On the matrix P of the web

Let n be a positive integer. Let $i = 1, 2, \dots, n$ be the sites of the web. For each site i, let μ_i and $\nu(i)$ be the number of sites pointed by i and the set whose elements are the sites pointed by i, respectively ($|\nu(i)| = \mu_i$).

Moreover, for each site i let $\rho(i)$ be the set whose elements are the sites which point to i. Note that the $\rho(i)$ can be computed from the $\nu(i)$ via the algorithm:

For $i = 1, 2, \dots, n$

  $\rho(i) := \emptyset$;

  for $k = 1, 2, \dots, n$

    if $i \in \nu(k)$ then $\rho(i) := \rho(i) \cup \{k\}$

Now let P be the $n \times n$ matrix whose ith row, $i = 1, \dots, n$, has all entries equal to zero except those whose indeces are in $\nu(i)$ which are set equal to $1/\mu_i$. Note that

$$(P\mathbf{x})_k = \begin{cases} \frac{1}{\mu_k} \sum_{s\in\nu(k)} x_s & \nu(k) \neq \emptyset \\ 0 & \nu(k) = \emptyset \end{cases}, \quad k = 1, 2, \dots, n;$$

$$(P^T\mathbf{x})_k = \begin{cases} \sum_{s\in\rho(k)} \frac{1}{\mu_s} x_s & \rho(k) \neq \emptyset \\ 0 & \rho(k) = \emptyset \end{cases}, \quad k = 1, 2, \dots, n;$$

$$\mathbf{x}^* P\mathbf{x} = \sum_{k=1}^{n} \overline{x_k}(P\mathbf{x})_k = \sum_{k:\nu(k)\neq\emptyset} \overline{x_k}\frac{1}{\mu_k} \sum_{s\in\nu(k)} x_s$$

$$= (P^T\mathbf{x})^*\mathbf{x} = \sum_{k=1}^{n} \overline{(P^T\mathbf{x})_k} x_k = \sum_{k:\rho(k)\neq\emptyset} x_k \sum_{s\in\rho(k)} \frac{1}{\mu_s}\overline{x_s}.$$

Example: $n = 5$,

$$\begin{array}{ll} \mu_1 = 0 & \nu(1) = \emptyset \\ \mu_2 = 3 & \nu(2) = \{1,3,5\} \\ \mu_3 = 2 & \nu(3) = \{2,5\} \\ \mu_4 = 0 & \nu(4) = \emptyset \\ \mu_5 = 2 & \nu(5) = \{1,3\} \end{array}$$

Compute $\rho(i)$, $i = 1, 2, 3, 4, 5$, and write the 5×5 matrix P.

On $\rho(P^*P)$ and $\mathbf{v}$ such that $P^*P\mathbf{v} = \rho(P^*P)\mathbf{v}$

Upper bounds for $\rho(P^*P)$:

$$\rho(P^*P) = \rho(PP^*) = \|P^*P\|_2 = \|PP^*\|_2 \leq \|P^*P\|_F = \|PP^*\|_F,$$
$$\rho(P^*P) \leq \min\{\min\{\|PP^*\|_1, \|P^*P\|_1\}, \min\{\|PP^*\|_\infty, \|P^*P\|_\infty\}\}$$

(note that $\|AA^*\|_{2,F} = \|A^*A\|_{2,F}$, $\forall A$; question: $\|AA^*\|_1 = \|A^*A\|_1$?). One can easily prove the identities

$$(PP^*)_{ij} = \frac{1}{\mu_i\mu_j}|\nu(i)\cap\nu(j)|, \ (P^*P)_{ij} = \sum_{k\in\rho(i)\cap\rho(j)} \frac{1}{\mu_k^2}.$$

So, we have the bounds

$$\rho(P^*P) \le \frac{|\{i : \nu(i) \neq \emptyset\}|}{\min_{i:\nu(i)\neq\emptyset} \mu_i}$$

and

$$\rho(P^*P) \le \frac{|\{i : \rho(i) \neq \emptyset\}| \max_i |\rho(i)|}{\min_{k:\nu(k)\neq\emptyset} \mu_k^2}.$$

Lower bounds for $\rho(P^*P)$:

$$\rho(P^*P) = \|P\|_2^2 = \max_{\mathbf{x},\|\mathbf{x}\|_2=1} \|P\mathbf{x}\|_2^2.$$

Thus we have

$$\rho(P^*P) \ge \max_{i:\rho(i)\neq\emptyset} \|P\mathbf{e}_i\|_2^2 = \max_{i:\rho(i)\neq\emptyset} \sum_{k\in\rho(i)} \frac{1}{\mu_k^2},$$

$$\rho(P^*P) \ge \max_{i,j:i\neq j;\rho(i),\rho(j)\neq\emptyset} \|P\frac{1}{\sqrt{2}}(\mathbf{e}_i + \mathbf{e}_j)\|_2^2$$

$$= \frac{1}{2} \max_{i,j:i\neq j;\rho(i),\rho(j)\neq\emptyset} \Big[\sum_{k\in(\rho(i)\cup\rho(j))\setminus(\rho(i)\cap\rho(j))} \frac{1}{\mu_k^2} + \sum_{k\in\rho(i)\cap\rho(j)} \frac{4}{\mu_k^2}\Big]$$

(check the latter!).

On the other side, from the equality

$$\rho(P^*P) = \|P\|_2^2 = \|P^*\|_2^2 = \max_{\mathbf{x},\|\mathbf{x}\|_2=1} \|P^T\mathbf{x}\|_2^2$$

we obtain

$$\rho(P^*P) \ge \max_{i:\nu(i)\neq\emptyset} \|P^T\mathbf{e}_i\|_2^2 = \max_{i:\nu(i)\neq\emptyset} \frac{1}{\mu_i} = \frac{1}{\min_{i:\nu(i)\neq\emptyset} \mu_i},$$

$$\rho(P^*P) \ge \max_{i,j:i\neq j;\nu(i),\nu(j)\neq\emptyset} \|P^T\frac{1}{\sqrt{2}}(\mathbf{e}_i + \mathbf{e}_j)\|_2^2$$

$$= \frac{1}{2} \max_{i,j:i\neq j;\nu(i),\nu(j)\neq\emptyset} \Big[\frac{1}{\mu_i} + \frac{1}{\mu_j} + \frac{2|\nu(i)\cap\nu(j)|}{\mu_i\mu_j}\Big]$$

(check the latter!).

Check the lower and upper bounds obtained for $\rho(P^*P)$ when $P = \mathbf{e}\mathbf{e}_i^T$, $P = \frac{1}{n}\mathbf{e}_i\mathbf{e}^T$, $P = \frac{1}{2}\mathbf{e}(\mathbf{e}_i + \mathbf{e}_j)^T$, $P = \frac{1}{n}(\mathbf{e}_i + \mathbf{e}_j)\mathbf{e}^T$, $\mathbf{e} = [1\,1\,\cdots\,1]^T$, noting that in all cases the best lower bound can be equal to the upper bound.

Let us associate to P its best circulant approximation:

$$C_P = F\,\mathrm{diag}\,((F^*PF)_{ii})F^* = \sqrt{n}Fd(FC_P^T\mathbf{e}_1)F^*.$$

It is clear that there are two formulas for the eigenvalues of C_P. Let us explicit these formulas.

(a) Note that

$$(F^*PF)_{jj} = \frac{1}{n}\sum_{k:\nu(k)\neq\emptyset}\sum_{s\in\nu(k)}\frac{1}{\mu_k}\omega^{(s-k)(j-1)} = \frac{1}{n}\sum_{k:\rho(k)\neq\emptyset}\sum_{s\in\rho(k)}\frac{1}{\mu_s}\omega^{(k-s)(j-1)}$$

(use the two expressions obtained above for $\mathbf{x}^*P\mathbf{x}$:

$$\begin{array}{rcl}(F\mathbf{e}_j)^*(PF\mathbf{e}_j) & = & \sum_{k:\nu(k)\neq\emptyset}\overline{(F\mathbf{e}_j)_k}\frac{1}{\mu_k}\sum_{s\in\nu(k)}(F\mathbf{e}_j)_s \\ & = & \sum_{k:\nu(k)\neq\emptyset}\frac{1}{\sqrt{n}}\overline{\omega}^{(k-1)(j-1)}\frac{1}{\mu_k}\sum_{s\in\nu(k)}\frac{1}{\sqrt{n}}\omega^{(s-1)(j-1)};\end{array}$$

$$\begin{array}{rcl}(P^TF\mathbf{e}_j)^*(F\mathbf{e}_j) & = & \sum_{k:\rho(k)\neq\emptyset}(\sum_{s\in\rho(k)}\frac{1}{\mu_s}\overline{(F\mathbf{e}_j)_s})(F\mathbf{e}_j)_k \\ & = & \sum_{k:\rho(k)\neq\emptyset}(\sum_{s\in\rho(k)}\frac{1}{\mu_s}\frac{1}{\sqrt{n}}\overline{\omega}^{(s-1)(j-1)})\frac{1}{\sqrt{n}}\omega^{(k-1)(j-1)}.\end{array}$$

)

(b) Moreover, note that, if s_i, $i = -n+1, \ldots, -1, 0, 1, \ldots, n-1$, denote the sums of the entries on the ith diagonal of P (f.i. …, $s_{-1} = \sum_{i=1}^{n-1}(P)_{i+1,i}$, $s_0 = \sum_{i=1}^{n}(P)_{ii}$, $s_1 = \sum_{i=1}^{n-1}(P)_{i,i+1}$, …), then we have $s_0 = 0$ and, for $i = 1, \ldots, n-1$,

$$s_i = \sum_{t=1\ldots n-i,\, t\in\rho(t+i)}\frac{1}{\mu_t} = \sum_{t=1\ldots n-i,\, t+i\in\nu(t)}\frac{1}{\mu_t},$$

$$s_{-i} = \sum_{t=1\ldots n-i,\, t\in\nu(t+i)}\frac{1}{\mu_{t+i}} = \sum_{t=1\ldots n-i,\, t+i\in\rho(t)}\frac{1}{\mu_{t+i}}.$$

(In fact, for the s_i:

$$s_i = \sum_{t=1}^{n-i}P_{t,t+i} = \sum_{t=1}^{n-i}(P\mathbf{e}_{t+i})_t = \sum_{t=1\ldots n-i,\, \nu(t)\neq\emptyset,\, t+i\in\nu(t)}\frac{1}{\mu_t}.$$

The latter equality holds because of the following remark:

$$(P\mathbf{e}_{t+i})_k = \left\{\begin{array}{ll}\frac{1}{\mu_k}\sum_{s\in\nu(k)}(\mathbf{e}_{t+i})_s & \nu(k)\neq\emptyset \\ 0 & \nu(k)=\emptyset\end{array}\right. = \left\{\begin{array}{ll}\frac{1}{\mu_k} & \nu(k)\neq\emptyset,\ t+i\in\nu(k) \\ 0 & \text{otherwise}\end{array}\right.$$

$$(P\mathbf{e}_{t+i})_t = \left\{\begin{array}{ll}\frac{1}{\mu_t} & \nu(t)\neq\emptyset,\ t+i\in\nu(t) \\ 0 & \text{otherwise}\end{array}\right.$$

Analogously,

$$\begin{array}{rcl}s_i & = & \sum_{t=1}^{n-i}P_{t,t+i} = \sum_{t=1}^{n-i}(P^T\mathbf{e}_t)_{t+i} \\ & = & \sum_{t=1\ldots n-i,\, \rho(t+i)\neq\emptyset}\sum_{s\in\rho(t+i)}\frac{1}{\mu_s}(\mathbf{e}_t)_s \\ & = & \sum_{t=1\ldots n-i,\, \rho(t+i)\neq\emptyset,\, t\in\rho(t+i)}\frac{1}{\mu_t}.\end{array}$$

Example: $s_1 = \sum_{t=1\ldots n-1,\, t:t\to t+1}\frac{1}{\mu_t}$ ….

For the s_{-i}:

$$\begin{array}{rcl}s_{-i} & = & \sum_{t=1}^{n-i}P_{t+i,t} = \sum_{t=1}^{n-i}(P\mathbf{e}_t)_{t+i} \\ & = & \sum_{t=1\ldots n-i,\, \nu(t+i)\neq\emptyset}\frac{1}{\mu_{t+i}}\sum_{s\in\nu(t+i)}(\mathbf{e}_t)_s, \\ s_{-i} & = & \sum_{t=1}^{n-i}P_{t+i,t} = \sum_{t=1}^{n-i}(P^T\mathbf{e}_{t+i})_t \\ & = & \sum_{t=1\ldots n-i,\, \rho(t)\neq\emptyset}\sum_{s\in\rho(t)}\frac{1}{\mu_s}(\mathbf{e}_{t+i})_s.\end{array}$$

)

It can be shown that the first row of C_P, say $\mathbf{c}^T = [c_0\, c_1\, \cdots\, c_{n-1}]$, can be computed via the identities

$$c_0 = s_0/n = 0, \quad c_i = (s_i + s_{-n+i})/n,\ i = 1, \dots, n-1.$$

So, $C_P = Fd(\mathbf{z})F^*$, $\mathbf{z} = \sqrt{n}F\mathbf{c}$.

Any time something in the web changes, we have to update P and C_P. In particular, we have to compute the new s_i in order to define the new c_i and thus the new vector $\mathbf{z}$ (of the eigenvalues of C_P). Now we show how such new s_i can be computed from the old s_i, in the main situations that may occur.

Case 1*: nascita di un sito*

$$P_{old} \quad \rightarrow \quad P_{new} = \begin{bmatrix} P_{old} & \mathbf{0} \\ \cdots \frac{1}{\mu_{n+1}} \cdots & 0 \end{bmatrix}$$

We call the new site $n+1$, and we associate to it the new objects $\mu_{n+1} = ?$, $\nu(n+1) = \{?\} \subset \{1, 2, \dots, n\}$, $\rho(n+1) = \emptyset$ (we need new memory allocations for them). We assume $\mu_{n+1} = |\nu(n+1)|$ small ($\mu_{n+1} \geq 1$ (?)).

Then we introduce two new numbers s_n, s_{-n} (we need other two new cells in memory for them), and we set:

$$s_n := 0,\ s_{-n} := 0,\ s_{j-n-1} := s_{j-n-1} + \frac{1}{\mu_{n+1}},\ j \in \nu(n+1).$$

After this, we introduce another new number c_n (we need a further new cell in memory for it), and we set:

$$c_n := 0,\ c_j := (s_j + s_{-(n+1)+j})/n,\ j \in \nu(n+1).$$

Finally, set $n := n+1$.

Case 2*: morte di un sito contemporanea alla nascita di un altro*

$$P_{old} = \begin{bmatrix} \cdot \frac{1}{\mu_r} \cdot & 0 & \cdot \frac{1}{\mu_r} \cdot \end{bmatrix} \quad \rightarrow \quad P_{new} = \begin{bmatrix} & \mathbf{0} & \\ \cdot \frac{1}{\mu_r^{new}} \cdot & 0 & \cdot \frac{1}{\mu_r^{new}} \cdot \\ & \mathbf{0} & \end{bmatrix}$$

Assume that the site r dies. Then we have to set

$$s_{k-r} := s_{k-r} - \frac{1}{\mu_r},\ k \in \nu(r),$$

and apply (4) (see below) for $j = r$ and $\forall i \in \rho(r)$. Assume now that at the same time a new site is created. We call it r and we associate to it $\mu_r = ?$, $\nu(r) = \{?\} \subset \{1, 2, \dots, n\} \backslash \{r\}$, $\rho(r) = \emptyset$ (i.e. we use the memory allocations of the died site). We assume $\mu_r = |\nu(r)|$ small ($\mu_r \geq 1$ (?)). Then we have to set

$$s_{k-r} := s_{k-r} + \frac{1}{\mu_r},\ k \in \nu(r).$$

After this, we set

$$c_{k-r} = (s_{k-r} + s_{-n+k-r})/n,\ k \in \nu(r),\ k - r > 0,$$

$$c_{n+k-r} = (s_{n+k-r} + s_{k-r})/n,\ k \in \nu(r),\ k - r < 0.$$

3: Site i decides to point to the site j

$$P_{old} = \begin{bmatrix} \cdot\frac{1}{\mu_i}\cdot & 0 & \cdots \end{bmatrix} \quad \rightarrow \quad P_{new} = \begin{bmatrix} \cdot\frac{1}{\mu_i+1}\cdot & \frac{1}{\mu_i+1} & \cdots \end{bmatrix}$$

First we have to set:

$$s_{j-i} := s_{j-i} + \frac{1}{\mu_i + 1},\ s_{k-i} := s_{k-i} + \frac{1}{\mu_i + 1} - \frac{1}{\mu_i},\ k \in \nu(i).$$

Then set

$$\begin{array}{l} c_{j-i} = (s_{j-i} + s_{-n+j-i})/n,\ \text{if } j - i > 0,\ \text{or} \\ c_{n+j-i} = (s_{n+j-i} + s_{j-i})/n,\ \text{if } j - i < 0, \end{array}$$

and

$$\begin{array}{l} c_{k-i} = (s_{k-i} + s_{-n+k-i})/n,\ k \in \nu(i),\ k - i > 0, \\ c_{n+k-i} = (s_{n+k-i} + s_{k-i})/n,\ k \in \nu(i),\ k - i < 0. \end{array}$$

Finally, we set $\mu_i := \mu_i + 1$, $\nu(i) := \nu(i) \cup \{j\}$, $\rho(j) := \rho(j) \cup \{i\}$. The latter identities require reordering (forward shift).

4: Site i decides not to point to the site j anymore

$$P_{old} = \begin{bmatrix} \cdot\frac{1}{\mu_i}\cdot & \frac{1}{\mu_i} & \cdots \end{bmatrix} \quad \rightarrow \quad P_{new} = \begin{bmatrix} \cdot\frac{1}{\mu_i-1}\cdot & 0 & \cdots \end{bmatrix}$$

First we have to set:

$$s_{j-i} := s_{j-i} - \frac{1}{\mu_i},\ s_{k-i} := s_{k-i} + \frac{1}{\mu_i - 1} - \frac{1}{\mu_i},\ k \in \nu(i)\backslash\{j\}.$$

Then set

$$\begin{array}{l} c_{j-i} = (s_{j-i} + s_{-n+j-i})/n,\ \text{if } j - i > 0,\ \text{or} \\ c_{n+j-i} = (s_{n+j-i} + s_{j-i})/n,\ \text{if } j - i < 0, \end{array}$$

and

$$\begin{array}{l} c_{k-i} = (s_{k-i} + s_{-n+k-i})/n,\ k \in \nu(i)\backslash\{j\},\ k - i > 0, \\ c_{n+k-i} = (s_{n+k-i} + s_{k-i})/n,\ k \in \nu(i)\backslash\{j\},\ k - i < 0. \end{array}$$

Finally, we set $\mu_i := \mu_i - 1$, $\nu(i) := \nu(i)\backslash\{j\}$, $\rho(j) := \rho(j)\backslash\{i\}$. The latter identities require reordering (backward shift).

Remark Note that, after case 1 or case 2 has been run, the site i^* which has created the new site (n in Case 1 and r in case 2) must have a link to such new site. In other words, after case 1 or case 2 we have to run (3) for $j = n$ and $i = i^* \in \{1, \dots, n-1\}$ or for $j = r$ and $i = i^* \in \{1, \dots, n\}\backslash\{r\}$.

An algorithm generating the test matrix P and the corresponding approximation C_P

Given a natural number N one could generate a test matrix P of order N and, simultaneously, its best circulant approximation C_P (or, more precisely: non negative integers $\mu_1, \dots, \mu_N$ and sets $\nu(1), \dots, \nu(N)$ defining P; and real numbers $s_{-N+1}, \dots, s_0, \dots, s_{N-1}$, $c_0, \dots, c_{N-1}$, $z_1, \dots, z_N$ defining $C_P = Fd(\mathbf{z})F^*$) by the following procedure:

- Consider an initial matrix matrix P of small size (i.e. choose a small n and define μ_i,$\nu(i)$, $i = 1, \dots, n$)
- Apply to P the operators (Case 1)-(3),(Case 2)-(3),(3),(4) repeatedly, in suitable order, each possibly more times, until n is equal to N. During this phase, for what concerns C_P, update only the s_i. (Note that the operator (Case 2)-(3) requires $|\rho(r)|$ applications of (4); so it is expensive if $|\rho(r)|$ is large. However, the death of r means generally a small $|\rho(r)|$)
- When $n = N$ compute also the first row $\mathbf{c}^T$ of C_P and the vector $\mathbf{z} = \sqrt{n}F\mathbf{c}$ of the eigenvalues of C_P

Note that it is advisable to choose N as a power of 2, so that the FFT involved (in computing the vector $\mathbf{z}$ defining the preconditioner C_P, and in each step of the Richardson method preconditioned by C_P) are more efficient.

Upper and lower bounds for $\rho(P^*P)$

A first upper bound is obtained as follows:

$$\begin{array}{rcl}
\sqrt{\rho(P^*P)} & = & \|P\|_2 = \|P^*\|_2 = \max_{\mathbf{x}:\|\mathbf{x}\|_2=1} \|P^T\mathbf{x}\|_2 \\
& \leq & \max_{\mathbf{x}:\|\mathbf{x}\|_2=1} \|P^T\mathbf{x}\|_1 \\
& = & \max_{\mathbf{x}:\|\mathbf{x}\|_2=1} \sum_i |(\sum_{s:\nu(s)\neq\emptyset} x_s P^T\mathbf{e}_s)_i| \\
& = & \max_{\mathbf{x}:\|\mathbf{x}\|_2=1} \sum_i |\sum_{s:\nu(s)\neq\emptyset} x_s (P^T)_{i,s}| \\
& \leq & \max_{\mathbf{x}:\|\mathbf{x}\|_2=1} \sum_i \sum_{s:\nu(s)\neq\emptyset} |x_s|(P^T)_{i,s} \\
& = & \max_{\mathbf{x}:\|\mathbf{x}\|_2=1} \sum_{s:\nu(s)\neq\emptyset} |x_s| \sum_i (P^T)_{i,s} \\
& = & \max_{\mathbf{x}:\|\mathbf{x}\|_2=1} \sum_{s:\nu(s)\neq\emptyset} |x_s| \\
& = & \max_{\mathbf{x}:\|\mathbf{x}\|_2=1 \& x_k=0, \forall k\, \nu(k)=\emptyset} \sum_{s:\nu(s)\neq\emptyset} |x_s| \\
& = & \sqrt{|\{s : \nu(s) \neq \emptyset\}|}.
\end{array}$$

However, a lower upper bound can be obtained. In fact, we have that $\rho(P^*P) \leq d$ where

$$d = \min\{\frac{|\{i : \nu(i) \neq \emptyset\}|}{\min_{i:\nu(i)\neq\emptyset} \mu_i},\ \frac{|\{i : \rho(i) \neq \emptyset\}| \max_i |\rho(i)|}{\min_{k:\nu(k)\neq\emptyset} \mu_k^2}\}.$$

Proof.

$$\begin{array}{rcl}
\|PP^*\|_\infty & = & \max_i \sum_j |(PP^*)_{ij}| = \max_i \sum_j \frac{1}{\mu_i\mu_j}|\nu(i) \cap \nu(j)| \\
& = & \max_i \frac{1}{\mu_i} \sum_j \frac{1}{\mu_j}|\nu(i) \cap \nu(j)| \\
& = & \max_i \frac{1}{\mu_i} \sum_{j:\nu(j)\neq\emptyset} \frac{1}{\mu_j}|\nu(i) \cap \nu(j)| \\
& \leq & \max_i \frac{1}{\mu_i} \sum_{j:\nu(j)\neq\emptyset} \frac{1}{\mu_j}\mu_j = \max_i \frac{1}{\mu_i}|\{j : \nu(j) \neq \emptyset\}| \\
& = & |\{j : \nu(j) \neq \emptyset\}| \frac{1}{\min_{i:\nu(i)\neq\emptyset} \mu_i},
\end{array}$$

$$\begin{array}{rcl}
\|P^*P\|_\infty & = & \max_i \sum_j |(P^*P)_{ij}| = \max_i \sum_j \sum_{k\in\rho(i)\cap\rho(j)} \frac{1}{\mu_k^2} \\
& \leq & \max_i \frac{1}{\min_{k:\nu(k)\neq\emptyset} \mu_k^2} \sum_j \sum_{k\in\rho(i)\cap\rho(j)} 1 \\
& = & \max_i \frac{1}{\min_{k:\nu(k)\neq\emptyset} \mu_k^2} \sum_j |\rho(i) \cap \rho(j)| \\
& = & \max_i \frac{1}{\min_{k:\nu(k)\neq\emptyset} \mu_k^2} \sum_{j:\rho(j)\neq\emptyset} |\rho(i) \cap \rho(j)| \\
& \leq & |\{j : \rho(j) \neq \emptyset\}| \max_k |\rho(k)| \frac{1}{\min_{k:\nu(k)\neq\emptyset} \mu_k^2}. \quad \square
\end{array}$$

(Question: is the following inequality

$$\frac{|\{j : \nu(j) \neq \emptyset\}|}{\min_{i:\nu(i)\neq\emptyset} \mu_i} \leq \frac{|\{j : \rho(j) \neq \emptyset\}| \max_k |\rho(k)|}{\min_{k:\nu(k)\neq\emptyset} \mu_k^2}$$

or, equivalently,

$$|\{j : \rho(j) \neq \emptyset\}| \max_k |\rho(k)| \geq |\{j : \nu(j) \neq \emptyset\}| \min_{k:\nu(k)\neq\emptyset} |\nu(k)|$$

true? Note that

$$\begin{array}{l} \max_k |\rho(k)| \leq |\{j : \nu(j) \neq \emptyset\}|, \\ |\{j : \rho(j) \neq \emptyset\}| \geq \max_k |\nu(k)| \geq \min_{k:\nu(k)\neq\emptyset} |\nu(k)|. \end{array}$$

)

Let us now obtain lower bounds for $\rho(P^*P)$ more general than those obtained in the previous section.

Choose $\mathbf{x} = \frac{1}{\sqrt{3}}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k)$, i, j, k distinct, in

$$\rho(P^*P) \geq \|P^T\mathbf{x}\|_2^2, \ \|\mathbf{x}\|_2 = 1,$$

in order to obtain

$$\begin{array}{rcl} \rho(P^*P) \geq r_3 & = & \frac{1}{3} \max_{i,j,k:i,j,k\, distinct,\, \nu(i),\nu(j),\nu(k)\neq\emptyset} \\ & & \Big[\frac{1}{\mu_i} + \frac{1}{\mu_j} + \frac{1}{\mu_k} \\ & & + \frac{2|\nu(i)\cap\nu(j)|}{\mu_i\mu_j} + \frac{2|\nu(i)\cap\nu(k)|}{\mu_i\mu_k} + \frac{2|\nu(j)\cap\nu(k)|}{\mu_j\mu_k} \Big]. \end{array}$$

Note that if

$$P = \frac{1}{n}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k)\mathbf{e}^T,$$

then in the above inequality the equal sign holds, i.e. $d = \rho(P^*P) = r_3 = 3/n$.

Choose $\mathbf{x} = \frac{1}{\sqrt{3}}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k)$, i, j, k distinct, in

$$\rho(P^*P) \geq \|P\mathbf{x}\|_2^2, \ \|\mathbf{x}\|_2 = 1,$$

in order to obtain

$$\begin{array}{rcl} \rho(P^*P) \geq c_3 & = & \frac{1}{3} \max_{\mathcal{A}_3 \subset \{i:\rho(i)\neq\emptyset\},\, |\mathcal{A}_3|=3} \\ & & \Big[\sum_{i\in\mathcal{A}_3} \sum_{s\in\rho(i)\setminus\cup_{j\in\mathcal{A}_3\setminus\{i\}}\rho(j)} \frac{1}{\mu_s^2} \\ & & + \sum_{i\in\mathcal{A}_3} \sum_{s\in(\cap_{j\in\mathcal{A}_3\setminus\{i\}}\rho(j))\setminus\rho(i)} \frac{4}{\mu_s^2} \\ & & + \sum_{s\in\cap_{i\in\mathcal{A}_3}\rho(i)} \frac{9}{\mu_s^2} \Big]. \end{array}$$

Note that if

$$P = \frac{1}{3}\mathbf{e}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k)^T,$$

then in the above inequality the equal sign holds, i.e. $d = \rho(P^*P) = c_3 = n/3$.

We guess that the following results (r) and (c) hold:

(r) Set

$$r_k = \frac{1}{k} \max_{\mathcal{A}_k \subset \{i:\nu(i)\neq\emptyset\}, |\mathcal{A}_k|=k} \Big[\sum_{r\in\mathcal{A}_k} \frac{1}{\mu_r} + \sum_{r,s\in\mathcal{A}_k, r<s} \frac{2}{\mu_r\mu_s} |\nu(r)\cap\nu(s)| \Big].$$

Then

$$r_k \leq \rho(P^*P) \leq d, \ \forall k \in \{1, \ldots, |\{i : \nu(i) \neq \emptyset\}|\}.$$

(Proof: apply the inequality

$$\rho(P^*P) = \|P^*\|_2^2 \geq \|P^T\mathbf{x}\|_2^2, \quad \|\mathbf{x}\|_2 = 1,$$

for $\mathbf{x} = \frac{1}{\sqrt{k}}(\mathbf{e}_{i_1} + \ldots + \mathbf{e}_{i_k})$, $1 \leq i_1 < i_2 < \ldots < i_k \leq n$).

Note that if

$$P = \frac{1}{n}(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \ldots + \mathbf{e}_{i_k})\mathbf{e}^T, \quad 1 \leq i_1 < i_2 < \ldots < i_k \leq n,$$

then $r_k = \rho(P^*P) = d = \frac{k}{n}$.

(c) Now set

$$c_k = \frac{1}{k} \max_{\mathcal{A}_k \subset \{i:\rho(i)\neq\emptyset\}, |\mathcal{A}_k|=k} \Big[\sum_{r=1}^{k} \sum_{\gamma_r \subset \mathcal{A}_k, |\gamma_r|=r} \sum_{s \in \cap_{j\in\gamma_r}\rho(j) \setminus \cup_{j\in\mathcal{A}_k\setminus\gamma_r}\rho(j)} \frac{r^2}{\mu_s^2}\Big].$$

Then

$$c_k \leq \rho(P^*P) \leq d, \ \forall k \in \{1, \ldots, |\{i : \rho(i) \neq \emptyset\}|\}.$$

(Proof: apply the inequality

$$\rho(P^*P) = \|P\|_2^2 \geq \|P\mathbf{x}\|_2^2, \quad \|\mathbf{x}\|_2 = 1,$$

for $\mathbf{x} = \frac{1}{\sqrt{k}}(\mathbf{e}_{i_1} + \ldots + \mathbf{e}_{i_k})$, $1 \leq i_1 < i_2 < \ldots < i_k \leq n$).

Note that if

$$P = \frac{1}{k}\mathbf{e}(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \ldots + \mathbf{e}_{i_k})^T, \quad 1 \leq i_1 < i_2 < \ldots < i_k \leq n,$$

then $c_k = \rho(P^*P) = d = \frac{n}{k}$.

Remark. By choosing $k = 1$, $k = 2$ and $k = 3$ in the above statements we retrieve the lower bounds for $\rho(P^*P)$ obtained explicitly.

If the lower bounds in (r) and (c) for $\rho(P^*P)$ are true, then we can say that

$$\max\{\max_{1,\ldots,|\{i:\nu(i)\neq\emptyset\}|} r_k, \max_{1,\ldots,|\{i:\rho(i)\neq\emptyset\}|} c_k\} \leq \rho(P^*P) \leq d$$

$$d = \min\{\frac{|\{i : \nu(i) \neq \emptyset\}|}{\min_{i:\nu(i)\neq\emptyset} \mu_i}, \ \frac{|\{i : \rho(i) \neq \emptyset\}| \max_i |\rho(i)|}{\min_{k:\nu(k)\neq\emptyset} \mu_k^2}\}.$$

Question: is the sequence $\max\{r_k, c_k\}$, $k = 1, \ldots,$ non decreasing? If yes, then one could compute its terms to obtain better and better lower bounds for $\rho(P^*P)$.

Examples.

c) $P = \mathbf{e}\mathbf{e}_i^T$, $d = n$
$r_1 = 1$, $c_1 = n$
$r_2 = 2$, $c_2 =$und
$r_3 = 3$, $c_3 =$und

cc) $P = \frac{1}{2}\mathbf{e}(\mathbf{e}_i + \mathbf{e}_j)^T$, $d = n/2$
$r_1 = 1/2$, $c_1 = n/4$
$r_2 = 1$, $c_2 = n/2$

$r_3 = 3/2$, $c_3 =$und

r) $P = \frac{1}{n}\mathbf{e}_i\mathbf{e}^T$, $d = 1/n$
$r_1 = 1/n$, $c_1 = 1/n^2$
$r_2 =$und, $c_2 = 2/n^2$
$r_3 =$und, $c_3 = 3/n^2$

rr) $P = \frac{1}{n}(\mathbf{e}_i + \mathbf{e}_j)\mathbf{e}^T$, $d = 2/n$
$r_1 = 1/n$, $c_1 = 2/n^2$
$r_2 = 2/n$, $c_2 = 4/n^2$
$r_3 =$und, $c_3 = 6/n^2$

Set $\tilde{\mathbf{e}} = \sum_{i=1}^{t} \mathbf{e}_i$.

c1) $P = \tilde{\mathbf{e}}\mathbf{e}_i^T$, $d = t$
$r_1 = 1$, $c_1 = t$
$r_2 = 2$, $c_2 =$und
$r_3 = 3$, $c_3 =$und

cc1) $P = \frac{1}{2}\tilde{\mathbf{e}}\mathbf{e}_i^T + (\mathbf{e} - \frac{1}{2}\tilde{\mathbf{e}})\mathbf{e}_j^T$, $d = n$
$r_1 = 1$, $c_1 = t/4 + (n - t)$
$r_2 = 2$, $c_2 = n/2$
$r_3 = 3$, $c_3 =$und

r1) $P = \frac{1}{t}\mathbf{e}_i\tilde{\mathbf{e}}^T$, $d = 1/t$
$r_1 = 1/t$, $c_1 = 1/t^2$
$r_2 =$und, $c_2 = 2/t^2$
$r_3 =$und, $c_3 = 3/t^2$

rr1) $P = \frac{1}{n}\mathbf{e}_i\mathbf{e}^T + \frac{1}{t}\mathbf{e}_j\tilde{\mathbf{e}}^T$, $d = 2/t$
$r_1 = 1/t$, $c_1 = 1/t^2 + 1/n^2$
$r_2 = \frac{1}{2}(3/n + 1/t)$, $c_2 = 2/t^2 + 2/n^2$
$r_3 =$und, $c_3 = 3/t^2 + 3/n^2(?)$

Question: from the equality

$$\rho(P^*P) = \rho(PP^*) = \inf_k (\|(PP^*)^k\|_\infty)^{1/k} = \lim_k (\|(PP^*)^k\|_\infty)^{1/k}$$

is it possible to define a non increasing sequence d_k such that $d_0 = d = |\{i : \nu(i) \neq \emptyset\}|/\min \mu_k$ (recall that $\|PP^*\|_\infty \leq d$) and $\rho(PP^*) \leq d_k \leq d$, $\forall\, k$? If yes, then one could compute its terms to obtain better and better upper bounds for $\rho(P^*P)$.

The proof of an inequality

We now prove the inequality:

$$(\min_{k:\nu(k)\neq\emptyset} |\nu(k)|)\ |\{j : \nu(j) \neq \emptyset\}| \leq (\max_k |\rho(k)|)\ |j : \rho(j) \neq \emptyset|.$$

So, for the d in the upper bound $\rho(P^*P) \leq d$ (see above) we simply have

$$d = \frac{|\{k : \nu(k) \neq \emptyset\}|}{\min_{k:\nu(k)\neq\emptyset} \mu_k}.$$

Without loss of generality, assume that

$$\{j : \nu(j) \neq \emptyset\} = \{1, 2, \ldots, x\},\ x := |\{j : \nu(j) \neq \emptyset\}|,$$
$$\{j : \rho(j) \neq \emptyset\} = \{1, 2, \ldots, y\},\ y := |\{j : \rho(j) \neq \emptyset\}|.$$

Thus, the $x \times y$ upper left submatrix of P, which we call W, has no null row and no null column, whereas the entries of the remaining part of P are all zeroes:

$$P = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}.$$

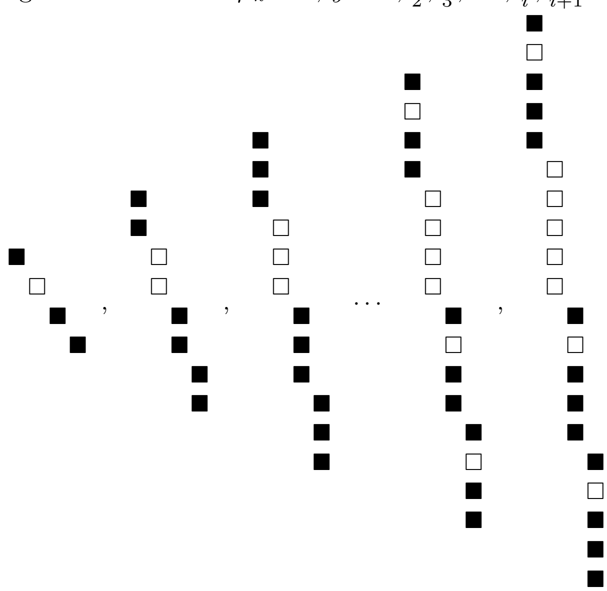
(1) If $\min_{k:\nu(k)\neq\emptyset} |\nu(k)| = 1$, then $x \leq (\max_k |\rho(k)|)y$.

Proof. We prove the thesis considering the two different cases $y \geq x$ and $y < x$. It is useful to observe that, by the hypotheses, the nonzero entries of the matrix W for $y = x, \frac{x}{2}, \frac{x}{3}, \ldots, \frac{x}{i}, \frac{x}{i+1}$ must be (modulo permutations) in the positions shown in Figure 1.

$y \geq x$: In this case $\max_k |\rho(k)| \geq 1$ (see Figure 1, $y = x$). Thus $1 \cdot x \leq 1 \cdot y \leq (\max_k |\rho(k)|)y$.

$y < x$: If $y < x$, then $\max_k |\rho(k)| \geq 2$ (see Figure 1, $y = x, \frac{x}{2}$). If $\frac{x}{2} \leq y$ too, then $x \leq 2y \leq (\max_k |\rho(k)|)y$. If $y < \frac{x}{2}$, then $\max_k |\rho(k)| \geq 3$ (see Figure 1, $y = \frac{x}{2}, \frac{x}{3}$). If $\frac{x}{3} \leq y$ too, then $x \leq 3y \leq (\max_k |\rho(k)|)y$. In general, let $i \in \{1, 2, \ldots\}$. If $y < \frac{x}{i}$, then $\max_k |\rho(k)| \geq i + 1$ (see Figure 1, $y = \frac{x}{i}, \frac{x}{i+1}$). If $\frac{x}{i+1} \leq y$ too, then $x \leq (i + 1)y \leq (\max_k |\rho(k)|)y$.

Figure 1: W for $\min \mu_k = 1$, $y = x, \frac{x}{2}, \frac{x}{3}, \ldots, \frac{x}{i}, \frac{x}{i+1}$



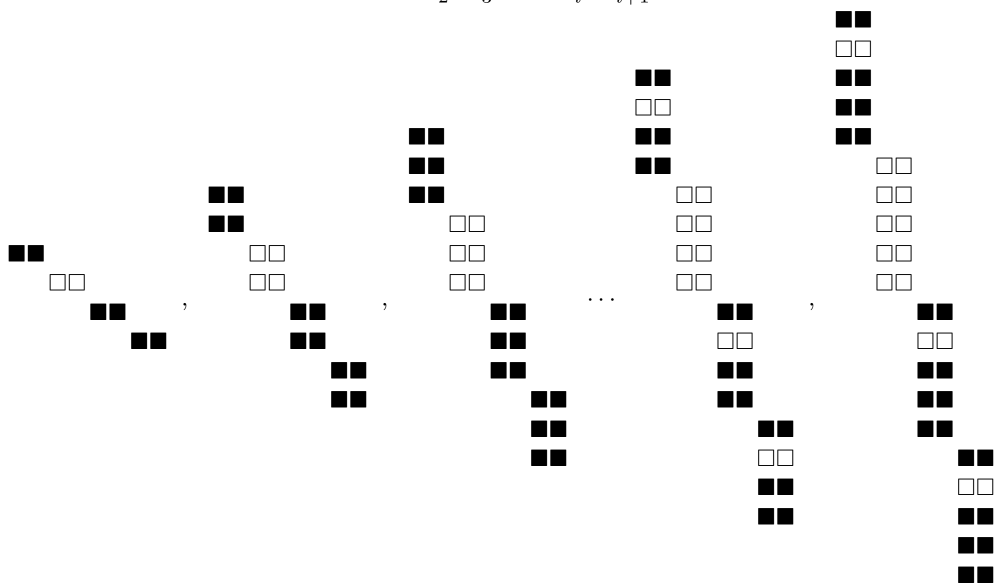
(2) If $\min_{k:\nu(k)\neq\emptyset} |\nu(k)| = 2$, then $2x \leq (\max_k |\rho(k)|)y$.

Proof. We prove the thesis considering the two different cases $y \geq 2x$ and $y < 2x$. It is useful to observe that, by the hypotheses, the nonzero entries of the matrix W for $y = 2x, 2\frac{x}{2}, 2\frac{x}{3}, \ldots, 2\frac{x}{i}, 2\frac{x}{i+1}$ must be (modulo permutations) in the positions shown in Figure 2.

$y \geq 2x$: In this case $\max_k |\rho(k)| \geq 1$ (see Figure 2, $y = 2x$). Thus $2x \leq 1 \cdot y \leq (\max_k |\rho(k)|)y$.

$y < 2x$: If $y < 2x$, then $\max_k |\rho(k)| \geq 2$ (see Figure 2, $y = 2x, 2\frac{x}{2}$). If $2\frac{x}{2} \leq y$ too, then $2x \leq 2y \leq (\max_k |\rho(k)|)y$. If $y < 2\frac{x}{2}$, then $\max_k |\rho(k)| \geq 3$ (see Figure 2, $y = 2\frac{x}{2}, 2\frac{x}{3}$). If $2\frac{x}{3} \leq y$ too, then $2x \leq 3y \leq (\max_k |\rho(k)|)y$. In general, let $i \in \{1, 2, \ldots\}$. If $y < 2\frac{x}{i}$, then $\max_k |\rho(k)| \geq i + 1$ (see Figure 2, $y = 2\frac{x}{i}, 2\frac{x}{i+1}$). If $2\frac{x}{i+1} \leq y$ too, then $2x \leq (i + 1)y \leq (\max_k |\rho(k)|)y$.

Figure 2: W for $\min \mu_k = 2$, $y = 2x, 2\frac{x}{2}, 2\frac{x}{3}, \ldots, 2\frac{x}{i}, 2\frac{x}{i+1}$



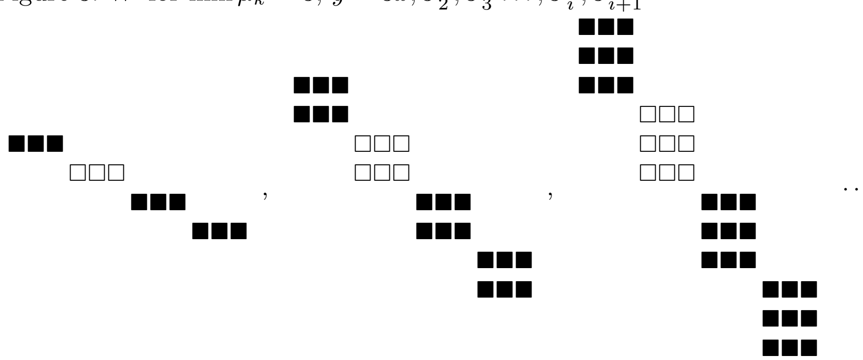
(3) If $\min_{k:\nu(k)\neq\emptyset} |\nu(k)| = 3$, then $3x \leq (\max_k |\rho(k)|)y$.

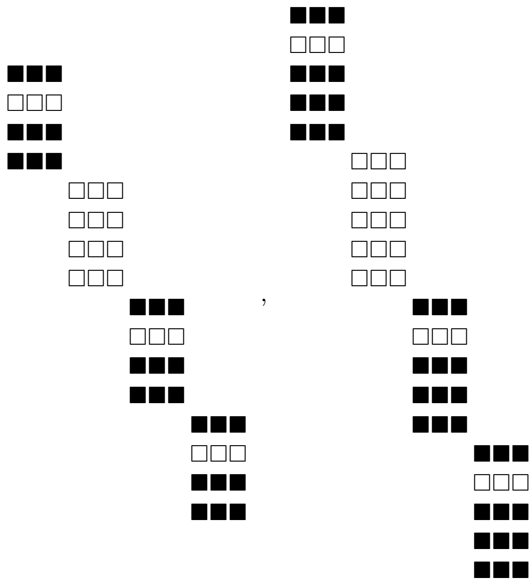
Proof. We prove the thesis considering the two different cases $y \geq 3x$ and $y < 3x$. It is useful to observe that, by the hypotheses, the nonzero entries of the matrix W for $y = 3x, 3\frac{x}{2}, 3\frac{x}{3}, \ldots, 3\frac{x}{i}, 3\frac{x}{i+1}$ must be (modulo permutations) in the positions shown in Figure 3.

$y \geq 3x$: In this case $\max_k |\rho(k)| \geq 1$ (see Figure 3, $y = 3x$). Thus $3x \leq 1 \cdot y \leq (\max_k |\rho(k)|)y$.

$y < 3x$: If $y < 3x$, then $\max_k |\rho(k)| \geq 2$ (see Figure 3, $y = 3x, 3\frac{x}{2}$). If $3\frac{x}{2} \leq y$ too, then $3x \leq 2y \leq (\max_k |\rho(k)|)y$. If $y < 3\frac{x}{2}$, then $\max_k |\rho(k)| \geq 3$ (see Figure 3, $y = 3\frac{x}{2}, 3\frac{x}{3}$). If $3\frac{x}{3} \leq y$ too, then $3x \leq 3y \leq (\max_k |\rho(k)|)y$. In general, let $i \in \{1, 2, \ldots\}$. If $y < 3\frac{x}{i}$, then $\max_k |\rho(k)| \geq i + 1$ (see Figure 3, $y = 3\frac{x}{i}, 3\frac{x}{i+1}$). If $3\frac{x}{i+1} \leq y$ too, then $3x \leq (i + 1)y \leq (\max_k |\rho(k)|)y$.

Figure 3: W for $\min \mu_k = 3$, $y = 3x, 3\frac{x}{2}, 3\frac{x}{3} \ldots, 3\frac{x}{i}, 3\frac{x}{i+1}$





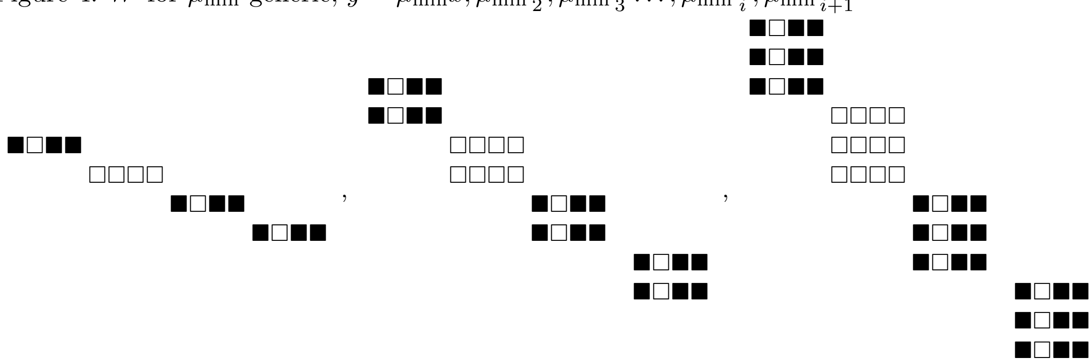
(4) $(\min_{k:\nu(k)\neq\emptyset} |\nu(k)|)x \leq (\max_k |\rho(k)|)y$.

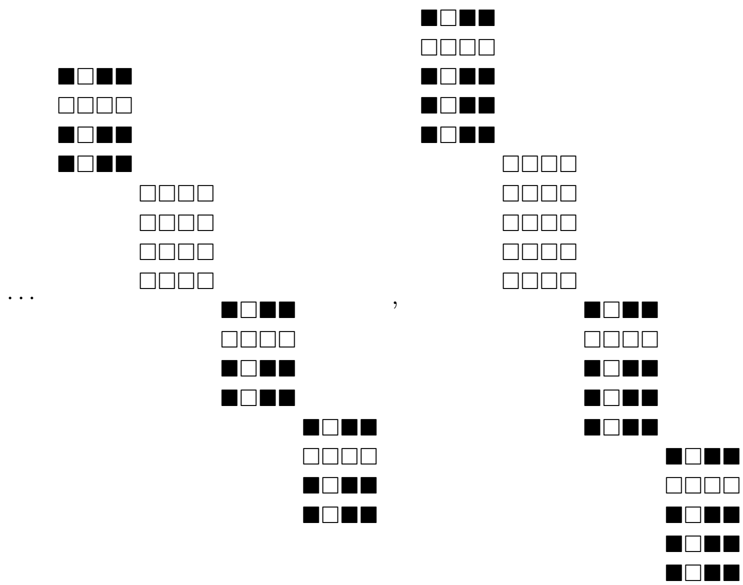
Proof. Set $\mu_{\min} = \min_{k:\mu_k>0} \mu_k$. We prove the thesis considering the two different cases $y \geq \mu_{\min}x$ and $y < \mu_{\min}x$. It is useful to observe that, by the hypotheses, the nonzero entries of the matrix W for $y = \mu_{\min}x, \mu_{\min}\frac{x}{2}, \mu_{\min}\frac{x}{3}$, …, $\mu_{\min}\frac{x}{i}, \mu_{\min}\frac{x}{i+1}$ must be (modulo permutations) in the positions shown in Figure 4.

$y \geq \mu_{\min}x$: In this case $\max_k |\rho(k)| \geq 1$ (see Figure 4, $y = \mu_{\min}x$). Thus $\mu_{\min}x \leq 1 \cdot y \leq (\max_k |\rho(k)|)y$.

$y < \mu_{\min}x$: If $y < \mu_{\min}x$, then $\max_k |\rho(k)| \geq 2$ (see Fig.4, $y = \mu_{\min}x, \mu_{\min}\frac{x}{2}$). If $\mu_{\min}\frac{x}{2} \leq y$ too, then $\mu_{\min}x \leq 2y \leq (\max_k |\rho(k)|)y$. If $y < \mu_{\min}\frac{x}{2}$, then $\max_k |\rho(k)| \geq 3$ (see Figure 4, $y = \mu_{\min}\frac{x}{2}, \mu_{\min}\frac{x}{3}$). If $\mu_{\min}\frac{x}{3} \leq y$ too, then $\mu_{\min}x \leq 3y \leq (\max_k |\rho(k)|)y$. In general, let $i \in \{1, 2, \ldots\}$. If $y < \mu_{\min}\frac{x}{i}$, then $\max_k |\rho(k)| \geq i + 1$ (see Figure 4, $y = \mu_{\min}\frac{x}{i}, \mu_{\min}\frac{x}{i+1}$). If $\mu_{\min}\frac{x}{i+1} \leq y$ too, then $\mu_{\min}x \leq (i + 1)y \leq (\max_k |\rho(k)|)y$.

Figure 4: W for $\mu_{\min}$ generic, $y = \mu_{\min}x, \mu_{\min}\frac{x}{2}, \mu_{\min}\frac{x}{3} \ldots, \mu_{\min}\frac{x}{i}, \mu_{\min}\frac{x}{i+1}$





Better upper bounds for $\rho(P^*P)$?

Let $i,j \in \{1,2,\ldots,n\}$. If $\nu(i)=\emptyset$ or $\nu(j)=\emptyset$, then $((PP^*)^k)_{ij}=0$, for all $k \geq 1$. Otherwise, i.e. if both $\nu(i)$ and $\nu(j)$ are non empty, then

$$(PP^*)_{ij} = \frac{1}{\mu_i \mu_j} |\nu(i) \cap \nu(j)|,$$

$$((PP^*)^2)_{ij} = \frac{1}{\mu_i \mu_j} \sum_{k:\nu(k)\neq\emptyset} \frac{|\nu(i)\cap\nu(k)|\,|\nu(k)\cap\nu(j)|}{\mu_k^2},$$

$$((PP^*)^3)_{ij} = \frac{1}{\mu_i \mu_j} \sum_{k,s:\nu(k),\nu(s)\neq\emptyset} \frac{|\nu(i)\cap\nu(k)|\,|\nu(k)\cap\nu(s)|\,|\nu(s)\cap\nu(j)|}{\mu_k^2 \mu_s^2},$$

$\ldots$

$$((PP^*)^k)_{ij} = \frac{1}{\mu_i \mu_j} \sum_{r_1,\ldots,r_{k-1}:\nu(r_s)\neq\emptyset,\,\forall s} \frac{|\nu(i)\cap\nu(r_1)||\nu(r_1)\cap\nu(r_2)|\cdots|\nu(r_{k-1})\cap\nu(j)|}{\mu_{r_1}^2 \mu_{r_2}^2 \cdots \mu_{r_{k-1}}^2},$$

$$((PP^*)^{k+1})_{ij} = \frac{1}{\mu_j} \sum_{r_k:\nu(r_k)\neq\emptyset} \frac{|\nu(r_k)\cap\nu(j)|}{\mu_{r_k}} ((PP^*)^k)_{i,r_k}.$$

Aim: find a sequence d_k such that $\rho(P^*P) \leq d_{k+1} \leq d_k \leq d_0$, $d_k \to \rho(P^*P)$.

Recall that

$$\begin{aligned} \rho(P^*P) &= \rho(PP^*) = ((\rho(PP^*))^k)^{\frac{1}{k}} = (\rho((PP^*)^k))^{\frac{1}{k}} \\ &\leq \|(PP^*)^k\|^{\frac{1}{k}} \leq \|PP^*\|,\ k=1,2,\ldots, \end{aligned}$$

$$\rho(PP^*) = \inf_k \|(PP^*)^k\|^{\frac{1}{k}} = \lim_k \|(PP^*)^k\|^{\frac{1}{k}},$$

$$\|PP^*\|_\infty = \max_{i:\mu_i>0} \sum_{j:\mu_j>0} \frac{|\nu(i)\cap\nu(j)|}{\mu_i\mu_j} \leq d_0 := d = \frac{|\{j:\nu(j)\neq\emptyset\}|}{\mu_{min}}.$$

Thus, from the inequality

$$\begin{aligned}\|(PP^*)^2\|_\infty &= \max_{i:\mu_i>0} \tfrac{1}{\mu_i} \textstyle\sum_{j:\mu_j>0} \tfrac{1}{\mu_j} \sum_{k:\mu_k>0} \tfrac{|\nu(i)\cap\nu(k)||\nu(k)\cap\nu(j)|}{\mu_k^2} \\ &\leq \max_{i:\mu_i>0} \tfrac{1}{\mu_i^2} \big(\textstyle\sum_{j:\mu_j>0} \tfrac{|\nu(i)\cap\nu(j)|}{\mu_j}\big)^2 = \|PP^*\|_\infty^2 \leq d_0^2,\end{aligned}$$

we have that a first step towards our aim, could be the following: look for d_1 such that

$$\|(PP^*)^2\|_\infty = \max_{i:\mu_i>0} \frac{1}{\mu_i} \sum_{j:\mu_j>0} \frac{1}{\mu_j} \sum_{k:\mu_k>0} \frac{|\nu(i)\cap\nu(k)|\,|\nu(k)\cap\nu(j)|}{\mu_k^2} \leq d_1^2,\ d_1 < d_0.$$

(Note that

$$\begin{aligned}&\rho(PP^*) \leq \ldots \leq \|(PP^*)^8\|_\infty^{\frac{1}{8}} \leq \|(PP^*)^4\|_\infty^{\frac{1}{4}} \leq \|(PP^*)^2\|_\infty^{\frac{1}{2}} \leq \|(PP^*)\|_\infty, \\ &\|(PP^*)^{2^k}\|_\infty^{\frac{1}{2^k}} \ \rightarrow \ \rho(PP^*).\end{aligned}$$

So, the ideal result we would like to obtain is the following: define an easily computable number d_k such that

$$\|(PP^*)^{2^k}\|_\infty^{\frac{1}{2^k}} \leq d_k \leq \|(PP^*)^{2^{k-1}}\|_\infty^{\frac{1}{2^{k-1}}},\ k = 1, 2, \ldots.$$

)

Computing $\nu(i)\cap\nu(j)$, $|\nu(i)\cap\nu(j)|$ *and* $(PP^*)_{ij} = \frac{|\nu(i)\cap\nu(j)|}{\mu_i\mu_j}$

The following algorithm

for $i = 1, \ldots, n$ {
  $m_i := null$; $(m_i(j) := (\emptyset, 0, 0)\ \forall\, j = 1, \ldots, n)$
  if $\nu(i) \neq \emptyset$ $(\mu_i > 0)$ then {
    $\eta_i := \nu(i)$; $c_i := \mu_i$; $f_i := 1/\mu_i$;
    for $j = i+1, \ldots, n$ {
      $\eta_j = \emptyset$; $c_j = 0$;
      for $k \in \nu(i)$ {
        if $k \in \nu(j)$ then {
          $\eta_j := \eta_j \cup \{k\}$;
          $c_j := c_j + 1$ } } $(\eta_j = \nu(i)\cap\nu(j),\ c_j = |\nu(i)\cap\nu(j)|)$
      if $c_j > 0$ then { $f_j := c_j/(\mu_i\mu_j)$
    }
    for $j = i, \ldots, n$ { $(\eta_j \neq \emptyset$ for at least $j = i)$
      $m_i(j) := (\eta_j, c_j, f_j)$ }
    for $j = 1, \ldots, i-1$ { $m_i(j) := m_j(i)$ } $(m_i \neq null)$
  }
}
yields the vectors m_i here below:

$m_1 = [(\ ,\ ) \ldots]$
$\ldots$
$m_i = [(\ ,\ ) \ldots (\nu(i)\cap\nu(j), |\nu(i)\cap\nu(j)|, \frac{|\nu(i)\cap\nu(j)|}{\mu_i\mu_j}) \ldots (\ ,\ )]$
$\ldots$
$m_n = [(\ ,\ ) \ldots]$.

Question: is it possible to compute $\|(PP^*)^{2^k}\|_\infty$ from $\|(PP^*)^{2^{k-1}}\|_\infty$ with less than $|\{j : \nu(j) \neq \emptyset\}|^3$ arithmetic operations ?

$\rho(H)$ as the limit of a sequence (written about two years ago)

(Given $\gamma_k \in S \subset X$ and $\gamma \in X$, by writing

$$\gamma_k \to \gamma$$

we mean the convergence to zero of the sequence of non negative real numbers $\xi_k = \|\gamma_k - \gamma\|$, where $\|$ may be the absolute value ($X = \mathbb{R}, \mathbb{C}$), a vector norm ($X = \mathbb{R}^n, \mathbb{C}^n$), a matrix norm ($X = \mathbb{R}^{n\times n}, \mathbb{C}^{n\times n}$), a functional norm ($X = C^0, L^2$).)

Let H be a $n \times n$ matrix and ρ_k the sequence of non negative real numbers

$$\rho_k = (\|H^k\|)^{1/k},\ k \geq 1.$$

It is simple to verify that $\rho(H) \leq \rho_k \leq \rho_1 = \|H\|$, $\forall k$ and that $\ldots \rho_8 \leq \rho_4 \leq \rho_2 \leq \rho_1; \ldots \rho_6 \leq \rho_3 \leq \rho_1; \ldots$. It is not simple to establish if the sequence ρ_k is non increasing or not. In particular, is it true that

$$\rho_3 \leq \rho_2 \text{ ? (surely we have } \rho_3 \leq \rho_2 \frac{\|H\|^{1/3}}{\|H^2\|^{1/6}})$$

However, even if we succeed in proving the non increasing behaviour of ρ_k, or, more simply, the convergence of ρ_k, we could only conclude that

$$\rho(H) \leq \lim_k \rho_n \leq \rho_s.$$

In the following theorem, instead, it is noticed that a stronger result holds: the sequence ρ_k *converges (from above) to* $\rho(H)$.

Lemma *Given $H \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$ there exists a matrix norm $\hat{\|} = \hat{\|}_{H,\varepsilon}$ such that $\hat{\|}H\hat{\|} < \rho(H) + \varepsilon$.*

Proof. The thesis is verified, for example, by setting $\hat{\|}A\hat{\|} = \|SAS^{-1}\|_\infty$ where $S = DQ$, with Q unitary such that $T = QAQ^{-1}$ is upper triangular, and D diagonal with $D_{ii} = 1/\delta^i$, δ suitable.

Theorem *Given $H \in \mathbb{C}^{n\times n}$, indipendently from the choice of the matrix norm $\|\cdot\|$, we have*

$$\rho(H) = \lim_k \left(\|H^k\|\right)^{\frac{1}{k}}.$$

Proof. Set $\rho = \inf \rho_k$, for a particular choice of $\|$. Then it can be shown that ρ_k, for any choice of $\|$, converges to ρ. It follows that the definition of ρ does not depend upon the choice of $\|$. Since $\rho_k \geq \rho(H)$, we have $\rho \geq \rho(H)$. But we also have $\varepsilon + \rho(H) > \rho$, for any $\varepsilon > 0$. In fact, by the Lemma there exists $\hat{\|}$ such that $(\hat{\|}H^k\hat{\|})^{1/k} \leq \hat{\|}H\hat{\|} < \rho(H) + \varepsilon$, and $\rho = \inf(\hat{\|}H^k\hat{\|})^{1/k}$. Thus ρ must be equal to $\rho(H)$. QED

Problem: $\rho_1 \geq \rho_2 \geq \rho_4 \geq \rho_8 \geq \ldots$, are better and better approximations of $\rho(H)$. Is it possible to introduce a norm $\|\cdot\|$ such that these approximations are easily computable and converge rapidly to $\rho(H)$?

Corollary *Given $H \in \mathbb{C}^{n\times n}$, $H^k \to 0$ if and only if $\rho(H) < 1$.*

Proof. $\rho(H) < 1 \Rightarrow (\|H^k\|)^{1/k} \leq \nu < 1$, $\forall k \geq N \Rightarrow \|H^k\| \leq \nu^k \Rightarrow \|H^k\| \to 0$. Viceversa, $\|H^k\| \to 0 \Rightarrow \exists k : \|H^k\| < 1 \Rightarrow \exists k : \rho(H)^k = \rho(H^k) < 1 \Rightarrow \rho(H) < 1$. QED