$ of the variational version of the Poisson differential problem $-\Delta u = f$, $x \in \Omega$, $u = \varphi$, $x \in \partial\Omega$.

Let $\tilde{A}$ be the coefficient matrix of such system. We know that it is positive definite (because $\int_\Omega \nabla u \nabla v$ is coercive on $H_0^1(\Omega)$). Let us compute its entries $\tilde{a}_{ik}$,

$$\tilde{a}_{ik} = \int_\Omega \nabla\tilde{\varphi}_{j,k}\nabla\tilde{\varphi}_{j,i} = \int_{s(\tilde{\varphi}_{j,k})\cap s(\tilde{\varphi}_{j,i})} \nabla\tilde{\varphi}_{j,k}\nabla\tilde{\varphi}_{j,i}.$$

and, possibly, its eigenvalues.

$j = 1$: $\tilde{A}$ is a 1×1 matrix.
$\tilde{\varphi}_{11} = \varphi_{11} \Rightarrow \tilde{A} = A = [4]$. Eigenvalues: 4.

$j = 2$: $\tilde{A}$ is a 9×9 matrix.
Note that $\delta_2 = 2^{-1}\delta_1 = 2^{-2}\delta_0$, $\delta_0 = 1$. Assume $\tilde{\varphi}_{2,s} = \psi_{1,s} = \varphi_{2,s}$, $s = 1,2,3,4,6,7,8,9$, $\tilde{\varphi}_{2,5} = \varphi_{1,1}$. Then, for $r,s \in \{1,2,3,4,6,7,8,9\}$:

$$\tilde{a}_{rs} = \int_{s(\psi_{1,s})\cap s(\psi_{1,r})} \nabla\psi_{1,s}\nabla\psi_{1,r} = \int_{s(\varphi_{2,s})\cap s(\varphi_{2,r})} \nabla\varphi_{2,s}\nabla\varphi_{2,r} = a_{rs}.$$

Moreover, if $\varphi = \varphi_{11}$:

$$\begin{aligned}
\tilde{a}_{55} &= \int_{s(\varphi_{11})} \nabla\varphi_{11}\nabla\varphi_{11} \\
&= \int_{T_1^\varphi} \tfrac{1}{\delta_1}(1,0)\tfrac{1}{\delta_1}(1,0) + \int_{T_2^\varphi} \tfrac{1}{\delta_1}(0,-1)\tfrac{1}{\delta_1}(0,-1) \\
&\quad + \int_{T_3^\varphi} \tfrac{1}{\delta_1}(-1,-1)\tfrac{1}{\delta_1}(-1,-1) + \int_{T_4^\varphi} \tfrac{1}{\delta_1}(-1,0)\tfrac{1}{\delta_1}(-1,0) \\
&\quad + \int_{T_5^\varphi} \tfrac{1}{\delta_1}(0,1)\tfrac{1}{\delta_1}(0,1) + \int_{T_6^\varphi} \tfrac{1}{\delta_1}(1,1)\tfrac{1}{\delta_1}(1,1) \\
&= 4 = a_{55};
\end{aligned}$$

if $\varphi = \varphi_{11}$ and $\psi = \psi_{1,s}$, $s \in \{1,2,3,4,6,7,8,9\}$:

$$\tilde{a}_{5,s} = \int_{s(\psi)\cap s(\varphi)} \nabla\psi\nabla\varphi =$$

$$\begin{aligned}
s = 1: \quad &= \int_{T_1^\psi} \tfrac{1}{\delta_2}(1,0)\tfrac{1}{\delta_1}(0,0) + \int_{T_2^\psi} \tfrac{1}{\delta_2}(0,-1)\tfrac{1}{\delta_1}(1,1) \\
&\quad + \int_{T_3^\psi} \tfrac{1}{\delta_2}(-1,-1)\tfrac{1}{\delta_1}(1,1) + \int_{T_4^\psi} \tfrac{1}{\delta_2}(-1,0)\tfrac{1}{\delta_1}(1,1) \\
&\quad + \int_{T_5^\psi} \tfrac{1}{\delta_2}(0,1)\tfrac{1}{\delta_1}(0,0) + \int_{T_6^\psi} \tfrac{1}{\delta_2}(1,1)\tfrac{1}{\delta_1}(0,0) \\
&= -2\delta_2/\delta_1,
\end{aligned}$$

$$\begin{aligned}
s = 2: \quad &= \int_{T_1^\psi} \tfrac{1}{\delta_2}(1,0)\tfrac{1}{\delta_1}(1,1) + \int_{T_2^\psi} \tfrac{1}{\delta_2}(0,-1)\tfrac{1}{\delta_1}(1,1) \\
&\quad + \int_{T_3^\psi} \tfrac{1}{\delta_2}(-1,-1)\tfrac{1}{\delta_1}(0,1) + \int_{T_4^\psi} \tfrac{1}{\delta_2}(-1,0)\tfrac{1}{\delta_1}(0,1) \\
&\quad + \int_{T_5^\psi} \tfrac{1}{\delta_2}(0,1)\tfrac{1}{\delta_1}(0,1) + \int_{T_6^\psi} \tfrac{1}{\delta_2}(1,1)\tfrac{1}{\delta_1}(1,1) \\
&= \delta_2/\delta_1,
\end{aligned}$$

$$\begin{aligned}
s = 3: \quad &= \int_{T_1^\psi} \tfrac{1}{\delta_2}(1,0)\tfrac{1}{\delta_1}(0,1) + \int_{T_2^\psi} \tfrac{1}{\delta_2}(0,-1)\tfrac{1}{\delta_1}(-1,0) \\
&\quad + \int_{T_3^\psi} \tfrac{1}{\delta_2}(-1,-1)\tfrac{1}{\delta_1}(-1,0) + \int_{T_4^\psi} \tfrac{1}{\delta_2}(-1,0)\tfrac{1}{\delta_1}(-1,0) \\
&\quad + \int_{T_5^\psi} \tfrac{1}{\delta_2}(0,1)\tfrac{1}{\delta_1}(0,1) + \int_{T_6^\psi} \tfrac{1}{\delta_2}(1,1)\tfrac{1}{\delta_1}(0,1) \\
&= 2\delta_2/\delta_1,
\end{aligned}$$

$$\begin{aligned}
s = 4: \quad &= \int_{T_1^\psi} \tfrac{1}{\delta_2}(1,0)\tfrac{1}{\delta_1}(1,0) + \int_{T_2^\psi} \tfrac{1}{\delta_2}(0,-1)\tfrac{1}{\delta_1}(1,0) \\
&\quad + \int_{T_3^\psi} \tfrac{1}{\delta_2}(-1,-1)\tfrac{1}{\delta_1}(1,0) + \int_{T_4^\psi} \tfrac{1}{\delta_2}(-1,0)\tfrac{1}{\delta_1}(1,1) \\
&\quad + \int_{T_5^\psi} \tfrac{1}{\delta_2}(0,1)\tfrac{1}{\delta_1}(1,1) + \int_{T_6^\psi} \tfrac{1}{\delta_2}(1,1)\tfrac{1}{\delta_1}(1,1) \\
&= \delta_2/\delta_1,
\end{aligned}$$

$$
\begin{aligned}
s=6: \quad &= \int_{T_1^\psi} \tfrac{1}{\delta_2}(1,0)\tfrac{1}{\delta_1}(-1,-1) + \int_{T_2^\psi} \tfrac{1}{\delta_2}(0,-1)\tfrac{1}{\delta_1}(-1,-1) \\
&\quad + \int_{T_3^\psi} \tfrac{1}{\delta_2}(-1,-1)\tfrac{1}{\delta_1}(-1,-1) + \int_{T_4^\psi} \tfrac{1}{\delta_2}(-1,0)\tfrac{1}{\delta_1}(-1,0) \\
&\quad + \int_{T_5^\psi} \tfrac{1}{\delta_2}(0,1)\tfrac{1}{\delta_1}(-1,0) + \int_{T_6^\psi} \tfrac{1}{\delta_2}(1,1)\tfrac{1}{\delta_1}(-1,0) \\
&= \delta_2/\delta_1,
\end{aligned}
$$

$$
\begin{aligned}
s=7: \quad &= \int_{T_1^\psi} \tfrac{1}{\delta_2}(1,0)\tfrac{1}{\delta_1}(1,0) + \int_{T_2^\psi} \tfrac{1}{\delta_2}(0,-1)\tfrac{1}{\delta_1}(0,-1) \\
&\quad + \int_{T_3^\psi} \tfrac{1}{\delta_2}(-1,-1)\tfrac{1}{\delta_1}(0,-1) + \int_{T_4^\psi} \tfrac{1}{\delta_2}(-1,0)\tfrac{1}{\delta_1}(0,-1) \\
&\quad + \int_{T_5^\psi} \tfrac{1}{\delta_2}(0,1)\tfrac{1}{\delta_1}(1,0) + \int_{T_6^\psi} \tfrac{1}{\delta_2}(1,1)\tfrac{1}{\delta_1}(1,0) \\
&= 2\delta_2/\delta_1,
\end{aligned}
$$

$$
\begin{aligned}
s=8: \quad &= \int_{T_1^\psi} \tfrac{1}{\delta_2}(1,0)\tfrac{1}{\delta_1}(0,-1) + \int_{T_2^\psi} \tfrac{1}{\delta_2}(0,-1)\tfrac{1}{\delta_1}(0,-1) \\
&\quad + \int_{T_3^\psi} \tfrac{1}{\delta_2}(-1,-1)\tfrac{1}{\delta_1}(-1,-1) + \int_{T_4^\psi} \tfrac{1}{\delta_2}(-1,0)\tfrac{1}{\delta_1}(-1,-1) \\
&\quad + \int_{T_5^\psi} \tfrac{1}{\delta_2}(0,1)\tfrac{1}{\delta_1}(-1,-1) + \int_{T_6^\psi} \tfrac{1}{\delta_2}(1,1)\tfrac{1}{\delta_1}(0,-1) \\
&= \delta_2/\delta_1,
\end{aligned}
$$

$$
\begin{aligned}
s=9: \quad &= \int_{T_1^\psi} \tfrac{1}{\delta_2}(1,0)\tfrac{1}{\delta_1}(-1,-1) + \int_{T_2^\psi} \tfrac{1}{\delta_2}(0,-1)\tfrac{1}{\delta_1}(0,0) \\
&\quad + \int_{T_3^\psi} \tfrac{1}{\delta_2}(-1,-1)\tfrac{1}{\delta_1}(0,0) + \int_{T_4^\psi} \tfrac{1}{\delta_2}(-1,0)\tfrac{1}{\delta_1}(0,0) \\
&\quad + \int_{T_5^\psi} \tfrac{1}{\delta_2}(0,1)\tfrac{1}{\delta_1}(-1,-1) + \int_{T_6^\psi} \tfrac{1}{\delta_2}(1,1)\tfrac{1}{\delta_1}(-1,-1) \\
&= -2\delta_2/\delta_1.
\end{aligned}
$$

So, for $j=2$, the matrix $\tilde{A}$ differs from A only by the fifth row and the fifth column (note that the submatrix of $\tilde{A}$ formed by the entry $\tilde{a}_{55}$ is equal to $\tilde{A}$, $j=1$).

Here below the matrices A and $\tilde{A}$, for $j=2$, are compared:

$$
\begin{bmatrix}
4 & -1 & 0 & -1 & (-2\frac{1}{2}) & & & & \\
-1 & 4 & -1 & & -1\,(1\frac{1}{2}) & & & & \\
0 & -1 & 4 & & (2\frac{1}{2}) & -1 & & & \\
-1 & & & 4 & -1\,(1\frac{1}{2}) & 0 & -1 & & \\
(-2\frac{1}{2}) & -1\,(1\frac{1}{2}) & (2\frac{1}{2}) & -1\,(1\frac{1}{2}) & 4\,(4) & -1\,(1\frac{1}{2}) & (2\frac{1}{2}) & -1\,(1\frac{1}{2}) & (-2\frac{1}{2}) \\
 & & -1 & 0 & -1\,(1\frac{1}{2}) & 4 & & & -1 \\
 & & & -1 & (2\frac{1}{2}) & & 4 & -1 & 0 \\
 & & & & -1\,(1\frac{1}{2}) & & -1 & 4 & -1 \\
 & & & & (-2\frac{1}{2}) & -1 & 0 & -1 & 4
\end{bmatrix}.
$$

□ Eigenvalues and condition number of $\tilde{A}$: ??

Remark. Obviously, a different definition of $\tilde{\varphi}_{2,s}$, $s=1,\dots,9$, yields a different matrix $\tilde{A}$, however the spectrum of $\tilde{A}$ remains unchanged (why?).

$j=3$: $\tilde{A}$ is a 49×49 matrix.
Note that $\delta_3 = 2^{-1}\delta_2 = 2^{-2}\delta_1 = 2^{-3}\delta_0$, $\delta_0 = 1$. Assume $\tilde{\varphi}_{3,s} = \psi_{2,s} = \varphi_{3,s}$, $s \notin \{9,11,13,23,25,27,37,39,41\}$, $\tilde{\varphi}_{3,s} = \psi_{1,k_s} = \varphi_{2,k_s}$, $s = 9,11,13,23,27,37,39,41$, $k_s = 1,2,3,4,6,7,8,9$, $\tilde{\varphi}_{3,25} = \varphi_{1,1}$.

The hierarchical basis for $j=3$ (($j=2$) and [$j=1$]):

$$
\begin{bmatrix}
\varphi_{3,43} & \varphi_{3,44} & \varphi_{3,45} & \varphi_{3,46} & \varphi_{3,47} & \varphi_{3,48} & \varphi_{3,49} \\
\varphi_{3,36} & (\varphi_{2,7}) & \varphi_{3,38} & (\varphi_{2,8}) & \varphi_{3,40} & (\varphi_{2,9}) & \varphi_{3,42} \\
\varphi_{3,29} & \varphi_{3,30} & \varphi_{3,31} & \varphi_{3,32} & \varphi_{3,33} & \varphi_{3,34} & \varphi_{3,35} \\
\varphi_{3,22} & (\varphi_{2,4}) & \varphi_{3,24} & [(\varphi_{1,1})] & \varphi_{3,26} & (\varphi_{2,6}) & \varphi_{3,28} \\
\varphi_{3,15} & \varphi_{3,16} & \varphi_{3,17} & \varphi_{3,18} & \varphi_{3.19} & \varphi_{3,20} & \varphi_{3,21} \\
\varphi_{38} & (\varphi_{21}) & \varphi_{3,10} & (\varphi_{22}) & \varphi_{3,12} & (\varphi_{2,3}) & \varphi_{3,14} \\
\varphi_{31} & \varphi_{32} & \varphi_{33} & \varphi_{34} & \varphi_{35} & \varphi_{36} & \varphi_{37}
\end{bmatrix}.
$$

For $r, s \notin \{9, 11, 13, 23, 25, 27, 37, 39, 41\}$ we have

$$\tilde{a}_{rs} = \int_{s(\psi_{2,s})\cap s(\psi_{2,r})} \nabla\psi_{2,s}\nabla\psi_{2,r} = \int_{s(\varphi_{3,s})\cap s(\varphi_{3,r})} \nabla\varphi_{3,s}\nabla\varphi_{3,r} = a_{rs}.$$

The 9×9 submatrix of $\tilde{A}$ composed by the entries $\tilde{a}_{rs}$, $r, s \in \{9, 11, 13, 23, 25, 27, 37, 39, 41\}$, is equal to $\tilde{A}$, $j = 2$. The remaining entries of the 25th column (row) of $\tilde{A}$ are of type $\frac{\delta_3}{\delta_1}(\cdot) = \frac{1}{4}(\cdot)$. The remaining entries of the $9, 11, 13, 23, 27, 37, 39, 41$th columns (rows) of $\tilde{A}$ are of type $\frac{\delta_3}{\delta_2}(\cdot) = \frac{1}{2}(\cdot)$.

☐ Compute all entries of $\tilde{A}$, $j = 3$.

☐ Eigenvalues and condition number of $\tilde{A}$: ??

Cosine transform ?

Reading the proof of the fact that the sine transform is unitary, one also observes that

$$C^2 = 2(I+Q) = 2\begin{bmatrix} 2 & & & \\ & I & & J \\ & & 2 & \\ & J & & I \end{bmatrix},$$

$$C = \begin{bmatrix} C_{11} & C_{11}J \\ JC_{11} & JC_{11}J \end{bmatrix} + o_n \begin{bmatrix} 1 & \mathbf{e}^T & 1 & \mathbf{e}^T \\ \mathbf{e} & & \mathbf{v} & \\ 1 & \mathbf{v}^T & (-1)^{n+1} & \mathbf{v}^T J \\ \mathbf{e} & & J\mathbf{v} & \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ \vdots \\ (-1)^n \end{bmatrix},$$

$$\mathbf{x} + \mathbf{y} = \mathbf{e} = [1 \ 1 \ \cdots \ 1]^T, \quad \mathbf{y} - \mathbf{x} = \mathbf{v}, \ \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix},$$

$$\begin{aligned} &C_{11}^2 + o_n^2(\mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T) = I, \\ &(\mathbf{x}+\mathbf{y})^T C_{11} = -o_n \mathbf{y}^T, \\ &(\mathbf{x}-\mathbf{y})^T C_{11} = o_n \mathbf{y}^T J. \end{aligned}$$

☐ Check the previous remarks

☐ By using the previous remarks, try to introduce a unitary matrix $\hat{C}_{11}$, defining it in terms of C_{11}. Such $\hat{C}_{11}$ would define a fast cosine transform.