Abate: how to check if the eigenvalues of a matrix have negative real parts?
(Marco Abate was professor in Tor Vergata; now he is professor in SNS of Pisa)

Assume $n \geq 2$. Let D and C be the $(n-1) \times (n-1)$ matrices

$$D = \begin{bmatrix} n & & & & \\ & n+1 & & & \\ & & n+2 & & \\ & & & \ddots & \\ & & & & 2n-2 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & 1 & & & \\ -2 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ -2 & & & -1 & 1 \\ -2 & & & & -1 \end{bmatrix},$$

and set $A = DC$. What can one say about the eigenvalues of A ? Let us study the cases $n = 3, 4, 5, 6, 8$.

$$n = 3: \quad A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -9 & 3 \\ -8 & -4 \end{bmatrix}.$$

The eigenvalues of A are: $(-13 \pm \mathbf{i}\sqrt{71})/2$.

$$n = 4: \quad A = \begin{bmatrix} 4 & & \\ & 5 & \\ & & 6 \end{bmatrix} \begin{bmatrix} -3 & 1 & \\ -2 & -1 & 1 \\ -2 & & -1 \end{bmatrix} = \begin{bmatrix} -12 & 4 & \\ -10 & -5 & 5 \\ -12 & & -6 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned} p_A(\lambda) &= \det(\begin{bmatrix} \lambda+12 & -4 & \\ 10 & \lambda+5 & -5 \\ 12 & & \lambda+6 \end{bmatrix}) \\ &= (\lambda+12)[(\lambda+5)(\lambda+6)] + 4 \cdot 10[(\lambda+6)+6] \\ &= (\lambda+12)[\lambda^2 + 11\lambda + 70], \end{aligned}$$

therefore, one of the eigenvalues of A is $-12 = a_{11}$. The other two eigenvalues are $(-11 \pm \mathbf{i}\sqrt{159})/2$.

$$n = 5: \quad A = \begin{bmatrix} 5 & & & \\ & 6 & & \\ & & 7 & \\ & & & 8 \end{bmatrix} \begin{bmatrix} -3 & 1 & & \\ -2 & -1 & 1 & \\ -2 & & -1 & 1 \\ -2 & & & -1 \end{bmatrix} = \begin{bmatrix} -15 & 5 & & \\ -12 & -6 & 6 & \\ -14 & & -7 & 7 \\ -16 & & & -8 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned} p_A(\lambda) &= \det(\begin{bmatrix} \lambda+15 & -5 & & \\ 12 & \lambda+6 & -6 & \\ 14 & & \lambda+7 & -7 \\ 16 & & & \lambda+8 \end{bmatrix}) \\ &= (\lambda+15)[(\lambda+6)(\lambda+7)(\lambda+8)] \\ &\quad +5 \cdot 12[(\lambda+7)(\lambda+8) + 7(\lambda+8) + 7 \cdot 8]. \end{aligned}$$

What are the roots of this polynomial ? ...

$$n = 6: \quad A = \begin{bmatrix} 6 & & & & \\ & 7 & & & \\ & & 8 & & \\ & & & 9 & \\ & & & & 10 \end{bmatrix} \begin{bmatrix} -3 & 1 & & & \\ -2 & -1 & 1 & & \\ -2 & & -1 & 1 & \\ -2 & & & -1 & 1 \\ -2 & & & & -1 \end{bmatrix} = \begin{bmatrix} -18 & 6 & & & \\ -14 & -7 & 7 & & \\ -16 & & -8 & 8 & \\ -18 & & & -9 & 9 \\ -20 & & & & -10 \end{bmatrix}.$$

The characteristic polynomial of A is

$$
\begin{aligned}
p_A(\lambda) &= \det(\begin{bmatrix} \lambda+18 & -6 & & & \\ 14 & \lambda+7 & -7 & & \\ 16 & & \lambda+8 & -8 & \\ 18 & & & \lambda+9 & -9 \\ 20 & & & & \lambda+10 \end{bmatrix}) \\
&= (\lambda+18)\big[(\lambda+7)(\lambda+8)(\lambda+9)(\lambda+10)\big] \\
&\quad +6\cdot 14\big[(\lambda+8)(\lambda+9)(\lambda+10)+8(\lambda+9)(\lambda+10) \\
&\qquad +8\cdot 9(\lambda+10)+8\cdot 9\cdot 10\big].
\end{aligned}
$$

It can be shown that $p_A(-18)=0$, i.e. $-18=a_{11}$ is eigenvalue (see below).

We conjecture that for $n=8$ the matrix A (which is 7×7) has the following characteristic polynomial

$$
\begin{aligned}
p_A(\lambda) &= (\lambda+24)\big[(\lambda+9)(\lambda+10)(\lambda+11)(\lambda+12)(\lambda+13)(\lambda+14)\big] \\
&\quad +8\cdot 18\big[(\lambda+10)(\lambda+11)(\lambda+12)(\lambda+13)(\lambda+14) \\
&\qquad +10(\lambda+11)(\lambda+12)(\lambda+13)(\lambda+14) \\
&\qquad +10\cdot 11(\lambda+12)(\lambda+13)(\lambda+14) \\
&\qquad +10\cdot 11\cdot 12(\lambda+13)(\lambda+14) \\
&\qquad +10\cdot 11\cdot 12\cdot 13(\lambda+14) \\
&\qquad +10\cdot 11\cdot 12\cdot 13\cdot 14\big],
\end{aligned}
$$

and, for a generic n, where A is the following $(n-1)\times(n-1)$ matrix

$$
A = \begin{bmatrix} -3n & n & & & & \\ -2(n+1) & -(n+1) & (n+1) & & & \\ -2(n+2) & & -(n+2) & (n+2) & & \\ \vdots & & & \ddots & \ddots & \\ -2(2n-3) & & & & -(2n-3) & (2n-3) \\ -2(2n-2) & & & & & -(2n-2) \end{bmatrix},
$$

the characteristic polynomial of A has the following form

$$
\begin{aligned}
p_A(\lambda) &= (\lambda+3n)\big[(\lambda+n+1)(\lambda+n+2)\cdots(\lambda+2n-2)\big] \\
&\quad +n(2n+2)\big[(\lambda+n+2)\cdots(\lambda+2n-2) \\
&\quad +(n+2)(\lambda+n+3)\cdots(\lambda+2n-2) \\
&\quad +(n+2)(n+3)(\lambda+n+4)\cdots(\lambda+2n-2) \\
&\quad \cdots \\
&\quad +(n+2)(n+3)\cdots(2n-3)(\lambda+2n-2) \\
&\quad +(n+2)(n+3)\cdots(2n-2)\big] \\
&= (\lambda+3n)\ \prod_{j=1}^{n-2}(\lambda+n+j)+n(2n+2)q_{n-3}(\lambda)
\end{aligned}
$$

where q_{n-3} is the polynomial of degree $n-3$ here below:

$$
q_{n-3}(\lambda) = \sum_{k=1}^{n-2}\ \Pi_{j=1}^{k-1}(n+1+j)\ \Pi_{j=k+1}^{n-2}(\lambda+n+j).
$$

Note that if $t=\lambda+3n$, then

$$
\begin{aligned}
q_{n-3}(\lambda) &= \tilde{q}_{n-3}(t) = \sum_{k=1}^{n-2}\ \prod_{j=1}^{k-1}(n+1+j)\ \prod_{j=k+1}^{n-2}(t-2n+j) \\
&= \alpha t^0 + t(\ldots) = \alpha + (\lambda+3n)(\ldots)
\end{aligned}
$$

where

$$\begin{array}{rcl}\alpha & = & \sum_{k=1}^{n-2}\Pi_{j=1}^{k-1}(n+1+j)\Pi_{j=k+1}^{n-2}(-2n+j)=\sum_{k=1}^{n-2}(-1)^{n-k-2}a_k,\\ & & a_k=[(n+2)(n+3)\cdots(n+k)][(n+2)(n+3)\cdots(n+(n-k-1))].\end{array}$$

Moreover, since $a_k = a_{n-k-1}$, we have

$$\begin{array}{rcl}\alpha & = & \sum_{k=1}^{\lfloor n/2-1\rfloor}(-1)^{n-k-2}a_k+\sum_{k=\lceil n/2\rceil}^{n-2}(-1)^{n-k-2}a_k+\delta_{n,o}(-1)^{\frac{n-3}{2}}a_{\frac{n-1}{2}}\\ & = & \sum_{k=1}^{\lfloor n/2-1\rfloor}[(-1)^{n-k-2}+(-1)^{k-1}]a_k+\delta_{n,o}(-1)^{\frac{n-3}{2}}a_{\frac{n-1}{2}}.\end{array}$$

Thus,

$$\alpha=\left\{\begin{array}{ll}2\sum_{k=1}^{\frac{n-3}{2}}(-1)^{k-1}a_k+(-1)^{\frac{n-3}{2}}a_{\frac{n-1}{2}} & n \text{ odd}\\ 0 & n \text{ even}\end{array}\right. .$$

It follows that for n even $q_{n-3}(\lambda)=(\lambda+3n)(\ldots)$ and $p_A(\lambda)=(\lambda+3n)[\Pi_{j=1}^{n-2}(\lambda+n+j)+n(2n+2)(\ldots)]$. In other words, $-3n$ is eigenvalue of the $(n-1)\times(n-1)$ matrix A for all n even.

Let us look for the eigenvector $\mathbf{x}=[x_1\ x_2\ \cdots\ x_{n-1}]^T$ of A corresponding to the eigenvalue $-3n$. Note that $A\mathbf{x}=-3n\mathbf{x}$ iff $C\mathbf{x}=-3nD^{-1}\mathbf{x}$ iff

$$\begin{array}{l}-3x_1+x_2=-3n\frac{1}{n}x_1\\ -2x_1-x_2+x_3=-3n\frac{1}{n+1}x_2\\ -2x_1-x_3+x_4=-3n\frac{1}{n+2}x_3\\ \cdots\\ -2x_1-x_{n-2}+x_{n-1}=-3n\frac{1}{2n-3}x_{n-2}\\ -2x_1-x_{n-1}=-3n\frac{1}{2n-2}x_{n-1}\end{array}$$

if and only if

$$\begin{array}{l}x_1 \text{ arbitrary}\\ x_2=0\\ x_i=-\frac{2n-i+2}{n+i-2}x_{i-1}+2x_1,\ i=3,\ldots,n-1\\ x_{n-1}=4\frac{n-1}{n+2}x_1\end{array}$$

□ Prove that if the first $n-2$ equations are verified, then also the $(n-1)$th equation is verified, provided that n is even.

Abate, 15 Ottobre 1998: "Non è difficilissimo dimostrare che quando n è pari (per cui la matrice ha ordine dispari) allora $-3n$ è autovalore di A". Moreover, Abate thinks that "la matrice A [...] non ha autovalori con parte reale positiva.", but he is not able to prove his assertion.
How to prove Abate assertion?

(in a place, near our matrix A, n generic, I have written $\Re(\lambda_i)\leq n$ ...)

A sufficient condition for a matrix A to have $\Re(\lambda(A))<0$

We know that $A_h=\frac{1}{2}(A+A^*)$ p.d. implies $\Re(\lambda(A))>0$. Analogously, one can prove that $A_h=\frac{1}{2}(A+A^*)$ n.d. (negative definite) implies $\Re(\lambda(A))<0$.

But, for our matrix A, the matrix $A+A^*$ is not n.d. already for $n=4$. In fact:

$$n=3:\quad A+A^*=\left[\begin{array}{cc}-18 & -5\\ -5 & -8\end{array}\right],\quad \text{n.d.},$$

$$n = 4: \quad A + A^* = \begin{bmatrix} -24 & -6 & -12 \\ -6 & -10 & 5 \\ -12 & 5 & -12 \end{bmatrix}, \quad \text{not n.d..}$$

A necessary condition for a matrix A to have $\Re(\lambda(A)) < 0$ (see toe_1e)

Let A be a $n \times n$ real matrix and assume that $\Re(\lambda(A)) < 0$. Set $\mathbf{h}(\mathbf{x}) = \mathbf{b} + A\mathbf{x}$ where $\mathbf{b} \in \mathbb{R}^n$. Note that $\mathbf{h}(-A^{-1}\mathbf{b}) = \mathbf{0}$, $\mathbf{h} \in C^1(\mathbb{R}^n)$, and the eigenvalues of $J_{\mathbf{h}}(-A^{-1}\mathbf{b}) = J_{\mathbf{h}}(\mathbf{x}) = A$ have negative real parts. It follows that $\mathbf{x}(t)$, the solution of

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{b} + A\mathbf{x}(t), \ t > 0, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{p}$$

tends to $-A^{-1}\mathbf{b}$ (as $t \to +\infty$) for any choice of $\mathbf{x}_0$. Analogously, the Richardson-Eulero iterative scheme (Euler method applied to (p)) $\eta(t_{k+1}) = \eta(t_k) + \omega(\mathbf{b} + A\eta(t_k))$, $\omega > 0$, must converge to $-A^{-1}\mathbf{b}$ for all $\eta(0) = \mathbf{x}_0$ (if ω is sufficiently small).

One could check if the above Richardson-Eulero method converges when A is the Abate matrix ...

Necessary and necessary and sufficient conditions for a polynomial to have all roots with negative real part

Let $p(z)$ be a polynomial of degree n,

$$p(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n, \quad a_i \in \mathbb{R}, \ a_0 > 0,$$

and Δ_i, $i = 1, \ldots, n$, the determinant of the $i \times i$ upper left submatrix of the $n \times n$ matrix

$$M_p = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & & & \\ a_0 & a_2 & a_4 & \cdots & & & \\ 0 & a_1 & a_3 & a_5 & \cdots & & \\ 0 & a_2 & a_2 & a_4 & \cdots & & \\ & 0 & \cdots & \cdots & & & \\ & & & & & \ddots & \\ & & & & & & a_n \end{bmatrix}, \quad a_k = 0 \text{ if } k > n.$$

Then the roots of p have negative real parts if and only if $\Delta_i > 0$, $i = 1, \ldots, n$. Moreover, if the roots of p have negative real parts, then $a_i > 0$, $i = 1, \ldots, n$. (Check this well known result!).

For example, set $p(z) = z^3 + 23z^2 + 202z + 840$. The corresponding matrix M_p,

$$M_p = \begin{bmatrix} 23 & 840 & 0 \\ 1 & 202 & 0 \\ 0 & 23 & 840 \end{bmatrix},$$

has positive diagonal entries, so the necessary condition is verified. Moreover, $\Delta_1 = 23 > 0$, $\Delta_2 = 23 \cdot 202 - 840 \cdot 1 > 0$, $\Delta_3 = 840\Delta_2 > 0$, thus the roots of the polynomial p have negative real parts.

☐ Set $p(z) = z^4 + 2z^3 + 3z^2 + 4z + a$, $a \in \mathbb{R}$. Find the values of a for which the roots of p have negative real parts.

Write the matrix M_p:

$$M_p = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 1 & 3 & a & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 3 & a \end{bmatrix}.$$

In order to verify the necessary condition, the parameter a must be positive. Moreover, $\Delta_1 = 2 > 0$, $\Delta_2 = 2 \cdot 3 - 4 \cdot 1 = 2 > 0$, $\Delta_3 = 4 \cdot 2 - 4a$, $\Delta_4 = a\Delta_3$. So, the roots of the polynomial p have negative real parts iff $0 < a < 2$.

However the above criterium does not seem to be useful in our case because the coefficients of $p_A(\lambda)$ cannot be easily written (for n generic).

□ Is there a necessary and sufficient condition on the entries of A (instead of on the coefficients of $p_A(\lambda)$) in order to establish that the eigenvalues of A have negative real parts ?

□ Given a polynomial p it is easy to write a matrix whose characteristic polynomial is p. Given a matrix A, how to write a polynomial whose roots are the the eigenvalues of A ?

Some other remarks on the Abate matrix A

Remark 1. Set $D_{n-1} = D$ and $C_{n-1} = C$. Then

$$D_{n-1} = \begin{bmatrix} D_{n-2} + I & \mathbf{0} \\ \mathbf{0}^T & 2n-2 \end{bmatrix}, \quad C_{n-1} = \begin{bmatrix} \begin{matrix} C_{n-2} \end{matrix} & \begin{matrix} \mathbf{0} \\ 1 \end{matrix} \\ -2 \quad \mathbf{0}^T & -1 \end{bmatrix}.$$

It follows that

$$\begin{aligned} A = A_{n-1} = D_{n-1}C_{n-1} &= \begin{bmatrix} (D_{n-2}+I)C_{n-2} & \begin{matrix} \mathbf{0} \\ 2n-3 \end{matrix} \\ -2(2n-2) \quad \mathbf{0}^T & -(2n-2) \end{bmatrix} \\ &= \begin{bmatrix} A_{n-2} + C_{n-2} & \begin{matrix} \mathbf{0} \\ 2n-3 \end{matrix} \\ -2(2n-2) \quad \mathbf{0}^T & -(2n-2) \end{bmatrix}. \end{aligned}$$

Remark 2. The matrix C can be written as the product of two matrices whose eigenvalues are known. For example, if $n = 5$ then

$$\begin{bmatrix} -3 & 1 & & \\ -2 & -1 & 1 & \\ -2 & & -1 & 1 \\ -2 & & & -1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{bmatrix}.$$

For n generic:

$$\begin{bmatrix} -3 & 1 & & & \\ -2 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ -2 & & & -1 & 1 \\ -2 & & & & -1 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ \vdots & \vdots & \ddots & & \\ 1 & 1 & \cdots & 1 & \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}.$$

☐ Find the eigenvalues/eigenvectors of the tridiagonal matrix on the right.

Another decomposition of C is obtained here below. Note that

$$\begin{bmatrix} -3 & 1 & & \\ -2 & -1 & 1 & \\ -2 & & -1 & 1 \\ -2 & & & -1 \end{bmatrix} = \begin{bmatrix} -10 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 7 & 1 & & \\ 11 & 1 & 1 & \\ 13 & 1 & 1 & 1 \end{bmatrix}.$$

(We have found this decomposition by imposing that the nonzero non diagonal entries of the tridiagonal matrix and the diagonal entries of the triangular matrix are equal to 1). Moreover, for n generic:

$$\begin{bmatrix} -3 & 1 & & & \\ -2 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ -2 & & & -1 & 1 \\ -2 & & & & -1 \end{bmatrix} = \begin{bmatrix} -a & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix} \begin{bmatrix} u_0 = 1 & & & & & \\ u_1 & 1 & & & & \\ u_2 & 1 & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ u_{n-3} & 1 & 1 & \cdots & 1 & \\ u_{n-2} & 1 & 1 & \cdots & 1 & 1 \end{bmatrix},$$

where the latter equality holds if the following conditions are satisfied: $u_0 = 1$, $u_1 = a - 3$, $u_i = 2u_{i-1} - u_{i-2} - 2$, $i = 2, \ldots, n-2$, $u_{n-2} = u_{n-3} + 2$. Let us find the general solution of the difference equation

$$u_i - 2u_{i-1} + u_{i-2} = -2. \qquad \text{(de)}$$

Characteristic equation: $z^2 - 2z + 1 = 0$. General solution of the homogeneous equation associated to (de): $\alpha + \beta i$, $\alpha, \beta \in \mathbb{R}$. Particular solution of the equation (de): $-i^2$. General solution of (de): $\alpha + \beta i - i^2$, $\alpha, \beta \in \mathbb{R}$. The initial conditions imply $\alpha = 1$, $\beta = a - 3$. Set $u_i = 1 + (a-3)i - i^2$: then $u_{n-2} = u_{n-3} + 2$ iff $a = 2n$.

So, the claimed decomposition is obtained:

$$\begin{bmatrix} -3 & 1 & & & \\ -2 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ -2 & & & -1 & 1 \\ -2 & & & & -1 \end{bmatrix} = \begin{bmatrix} -2n & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & & & & & \\ u_1 & 1 & & & & \\ \vdots & \vdots & \ddots & & & \\ 1 + u_1 i - i^2 & 1 & & 1 & & \\ \vdots & \vdots & & & \ddots & \\ 1 + u_1(n-2) - (n-2)^2 & 1 & \cdots & 1 & \cdots & 1 \end{bmatrix},$$

$$u_1 = 2n - 3.$$

For example, if $n = 4$:

$$C = \begin{bmatrix} -3 & 1 & \\ -2 & -1 & 1 \\ -2 & & -1 \end{bmatrix} = \begin{bmatrix} -8 & 1 & \\ 1 & -2 & 1 \\ & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 5 & 1 & \\ 7 & 1 & 1 \end{bmatrix}.$$

Recall that we have to check the sign of the real part of the eigenvalues of $A = DC = DC_1C_2$, where C_1, C_2 can be taken from the above decompositions. Note that C_2DC_1 and C_1C_2D have the same eigenvalues of $A \ldots$

Solving the exercise on p.3

If x_i, $i = 3, \ldots, n-1$, are such that

$$\begin{aligned} x_i &= -\tfrac{2n-i+2}{n+i-2}x_{i-1} + 2x_1,\ i = 3, 4, \ldots, n-1, \\ x_{n-1} &= 4\tfrac{n-1}{n+2}x_1, \end{aligned}$$

where $x_2 = 0$, x_1 is an arbitrary nonzero number, and $\mathbf{x} = [x_1\ x_2\ \cdots\ x_{n-1}]^T$, then $A\mathbf{x} = -3n\mathbf{x}$.

Let us see if such x_i, $i = 3, \ldots, n-1$, exist. It is clear that they exist iff they are of type $\alpha_i x_1$ with the α_i, $i = 3, \ldots, n-1$, satisfying the identities

$$\begin{aligned} \alpha_i &= -\tfrac{2n-i+2}{n+i-2}\alpha_{i-1} + 2,\ i = 3, 4, \ldots, n-1, \\ \alpha_{n-1} &= 4\tfrac{n-1}{n+2}, \end{aligned}$$

where $\alpha_2 = 0$.

Now, first we show that there exist and are uniquely determined $\alpha_i \in \mathbb{R}$, $i = 3, \ldots, n-1$, satisfying the difference equation

$$y_i = -\frac{2n-i+2}{n+i-2}y_{i-1} + 2,\ i = 3, 4, \ldots. \tag{de}$$

and the condition $y_2 = 0$. Second we observe that they satisfy the condition $\alpha_{n-1} = 4\frac{n-1}{n+2}$ if and only if n is even.

If we look for a solution of type $y_i = ri + s$, $r, s \in \mathbb{R}$, of the equation

$$(n+i-2)y_i + (2n-i+2)y_{i-1} = 2(n+i-2)$$

then we obtain the conditions $r = 2/(3n+1)$, $s = 2(n-1)/(3n+1)$. So, we have a particular solution of (de):

$$y_i = \frac{2}{3n+1}i + \frac{2(n-1)}{3n+1}.$$

Now consider the homogeneous equation associated to (de):

$$y_i = -\frac{2n-i+2}{n+i-2}y_{i-1}.$$

We need its general solution. Observe that

$$\begin{aligned} y_2 &= -\tfrac{2n}{n}y_1,\ y_3 = -\tfrac{2n-1}{n+1}y_2 = \tfrac{(2n-1)2n}{(n+1)n}y_1, \\ y_4 &= -\tfrac{2n-2}{n+2}y_3 = -\tfrac{(2n-2)(2n-1)2n}{(n+2)(n+1)n}y_1,\ \ldots \end{aligned}$$

Thus,

$$y_i = c(-1)^{i-1}\frac{(2n-i+2)\cdots(2n-1)2n}{(n+i-2)\cdots(n+1)n},\ c \in \mathbb{R}.$$

Finally, the general solution of (de) is obtained by addition:

$$y_i = c(-1)^{i-1}\frac{(2n-i+2)\cdots(2n-1)2n}{(n+i-2)\cdots(n+1)n} + \frac{2}{3n+1}i + \frac{2(n-1)}{3n+1},\ c \in \mathbb{R}.$$

Among these, we are interested the one satisfying the condition $y_2 = 0$, which is obtained for $c = (n+1)/(3n+1)$. So, the first step is complete:

$$\alpha_i = \frac{n+1}{3n+1}(-1)^{i-1}\frac{(2n-i+2)\cdots(2n-1)2n}{(n+i-2)\cdots(n+1)n} + \frac{2}{3n+1}i + \frac{2(n-1)}{3n+1}. \quad (*)$$

For the second step note that the latter expression for $i = n-1$ implies

$$\begin{aligned}\alpha_{n-1} &= (-1)^{n-2}\tfrac{n+1}{3n+1}\tfrac{(n+3)\cdots(2n-1)2n}{(2n-3)\cdots(n+1)n} + \tfrac{2}{3n+1}(n-1) + \tfrac{2(n-1)}{3n+1} \\ &= \tfrac{4(n-1)(n+2+(-1)^{n-2}(2n-1))}{(3n+1)(n+2)}.\end{aligned}$$

Thus the further condition $\alpha_{n-1} = 4\frac{n-1}{n+2}$ is satisfied if and only if n is even.

Remarks

The first α_i:

$$\alpha_3 = 2,\ \alpha_4 = \frac{2(4-n)}{n+2},\ \alpha_5 = \frac{2(3n^2-6n+18)}{(n+2)(n+3)}.$$

A formula for the α_i alternative to (*):

$$\begin{aligned}\alpha_i &= \tfrac{2}{3n+1}\tfrac{(n+2)\cdots(n+i-2)(n+i-1)+(-1)^{i-1}(2n-i+2)\cdots(2n-2)(2n-1)}{(n+2)\cdots(n+i-2)} \\ &= \tfrac{2p_{i-3}(n)}{(n+2)\cdots(n+i-2)},\ i = 3,\dots,n-1,\end{aligned}$$

where the polynomials $p_j(n)$ are defined by $p_0(n) = 1$, $p_{i-3}(n) = (n+i-1)p_{i-4}(n)+(-1)^{i-1}(2n-i+3)\cdots(2n-1)$, $i = 4,\dots,n-1$. (Note that $p_{n-4}(n) = 2(n-1)(n+3)\cdots(2n-3)$ if n is even). The latter equality can be shown by induction: assume it true for $i = k$, i.e.

$$(3n+1)p_{k-3}(n) = (n+2)\cdots(n+k-2)(n+k-1)+(-1)^{k-1}(2n-k+2)\cdots(2n-2)(2n-1),$$

then prove it for $i = k+1$ (note that it is true for $i = 0$).

Deflation: a matrix whose eigenvalues are the remaining eigenvalues of A

We have proved that, if n is even, then there exist $\alpha_3,\dots,\alpha_{n-1}$ such that

$$A\mathbf{x} = A\begin{bmatrix} x_1 \\ 0 \\ \alpha_3 x_1 \\ \vdots \\ \alpha_{n-1}x_1\end{bmatrix} = -3n\begin{bmatrix} x_1 \\ 0 \\ \alpha_3 x_1 \\ \vdots \\ \alpha_{n-1}x_1\end{bmatrix} = -3n\mathbf{x},$$

where x_1 can be an arbitrary nonzero number. Thus, *if n is even, $-3n$ is eigenvalue of the $(n-1)\times(n-1)$ matrix A.*

Observe that the matrix $W = A - \frac{-3n}{\mathbf{w}^*\mathbf{x}}\mathbf{x}\mathbf{w}^*$, for any vector $\mathbf{w}$ such that $\mathbf{w}^*\mathbf{x} \neq 0$, has the same eigenvalues of A except $-3n$ which is replaced with 0. In particular, this is true for the matrix $W = A - \frac{1}{x_1}\mathbf{x}\mathbf{e}_1^T A$ (choose $\mathbf{w}^* = \mathbf{e}_1^T A$: $\mathbf{w}^*\mathbf{x} = -3nx_1 \neq 0$). Let us write such matrix W.

Since $(1/x_1)\mathbf{x} = [1\ 0\ \alpha_3\ \cdots\ \alpha_{n-1}]^T$ and $\mathbf{e}_1^T A = [-3n\ \ n\ \ 0\ \cdots\ 0]$, we have

$$W = \begin{bmatrix} -3n & n & & & \\ -2(n+1) & -(n+1) & n+1 & & \\ -2(n+2) & & -(n+2) & \ddots & \\ \vdots & & & \ddots & 2n-3 \\ -2(2n-2) & & & & -(2n-2)\end{bmatrix} - \begin{bmatrix} -3n & n & & & \\ 0 & 0 & 0 & \cdots & 0 \\ -3n\alpha_3 & n\alpha_3 & & & \\ \vdots & \vdots & & & \\ -3n\alpha_{n-1} & n\alpha_{n-1} & & & \end{bmatrix}$$

$$
= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -2(n+1) & -(n+1) & n+1 & & \\ -2(n+2)+3n\alpha_3 & -n\alpha_3 & -(n+2) & \ddots & \\ \vdots & \vdots & & \ddots & 2n-3 \\ -2(2n-2)+3n\alpha_{n-1} & -n\alpha_{n-1} & & & -(2n-2) \end{bmatrix}.
$$

It follows that *the remaining eigenvalues of the matrix A are the eigenvalues of the following $(n-2)\times(n-2)$ matrix B*:

$$
B = \begin{bmatrix} -(n+1) & n+1 & & \\ -n\alpha_3 & -(n+2) & \ddots & \\ \vdots & & \ddots & 2n-3 \\ -n\alpha_{n-1} & & & -(2n-2) \end{bmatrix}, \quad n \text{ even},
$$

$$
\alpha_3 = 2,\ \alpha_4 = 2\frac{4-n}{n+2},\ \alpha_5 = 2\frac{3n^2-6n+18}{(n+2)(n+3)},
$$

$$
\alpha_6 = 2\frac{-5n^3+33n^2-34n+96}{(n+2)(n+3)(n+4)},\ \alpha_7 = 2\frac{11n^4-77n^3+304n^2-208n+600}{(n+2)(n+3)(n+4)(n+5)},
$$

$$
\begin{aligned} \alpha_i &= \tfrac{2}{3n+1}\Big((-1)^{i-1}\tfrac{(2n-i+2)\cdot(2n-1)}{(n+2)\cdots(n+i-2)} + i + n - 1\Big) \\ &= \tfrac{2p_{i-3}(n)}{(n+2)\cdots(n+i-2)},\ i = 3,\dots,n-1, \end{aligned}
$$

$$
p_0(n) = 1,\ p_1(n) = 4-n,\ p_2(n) = 3n^2-6n+18,\ \dots
$$

$$
\begin{gathered} p_{i-3}(n) = (n+i-1)p_{i-4}(n) + (-1)^{i-1}(2n-i+3)\cdots(2n-1), \\ i = 4,\dots,n-1,\quad p_{n-4}(n) = 2(n-1)(n+3)\cdots(2n-3). \end{gathered}
$$

Now observe that if the matrix

$$
-(B+B^*) = \begin{bmatrix} 2(n+1) & n-1 & n\alpha_4 & \cdots & n\alpha_{n-1} \\ n-1 & 2(n+2) & -(n+2) & & \\ n\alpha_4 & -(n+2) & 2(n+3) & \ddots & \\ \vdots & & \ddots & \ddots & -(2n-3) \\ n\alpha_{n-1} & & & -(2n-3) & 2(2n-2) \end{bmatrix}
$$

is p.d., then $\Re(\lambda(-B))$ would be positive, that is, the remaining eigenvalues of A would have negative real part (and the conjecture of Abate would be proved).

Let us check for small even values of n if $-(B+B^*)$is effectively p.d.. For $n = 4$ we have:

$$
B = \begin{bmatrix} -5 & 5 \\ -4\alpha_3 & -6 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ -8 & -6 \end{bmatrix},\quad -(B+B^*) = \begin{bmatrix} 10 & 3 \\ 3 & 12 \end{bmatrix}.
$$

($\alpha_3 = 2$). It is clear that $-(B+B^*)$ is p.d.. (Note, on the contrary, that the matrix $-(A+A^*)$ for $n = 4$ is not p.d.:

$$
A = \begin{bmatrix} -12 & 4 & \\ -10 & -5 & 5 \\ -12 & & -6 \end{bmatrix},\quad -(A+A^*) = \begin{bmatrix} 24 & 6 & 12 \\ 6 & 10 & -5 \\ 12 & -5 & 12 \end{bmatrix}).
$$

For $n = 6$ the matrix B becomes:

$$B = \begin{bmatrix} -7 & 7 & & \\ -6\alpha_3 & -8 & 8 & \\ -6\alpha_4 & 0 & -9 & 9 \\ -6\alpha_5 & 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} -7 & 7 & & \\ -12 & -8 & 8 & \\ 3 & 0 & -9 & 9 \\ -15 & 0 & 0 & -10 \end{bmatrix}$$

($\alpha_3 = 2$, $\alpha_4 = -1/2$, $\alpha_5 = 5/2$). Thus,

$$-(B + B^*) = \begin{bmatrix} 14 & 5 & -3 & 15 \\ 5 & 16 & -8 & 0 \\ -3 & -8 & 18 & -9 \\ 15 & 0 & -9 & 20 \end{bmatrix}.$$

Note that the determinants of the upper left $i \times i$ submatrices of $-(B+B^*)$ are positive for $i = 1, 2, 3$, but $\det(-(B + B^*)) < 0$, so it is not p.d..

For $n = 8$ the matrix B becomes:

$$B = \begin{bmatrix} -9 & 9 & & & & \\ -8\alpha_3 & -10 & 10 & & & \\ -8\alpha_4 & 0 & -11 & 11 & & \\ -8\alpha_5 & 0 & 0 & -12 & 12 & \\ -8\alpha_6 & 0 & 0 & 0 & -13 & 13 \\ -8\alpha_7 & 0 & & & & -14 \end{bmatrix} = \begin{bmatrix} -9 & 9 & & & & \\ -16 & -10 & 10 & & & \\ 32/5 & 0 & -11 & 11 & & \\ -1296/55 & 0 & 0 & -12 & 12 & \\ 416/55 & 0 & 0 & 0 & -13 & 13 \\ -112/5 & 0 & & & & -14 \end{bmatrix}$$

($\alpha_4 = -4/5$, $\alpha_5 = 162/55$, $\alpha_6 = -52/55$, $\alpha_7 = 14/5$). Thus,

$$-(B + B^*) = \begin{bmatrix} 18 & 7 & -32/5 & 1296/55 & -416/55 & 112/5 \\ 7 & 20 & -10 & & & \\ -32/5 & -10 & 22 & -11 & & \\ 1296/55 & & -11 & 24 & -12 & \\ -416/55 & & & -12 & 26 & -13 \\ 112/5 & & & & -13 & 28 \end{bmatrix}.$$

It can be seen that $-(B + B^*)$ is not p.d..

Is the $(n-1) \times (n-1)$ *matrix* $-(A + A^*)$ *p.d. for all odd values of* n *?* No, in fact, for $n = 5$ we have

$$A = \begin{bmatrix} -15 & 5 & & \\ -12 & -6 & 6 & \\ -14 & & -7 & 7 \\ -16 & & & -8 \end{bmatrix}, \quad -(A + A^*) = \begin{bmatrix} 30 & 7 & 14 & 16 \\ 7 & 12 & -6 & \\ 14 & -6 & 14 & -7 \\ 16 & & -7 & 16 \end{bmatrix}$$

and the determinant of the 3×3 upper left submatrix of the latter matrix is negative.

So, the conjecture of Abate remains a conjecture :(

An appendix on the Abate matrix A

In toe_1h, in the case n even, we have introduced a $(n-2) \times (n-2)$ matrix B such that the eigenvalues of the $(n-1) \times (n-1)$ Abate matrix A are $\{-3n\} \cup$ {eigenvalues of B}.

Here we show that, by multiplying B on the left by a diagonal matrix, one obtains a matrix C' of type

$$C' = \begin{bmatrix} -1 & 1 & & & & \\ \beta_3 & -1 & \ddots & & & \\ \vdots & & \ddots & & & \\ \beta_{n/2+1} & & & -1 & & \\ \vdots & & & & \ddots & \\ \beta_3 & & & & & -1 \end{bmatrix}.$$

We first observe this fact in the cases $n = 4, 6, 8$:

$$\begin{bmatrix} 1/5 & \\ & 1/6 \end{bmatrix} \begin{bmatrix} -5 & 5 \\ -8 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4/3 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1/7 & & & \\ & 1/8 & & \\ & & 1/9 & \\ & & & 1/10 \end{bmatrix} \begin{bmatrix} -7 & 7 & & \\ -12 & -8 & 8 & \\ 3 & & -9 & 9 \\ -15 & & & -10 \end{bmatrix} = \begin{bmatrix} -1 & 1 & & \\ -3/2 & -1 & 1 & \\ 1/3 & & -1 & 1 \\ -3/2 & & & -1 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{9} & & & & & \\ & \frac{1}{10} & & & & \\ & & \frac{1}{11} & & & \\ & & & \frac{1}{12} & & \\ & & & & \frac{1}{13} & \\ & & & & & \frac{1}{14} \end{bmatrix} \begin{bmatrix} -9 & 9 & & & & \\ -16 & -10 & 10 & & & \\ \frac{32}{5} & & -11 & 11 & & \\ -\frac{48*27}{55} & & & -12 & 12 & \\ \frac{32*13}{55} & & & & -13 & 13 \\ -\frac{8*14}{5} & & & & & -14 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & & & & \\ -\frac{8}{5} & -1 & 1 & & & \\ \frac{32}{55} & & -1 & 1 & & \\ -\frac{108}{55} & & & -1 & 1 & \\ \frac{32}{55} & & & & -1 & 1 \\ -\frac{8}{5} & & & & & -1 \end{bmatrix}.$$

Then we prove it for each n (even), by the following two steps:

(1)

$$D'^{-1}B := \begin{bmatrix} \frac{1}{n+1} & & & \\ & \frac{1}{n+2} & & \\ & & \ddots & \\ & & & \frac{1}{2n-2} \end{bmatrix} \begin{bmatrix} -(n+1) & n+1 & & \\ -n\alpha_3 & -(n+2) & \ddots & \\ \vdots & & \ddots & 2n-3 \\ -n\alpha_{n-1} & & & -(2n-2) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & & \\ \frac{-n\alpha_3}{n+2} & -1 & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{-n\alpha_{n-1}}{2n-2} & & & -1 \end{bmatrix} =: C';$$

(2) observe that

$$\frac{\alpha_i}{n+i-1} = \frac{\alpha_{n-i+2}}{2n-i+1}, \; i = 3, \dots, n-1.$$

Thus (1) can be rewritten as follows

$$D'^{-1}B := \begin{bmatrix} \frac{1}{n+1} & & & \\ & \frac{1}{n+2} & & \\ & & \ddots & \\ & & & \frac{1}{2n-2} \end{bmatrix} \begin{bmatrix} -(n+1) & n+1 & & & & & \\ -n\alpha_3 & -(n+2) & n+2 & & & & \\ -n\alpha_4 & & & \ddots & & & \\ \vdots & & & & & & \\ -n\alpha_{n/2+1} & & & & -(n+\frac{n}{2}) & & \\ \vdots & & & & & & \\ -n\alpha_4 \frac{2n-3}{n+3} & & & & & \ddots & 2n-3 \\ -n\alpha_3 \frac{2n-2}{n+2} & & & & & & -(2n-2) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & & & & \\ \frac{-n\alpha_3}{n+2} & -1 & \ddots & & & \\ \vdots & & \ddots & & & \\ \frac{-n\alpha_{n/2+1}}{n+\frac{n}{2}} & & & -1 & & \\ \vdots & & & & \ddots & \\ \frac{-n\alpha_3}{n+2} & & & & & -1 \end{bmatrix} =: C'.$$

So, a decomposition for the matrix B, $B = D'C'$, analogous to the decomposition of A, $A = DC$, holds:

$$B = \begin{bmatrix} n+1 & & & \\ & n+2 & & \\ & & \ddots & \\ & & & 2n-2 \end{bmatrix} \begin{bmatrix} -1 & 1 & & & & \\ \frac{-n\alpha_3}{n+2} & -1 & \ddots & & & \\ \vdots & & \ddots & & & \\ \frac{-n\alpha_{n/2+1}}{n+\frac{n}{2}} & & & -1 & & \\ \vdots & & & & \ddots & \\ \frac{-n\alpha_3}{n+2} & & & & & -1 \end{bmatrix},$$

$$\begin{aligned} \alpha_i &= 2\frac{n+i-1}{3n+1}\left((-1)^{i-1}\frac{(2n-i+2)\cdots(2n-1)}{(n+2)\cdots(n+i-1)}+1\right) \\ &= \frac{2p_{i-3}(n)}{(n+2)\cdots(n+i-2)}, \ i = 3,\ldots,\frac{n}{2}+1. \end{aligned}$$

$p_0(n) = 1$,
$p_{i-3}(n) = (n+i-1)p_{i-4}(n) + (-1)^{i-1}(2n-i+3)\cdots(2n-1)$,
$i = 4,\ldots,\frac{n}{2}+1$,

$p_1(n) = 4-n$,
$p_2(n) = 3n^2 - 6n + 18$,
$p_3(n) = -5n^3 + 33n^2 - 34n + 96$,
$p_4(n) = 11n^4 - 77n^3 + 304n^2 - 208n + 600$,
$\ldots$