An exercise of matrix norms

It can be shown that, given any $n \times n$ matrix A and any matrix norm $\|\cdot\|$, and defined $\rho(A) = \max_i |\lambda_i(A)|$, then

$$\|A^n\|^{1/n} \to \rho(A) \quad n \to +\infty$$

(from above, in fact $\rho(A)^n = \rho(A^n) \leq \|A^n\|$ and the latter inequality implies $\rho(A) \leq \|A^n\|^{1/n}$).

Now, it is simple to verify that

$$\ldots \|A^8\|^{1/8} \leq \|A^4\|^{1/4} \leq \|A^2\|^{1/2} \leq \|A\|,$$

or that

$$\ldots \|A^6\|^{1/6} \leq \|A^3\|^{1/3} \leq \|A\|$$

(use the fourth property of matrix norms). But it is not clear if the sequence $\|A^n\|^{1/n}$ is not increasing, i.e. if the following inequality

$$\|A^n\|^{1/n} \leq \|A^{n-1}\|^{1/(n-1)}, \ \forall n \geq 2,$$

holds. In particular, is there a matrix A for which

$$\|A^3\|^{1/3} > \|A^2\|^{1/2} \ ?$$

Can Richardson-Eulero be improved?

Is there an $\varepsilon > 0$ such that

$$\rho((I - (1-\varepsilon)A)(I - (1+\varepsilon)A)) \leq \rho((I-A)^2) \quad ?$$

or, equivalently, is there a $\delta > 0$ ($\delta = \varepsilon^2$) such that

$$\rho((I-A)^2 - \delta A^2) \leq \rho((I-A)^2) \quad ?$$

Assume $A = I - \alpha P^T$, P row quasi-stochastic. Then the question becomes the following: is there a $\delta > 0$ ($\delta = \varepsilon^2$) such that

$$\rho(\alpha^2 (P^T)^2 - \delta(I - \alpha P^T)^2) \leq \rho(\alpha^2 (P^T)^2) \quad ?$$

EXAMPLE. Let us consider an example. Set

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ A = I - \alpha P^T = \begin{bmatrix} 1 & -\alpha \\ -\alpha & 1 \end{bmatrix}.$$

Then $(P^T)^2 = I$, and

$$(I - \alpha P^T)^2 = \begin{bmatrix} 1+\alpha^2 & -2\alpha \\ -2\alpha & 1+\alpha^2 \end{bmatrix},$$

$$\alpha^2 (P^T)^2 - \delta(I - \alpha P^T)^2 = \begin{bmatrix} \alpha^2 - \delta(1+\alpha^2) & 2\alpha\delta \\ 2\alpha\delta & \alpha^2 - \delta(1+\alpha^2) \end{bmatrix}.$$

So, the question becomes: is there a $\delta > 0$ such that

$$\rho_\delta := \max\{|\alpha^2 - \delta(1+\alpha)^2|, |\alpha^2 - \delta(1-\alpha)^2|\} \leq \alpha^2 \quad ?$$

By noting that

$$\frac{\alpha^2}{(1+\alpha)^2} < \frac{\alpha^2}{(1-\alpha)^2},$$

and observing the graphics of the functions $|\alpha^2 - \delta(1+\alpha)^2|$ and $|\alpha^2 - \delta(1-\alpha)^2|$, for $\delta > 0$, it is easy to conclude that

- $\rho_\delta < \alpha^2$ for $\delta \in (0, \frac{2\alpha^2}{(1+\alpha)^2})$,
- ρ_δ is minimum for $\delta = \delta_{ott} := \frac{\alpha^2}{\alpha^2+1}$, and
- $\rho_{\delta_{ott}} = \frac{2\alpha^3}{\alpha^2+1}$.

Thus the answer is yes. □

EXERCISE: Is the answer to the question yes if

$$P = \begin{bmatrix} 0 & a & 1-a \\ b & 0 & 1-b \\ c & 1-c & 0 \end{bmatrix}, \ a, b, c \in [0,1] \ ?$$

or if

$$P = \begin{bmatrix} 0 & a & 1-a \\ b & 0 & 1-b \\ 0 & 0 & 0 \end{bmatrix}, \ a, b \in [0,1] \ ?$$

or if ... .

Let us consider the second case:

$$P^T = \begin{bmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 1-a & 1-b & 0 \end{bmatrix}, \ (P^T)^2 = \begin{bmatrix} ba & 0 & 0 \\ 0 & ab & 0 \\ (1-b)a & (1-a)b & 0 \end{bmatrix}.$$

Note that $\rho(\alpha^2(P^T)^2) = ab\alpha^2$.

$$I - \alpha P^T = \begin{bmatrix} 1 & -\alpha b & 0 \\ -\alpha a & 1 & 0 \\ -\alpha(1-a) & -\alpha(1-b) & 1 \end{bmatrix},$$

$$(I - \alpha P^T)^2 = \begin{bmatrix} 1+\alpha^2 ab & -2\alpha b & 0 \\ -2\alpha a & \alpha^2 ab + 1 & 0 \\ -2\alpha(1-a) + \alpha^2 a(1-b) & \alpha^2 b(1-a) - 2\alpha(1-b) & 1 \end{bmatrix},$$

$$\alpha^2(P^T)^2 - \delta(I - \alpha P^T)^2 =$$

$$\begin{bmatrix} \alpha^2 ab - \delta(1+\alpha^2 ab) & -\delta(-2\alpha b) & 0 \\ -\delta(-2\alpha a) & \alpha^2 ab - \delta(\alpha^2 ab + 1) & 0 \\ \alpha^2(1-b)a - \delta(-2\alpha(1-a) + \alpha^2 a(1-b)) & \alpha^2(1-a)b - \delta(\alpha^2 b(1-a) - 2\alpha(1-b)) & -\delta \end{bmatrix}.$$

Let ρ_δ be the spectral radius of the latter matrix.

Is $\rho_\delta < ab\alpha^2$ for some $\delta > 0$?

$$\rho_\delta = \max\{|\alpha^2 ab - \delta(1-\alpha\sqrt{ab})^2|, |\alpha^2 ab - \delta(1+\alpha\sqrt{ab})^2|, |-\delta|\},$$

$$\delta_{ott} = \begin{cases} \frac{\alpha^2 ab}{1+(1-\alpha\sqrt{ab})^2} & ab < \frac{1}{4\alpha^2} \\ \frac{\alpha^2 ab}{1+\alpha^2 ab} & ab \geq \frac{1}{4\alpha^2} \end{cases},$$

$$\rho_{\delta_{ott}} = \begin{cases} \frac{\alpha^2 ab}{1+(1-\alpha\sqrt{ab})^2} & ab < \frac{1}{4\alpha^2} \\ \frac{2\alpha^3 ab\sqrt{ab}}{1+\alpha^2 ab} & ab \geq \frac{1}{4\alpha^2} \end{cases} .$$

Thus, also in the second case the answer is yes. □

Let us consider the first case:

$$I - \alpha P^T = \begin{bmatrix} 1 & -\alpha b & -\alpha c \\ -\alpha a & 1 & -\alpha(1-c) \\ -\alpha(1-a) & -\alpha(1-b) & 1 \end{bmatrix},$$

$$(I - \alpha P^T)^2 =$$

$$\begin{bmatrix} 1+\alpha^2 ba + \alpha^2 c(1-a) & -2\alpha b + \alpha^2 c(1-b) & -2\alpha c + \alpha^2 b(1-c) \\ -2\alpha a + \alpha^2(1-c)(1-a) & \alpha^2 ab + 1 + al^2(1-c)(1-b) & \alpha^2 ac - 2\alpha(1-c) \\ -2\alpha(1-a) + \alpha^2(1-b)a & \alpha^2(1-a)b - 2\alpha(1-b) & \alpha^2(1-a)c + \alpha^2(1-b)(1-c) + 1 \end{bmatrix}$$

. . .

Non stationary Richardson-Eulero methods

Is there a $z \in \mathbb{C}$ such that

$$\rho((I - (1-z)A)(I - (1+z)A)) < \rho((I-A)^2)$$

or, equivalently, such that

$$\rho((I-A)^2 - z^2 A^2) < \rho((I-A)^2) \quad ?$$

If yes, then a non stationary would be preferable with respect to a stationary Richardson-Eulero method.

Assume $A = I - \alpha P^T$, P row quasi-stochastic. Is there a $z \in \mathbb{C}$ such that

$$\rho((\alpha P^T)^2 - z^2(I - \alpha P^T)^2) < \rho((\alpha P^T)^2) \quad ?$$

If $\eta_j = \alpha\lambda_j$, where $\lambda_j =$ eigenvalues of P^T, then the required inequality becomes

$$\max_j |\eta_j^2 - z^2(1-\eta_j)^2| < \max_i |\eta_i^2|$$

$$|\eta_j^2 - z^2(1-\eta_j)^2|^2 < \max_i |\eta_i^2|^2, \ \forall j \quad (*)$$

$$|\eta_j|^4 - 2\Re(\overline{z}^2(1-\overline{\eta_j})^2\eta_j^2) + |z|^4|1-\eta_j|^4 < \max_i |\eta_i|^4, \ \forall j : \ \eta_j \neq 0.$$

In fact, if $j : \eta_j = 0$, then the inequality (*) is verified for any z, $|z| < \max_i |\eta_i|$. From now on, by writing $\forall j$ we will mean $\forall j : \eta_j \neq 0$.

If $z = \sqrt{\delta}e^{\mathbf{i}\varphi}$, $\delta > 0$, then the question is the following: are there $\varphi \in [0, 2\pi)$ and $\delta > 0$ such that

$$|\eta_j|^4 - 2\delta\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) + \delta^2|1-\eta_j|^4 < \max_i |\eta_i|^4, \ \forall j \quad ?$$

Call $p_{j,\varphi}(\delta)$ the parabola on the left of the latter inequality. Then

$$p'_{j,\varphi}(\delta) = -2\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) + 2\delta|1-\eta_j|^4,$$

$$p'_{j,\varphi}(0) = -2\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2).$$

If $\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) > 0$, then there exists $\delta_{j,\varphi} > 0$ such that $\forall\, \delta \in (0, \delta_{j,\varphi})$

$$|\eta_j|^4 - 2\delta\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) + \delta^2|1-\eta_j|^4 < |\eta_j|^4 \le \max_i |\eta_i|^4.$$

$(\delta_{j,\varphi} = 2\frac{\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2)}{|1-\eta_j|^4})$. Moreover, there exists $\delta_\varphi > 0$ such that $\forall\, \delta \in (0, \delta_\varphi)$

$$\max_{j:\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2)>0}\{|\eta_j|^4 - 2\delta\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) + \delta^2|1-\eta_j|^4\}$$
$$< \max_{j:\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2)>0} |\eta_j|^4 \le \max_i |\eta_i|^4.$$

In fact, let $j^* \in \{j : \Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) > 0\}$ be such that

$$|\eta_{j^*}|^4 = \max_{j:\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2)>0} |\eta_j|^4,$$
$\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_{j^*}})^2\eta_{j^*}^2)$ is minimum,
$|1-\eta_{j^*}|$ is maximum.

Then

$$\delta_\varphi = \min\{2\frac{\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_{j^*}})^2\eta_{j^*}^2)}{|1-\eta_{j^*}|^4}, \sigma^*\}$$

where σ^* is the minimum among the positive abscissas of the intersections between the parabola $p_{j^*,\varphi}$ and all the parabolas $p_{j,\varphi}$, with $j : \Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) > 0$, $p_{j,\varphi} \ne p_{j^*,\varphi}$.

However, $|\eta_j|^4 - 2\delta\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) + \delta^2|1-\eta_j|^4 > |\eta_j|^4$ in a right neighborhood of $\delta = 0$, for all j such that $\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) < 0$.

So, the inequality

$$\max_j\{|\eta_j|^4 - 2\delta\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta_j})^2\eta_j^2) + \delta^2|1-\eta_j|^4\} < \max_i |\eta_i|^4$$

(for some $\delta > 0$ and φ) remains unproved.

The region $\Re(e^{-\mathbf{i}2\varphi}(1-\overline{\eta})^2\eta^2) > 0$ *for* $|\eta| \le \alpha < 1$*:*

The region $\Re((1-\overline{\eta})^2\eta^2) > 0$ ($\varphi = 0$):
$\eta = re^{\mathbf{i}\theta} \Rightarrow (1-\overline{\eta})^2\eta^2 = r^2(e^{\mathbf{i}2\theta} - 2re^{\mathbf{i}\theta} + r^2) \Rightarrow \Re((1-\overline{\eta})^2\eta^2) = r^2(\cos(2\theta) - 2r\cos(\theta) + r^2)$. So, $\Re((1-\overline{\eta})^2\eta^2) > 0$ iff $r < \cos\theta - |\sin\theta|$ or $r > \cos\theta + |\sin\theta|$.

Exercise. Draw in $|\eta| \le \alpha$ the region $\{re^{\mathbf{i}\theta} : r < \cos\theta - |\sin\theta| \text{ or } r > \cos\theta + |\sin\theta|\}$.

The region $\Re(e^{-\mathbf{i}\pi}(1-\overline{\eta})^2\eta^2) > 0$ ($\varphi = \pi/2$):
$\Re(e^{-\mathbf{i}\pi}(1-\overline{\eta})^2\eta^2) > 0$ iff $\Re((1-\overline{\eta})^2\eta^2) < 0$. So, this region is the complementary of the previous one.

Note: the region $\Re(e^{-\mathbf{i}2(\psi+\frac{\pi}{2})}(1-\overline{\eta})^2\eta^2) > 0$ is the complementary of the region $\Re(e^{-\mathbf{i}2\psi}(1-\overline{\eta})^2\eta^2) > 0$.

By considering the cases $\varphi = 0$ and $\varphi = \pi/2$, we can say that there exists δ^* such that $\forall\, \delta \in (0, \delta^*)$

$$\max_{j:\Re((1-\overline{\eta_j})^2\eta_j^2)>0} |\eta_j^2 - \delta(1-\eta_j)^2| < \max_{j:\Re((1-\overline{\eta_j})^2\eta_j^2)>0} |\eta_j|^2 \le \max_i |\eta_i|^2,$$

$$\max_{j:\Re((1-\overline{\eta_j})^2\eta_j^2)<0} |\eta_j^2 + \delta(1-\eta_j)^2| < \max_{j:\Re((1-\overline{\eta_j})^2\eta_j^2)<0} |\eta_j|^2 \le \max_i |\eta_i|^2.$$

So, perhaps, by compensation, there exists a right neighborhood of $\delta = 0$ where

$$\max_j |\eta_j^2 - \delta(1-\eta_j)^2||\eta_j^2 + \delta(1-\eta_j)^2| < \max_i |\eta_i|^4.$$

This would be surely true if $\Re((1-\overline{\eta_i})^4\eta_i^4) > 0$, $\forall\, i$, or, equivalently, if $\Re((1-\overline{\eta})^4\eta^4) > 0$, $\forall\, \eta :\ |\eta| \leq \alpha$, or, equivalently, if

$$r^4[(\cos(2\theta) - 2r\cos\theta + r^2)^2 - (\sin(2\theta) - 2r\sin\theta + r^2)^2] > 0, \quad \forall\, \theta, \forall\, r \leq \alpha$$

($\eta = re^{\mathbf{i}\theta}$). But the latter inequality is not true for all required θ and r.

Exercise. Draw in $|\eta| \leq \alpha$ the region where the latter inequality i s verified.

Notes on work with Fra

Experimental tests show that the eigenvalues of P are all grouped in a circle with center in the origin and radius about 0.3 except one which is near 1. Moreover, they show that no eigenvalue of $C_P^{-1}P$ is outside the previous circle. (NOT CORRECT! P and C_P may be singular)

Experimental tests show that the eigenvalues of $I - \alpha P$ are all grouped in a circle with center in 1 and radius about 0.3α except one which is near $1-\alpha$. Moreover, they show that no eigenvalue of $C_{I-\alpha P}^{-1}(I-\alpha P)$ is outside the previous circle. (CORRECT)

One should give a theoretical justification of these observations.

We know that

$$C_{P^T} = F\,\mathrm{diag}\,((F^*P^TF)_{ii})F^*,\ C_P = F\,\mathrm{diag}\,((F^*PF)_{ii})F^*.$$

Moreover, $(F^*P^TF)_{ii} = (FP\overline{F})_{ii} = \overline{(F^*PF)_{ii}}$. Thus $C_{P^T} = C_P^* = C_P^T$, where the latter equality follows from the fact that C_P is real (because P is real and there exists a real basis for circulant matrices, see [maiop]).

So, in the notations $C_P = FD_PF^*$, $C_{P^T} = FD_{P^T}F^*$, we have the equality $D_{P^T} = \overline{D_P}$.

Note also that $(C_{P^T}^{-1}P^T)^T = PC_P^{-1}$. Thus, $C_{P^T}^{-1}P^T$ has the same eigenvalues of $C_P^{-1}P$. In other words, the spectra of P and $C_P^{-1}P$ coincide with the spectra of P^T and $C_{P^T}^{-1}P^T$, respectively. (NOT CORRECT! P and C_P may be singular)

Note also that $(C_{I-\alpha P^T}^{-1}(I-\alpha P^T))^T = (I-\alpha P)C_{I-\alpha P}^{-1}$. Thus, $C_{I-\alpha P^T}^{-1}(I-\alpha P^T)$ has the same eigenvalues of $C_{I-\alpha P}^{-1}(I-\alpha P)$. In other words, the spectra of $I-\alpha P$ and $C_{I-\alpha P}^{-1}(I-\alpha P)$ coincide with the spectra of $I-\alpha P^T$ and $C_{I-\alpha P^T}^{-1}(I-\alpha P^T)$, respectively. (CORRECT)

Another question is the following. How are the eigenvalues of $(I-\alpha P)^*(I-\alpha P)$ distributed on $(0,\infty)$? and the eigenvalues of $C_{(I-\alpha P)^*(I-\alpha P)}^{-1}(I-\alpha P)^*(I-\alpha P)$? This question in order to investigate the possibility of using preconditioned conjugate gradients to solve the google system.

A final question. Is there one event between 1,2,3,4 for which, after this event, the pagerank corresponding to P_{new} can be easily obtained from the pagerank corresponding to P_{old} ?

I level degree in Math, February 24, 2010, Matteo Ferrone, Giosi & a problem of Math Physics involving the columns of the sine transform: a more direct proof of their orthogonality

Set $S = \beta(\sin\frac{\pi jr}{n})_{j,r=1}^{n-1}$, $\beta \in \mathbb{R}$.

Denote by $a_{h,k}$ the inner product of the h and k columns of S ($A = S^T S$). Then we have:

$$\begin{array}{rcl} a_{h,k} & = & \beta^2 \sum_{j=1}^{n-1} \sin\frac{\pi jh}{n} \sin\frac{\pi jk}{n} \\ & = & \beta^2 \sum_{j=1}^{n-1} \frac{1}{2}[\cos\frac{\pi j(h-k)}{n} - \cos\frac{\pi j(h+k)}{n}] \\ & = & \frac{\beta^2}{2} \sum_{j=1}^{n-1} [\Re(e^{\mathbf{i}\frac{\pi j(h-k)}{n}} - e^{\mathbf{i}\frac{\pi j(h+k)}{n}})] \\ & = & \frac{\beta^2}{2} \Re(\sum_{j=0}^{n-1} e^{\mathbf{i}\frac{\pi j(h-k)}{n}} - \sum_{j=0}^{n-1} e^{\mathbf{i}\frac{\pi j(h+k)}{n}}) \end{array}$$

So, $a_{kk} = \frac{\beta^2}{2} n$ and if $h \neq k$:

$$\begin{array}{rcl} a_{h,k} & = & \frac{\beta^2}{2} \Re(\frac{1-e^{\mathbf{i}\pi j(h-k)}}{1-e^{\mathbf{i}\frac{\pi j(h-k)}{n}}} - \frac{1-e^{\mathbf{i}\pi j(h+k)}}{1-e^{\mathbf{i}\frac{\pi j(h+k)}{n}}}) \\ & = & \frac{\beta^2}{2}[1-e^{\mathbf{i}\pi j(h-k)}] \Re(\frac{1}{1-e^{\mathbf{i}\frac{\pi j(h-k)}{n}}} - \frac{1}{1-e^{\mathbf{i}\frac{\pi j(h+k)}{n}}}) \\ & = & \frac{\beta^2}{2}[1-e^{\mathbf{i}\pi j(h-k)}](0.5-0.5) = 0. \end{array}$$

The latter equality holds since

$$\Re(\frac{1}{1-e^{\mathbf{i}x}}) = \frac{1}{1-\cos x - \mathbf{i}\sin x} = \frac{(1-\cos x) + \mathbf{i}\sin x}{(1-\cos x)^2 + \sin^2 x} = \frac{1}{2}, \ \forall\, x \neq 2k\pi.$$

Thus $A = \frac{\beta^2}{2} nI$. In particular, S is unitary ($A = I$) for $\beta = \sqrt{2/n}$.

Claudia, Marcello, Andrea and the roots in $[0,1]$ *of Bernoulli polynomials*

Any odd degree Bernoulli polynomial $B_{2k+1}(x)$ is null for $x = 0, \frac{1}{2}, 1$. Assume that it is null also in $\hat{x} \in (0, \frac{1}{2})$. Then $B'_{2k+1}(x) = (2k+1)B_{2k}(x)$ is null in $\hat{x}_l \in (0, \hat{x})$ and in $\hat{x}_r \in (\hat{x}, \frac{1}{2})$. But then $B'_{2k}(x) = 2kB_{2k-1}(x)$ must assume the value zero in the open interval $(\hat{x}_l, \hat{x}_r) \subset (0, \frac{1}{2})$.

Thus we have proved the following

Result 1. Any time $B_{2k+1}(x)$ is zero in some point of the interval $(0, \frac{1}{2})$, also $B_{2k-1}(x)$ must be zero in $(0, \frac{1}{2})$.

It follows that if for some odd n the polynomial $B_n(x)$ is zero in $(0, \frac{1}{2})$, then $B_3(x)$ must be zero in $(0, \frac{1}{2})$. But the only zeros of B_3 are $0, \frac{1}{2}, 1$. Thus:

Result 2. For any n odd, the only zero in $(0,1)$ of B_n is $\frac{1}{2}$; for any n even, $n \neq 0$, the only stationary point in $(0,1)$ of B_n is $\frac{1}{2}$ (note that also $0, 1$ are stationary points for B_n for any n even, $n \neq 2$)

Result 3. The Bernoulli polynomials whose degree is even have two, and only two, roots in the interval $(0,1)$, say $\hat{x} \in (0, \frac{1}{2})$ and $1 - \hat{x}$.

Proof. The fact that in the interval $(0,1)$ there must be two distinct roots of B_{2k} of the form $\hat{x} \in (0, \frac{1}{2})$ and $1-\hat{x}$, follows from the equalities $\int_0^1 B_{2k}(x)dx = 0$ and $B_{2k}(x) = B_{2k}(1-x)$. Assume that B_{2k} has another pair of roots, say $\tilde{x} \in (0, \frac{1}{2}]$ and $1 - \tilde{x}$, $\tilde{x} \neq \hat{x}$. Then $B'_{2k} = 2kB_{2k-1}$ must have a root in the open interval $(\min\{\hat{x}, \tilde{x}\}, \max\{\hat{x}, \tilde{x}\}) \subset (0, \frac{1}{2})$, which is absurd by the Result 2.

As a consequence of the Result 2, if we prove that $|B_{2k}(\frac{1}{2})| \leq |B_{2k}(0)|$, then the inequality

$$|B_{2k}(x)| \leq |B_{2k}(0)|, \quad \forall\, x \in [0,1]$$

will be obtained.

I esonero AN3 - 17 Marzo 2010

Exercise 1. Consider the problem of approximating $I = \int_a^b f(x)dx$, being $f(x) = \frac{1}{x}$, $a = 1$, $b = 2$. Note that $I = \ln 2 = 0.69314718\ldots$.

One could use the Nicolaus Mercator series representation of $\ln 2$,

$$\begin{array}{l} \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots, \\ \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots, \end{array}$$

but too terms are required to obtain a sufficient accuracy (for example, one hundred terms give 0.6981..; one thousand terms give 0.69364.. (CASIO PB-200)).

A better method is approximating I by the trapezoidal quadrature formula, combined with the Romberg extrapolation method. Set $h = \frac{b-a}{n} = \frac{1}{n}$. Then the values

$$S_n = I_{\frac{1}{n}} = \frac{1}{n}\big(\frac{1}{2}f(1) + \frac{1}{2}f(2) + \sum_{i=1}^{n-1} f(1 + i\frac{1}{n})\big),$$

$n = 1, 2, 3, 4, \ldots$, approach I better and better (since $S_n \to I$ as $n \to +\infty$). Let us compute the first four such approximations:

$$\begin{array}{rcl} S_1 = I_1 &=& 1 \cdot (\frac{1}{2} + \frac{1}{4}) = \frac{3}{4} = 0.75, \\ S_2 = I_{\frac{1}{2}} &=& \frac{1}{2}(\frac{1}{2} + \frac{1}{4} + \frac{2}{3}) = \frac{17}{24} = 0.708\overline{3}, \\ S_3 = I_{\frac{1}{3}} &=& \frac{1}{3}(\frac{1}{2} + \frac{1}{4} + \frac{3}{4} + \frac{3}{5}) = \frac{7}{10} = 0.7, \\ S_4 = I_{\frac{1}{4}} &=& \frac{1}{4}(\frac{1}{2} + \frac{1}{4} + \frac{2}{3} + \frac{4}{5} + \frac{4}{7}) = \frac{1171}{1680} = 0.697023809, \end{array}$$

and from these, via the Romberg method, the following better quality approximations:

$$\begin{array}{rcl} \tilde{S}_2 &=& \frac{2^2 S_2 - S_1}{2^2 - 1} = \frac{25}{36} = 0.69\overline{4}, \\ \tilde{S}_4 &=& \frac{2^2 S_4 - S_2}{2^2 - 1} = \frac{1747}{2520} = 0.693253968, \\ \tilde{\tilde{S}}_4 &=& \frac{2^4 \tilde{S}_4 - \tilde{S}_2}{2^4 - 1} = \frac{4367}{6300} = 0.693174603. \end{array}$$

Let us prove an alternative extrapolation technique, where the intervals are divided by 3, instead of by 2. We know that

$$I = I_h + c_1 h^2 + c_2 h^4 + c_3 h^6 + \ldots \ .$$

Thus

$$\begin{array}{l} I = I_{3h} + c_1 3^2 h^2 + c_2 3^4 h^4 + c_3 3^6 h^6 + \ldots, \\ 3^2 I = 3^2 I_h + c_1 3^2 h^2 + c_2 3^2 h^4 + c_3 3^2 h^6 + \ldots, \\ (3^2 - 1)I = 3^2 I_h - I_{3h} + c_2(3^2 - 3^4)h^4 + c_3(3^2 - 3^6)h^6 + \ldots, \\ I = \frac{3^2 I_h - I_{3h}}{3^2 - 1} + \tilde{c}_2 h^4 + \tilde{c}_3 h^6 + \ldots. \end{array}$$

It follows that

$$\hat{I}_h = \frac{3^2 I_h - I_{3h}}{3^2 - 1}, \ I - \hat{I}_h = O(h^4).$$

By applying this formula (for $h = \frac{1}{3}$) to our particular problem, we obtain

$$\hat{S}_3 = \hat{I}_{\frac{1}{3}} = \frac{3^2 I_{\frac{1}{3}} - I_1}{3^2 - 1} = \frac{111}{160} = 0.69375.$$

Comparison of all approximations:

$$I < \tilde{\tilde{S}}_4 < \tilde{S}_4 < \hat{S}_3 < \tilde{S}_2 < S_4 < S_3 < S_2 < S_1.$$

Exercise 2. Higher order derivation rules for Bernoulli polynomials follow immediately from the identity $B_n'(x) = nB_{n-1}(x)$:

$$B_n'(x) = nB_{n-1}(x),\ B_n''(x) = nB_{n-1}'(x) = n(n-1)B_{n-2}(x),$$
$$B_n^{(j)}(x) = n(n-1)\cdots(n-j+1)B_{n-j}(x),$$

Thus, $B_8^{(5)}(x) = 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot B_3(x) = 6720x(x-\frac{1}{2})(x-1)$.

Let n be even. If $x^* > 1$ is such that $B_n(x^*) = 0$, then $B_{n-1}(\hat{x}) = 0$, for some $\hat{x} \in (1, x^*)$. Let us prove this fact.

Since $B_n(1) = B_n(x^*) = 0$, there exists $\hat{x} \in (1, x^*)$ such that $0 = B_n(x^*) - B_n(1) = B_n'(\hat{x})(x^*-1) = nB_{n-1}(\hat{x})(x^*-1)$. Alternatively,

$$0 = B_n(x^*) = B_n(0) + n\int_0^{x^*} B_{n-1}(t)dt = n\int_0^{x^*} B_{n-1}(t)dt = n\int_1^{x^*} B_{n-1}(t)dt,$$

thus B_{n-1} must become zero in some point of $(1, x^*)$.

Let $m \geq 1$, be a natural number. By the Euler-Maclaurin formula, for any $n \geq m$ we have

$$\sum_{r=m}^{n} \frac{1}{r} = \frac{1}{2}(\frac{1}{m} + \frac{1}{n}) + \ln n - \ln m + \sum_{j=1}^{k} \frac{B_{2j}(0)}{2j}[-\frac{1}{n^{2j}} + \frac{1}{m^{2j}}] + u_{k+1},$$

$$|u_{k+1}| \leq \frac{1}{k+1}|B_{2k+2}(0)||-\frac{1}{n^{2k+2}} + \frac{1}{m^{2k+2}}|.$$

It follows that

$$\sum_{r=1}^{n} \frac{1}{r} - \ln n = \sum_{r=1}^{m-1} \frac{1}{r} + \frac{1}{2}(\frac{1}{m} + \frac{1}{n}) - \ln m + \sum_{j=1}^{k} \frac{B_{2j}(0)}{2j}[-\frac{1}{n^{2j}} + \frac{1}{m^{2j}}] + u_{k+1},$$

and thus, if γ denotes $\lim_{n\to+\infty}(\sum_{r=1}^{n}\frac{1}{r} - \ln n)$, we have

$$\gamma = \sum_{r=1}^{m-1} \frac{1}{r} + \frac{1}{2m} - \ln m + \sum_{j=1}^{k} \frac{B_{2j}(0)}{2j}\frac{1}{m^{2j}} + u_{k+1}(\infty),$$

$$|u_{k+1}(\infty)| \leq \frac{|B_{2k+2}(0)|}{(k+1)m^{2k+2}}.$$

For instance, for $m = 1$ and $m = 10$ we obtain, respectively,

$$\gamma = \frac{1}{2} + \sum_{j=1}^{k} \frac{B_{2j}(0)}{2j} + u_{k+1}(\infty),\ |u_{k+1}(\infty)| \leq \frac{|B_{2k+2}(0)|}{k+1},$$

$$\gamma = \sum_{r=1}^{9} \frac{1}{r} + \frac{1}{20} - \ln 10 + \sum_{j=1}^{k} \frac{B_{2j}(0)}{2j}\frac{1}{10^{2j}} + u_{k+1}(\infty),\ |u_{k+1}(\infty)| \leq \frac{|B_{2k+2}(0)|}{(k+1)10^{2k+2}}.$$

As a consequence, two numbers that differ from the Euler-Mascheroni γ constant less that 0.01 are

$$\tfrac{1}{2}+\tfrac{1}{2}B_2(0)+\tfrac{1}{4}B_4(0)=\tfrac{69}{120}=0.575,$$

$$\tfrac{1}{2}+\tfrac{1}{2}B_2(0)+\tfrac{1}{4}B_4(0)+\tfrac{1}{6}B_6(0)=\tfrac{1459}{2520}=0.578968,$$

and two numbers that differ from γ less than $1/10^8$ and $1/10^{10}$, respectively, are

$$\textstyle\sum_{r=1}^{9}\frac{1}{r}+\frac{1}{20}-\ln 10+\frac{B_2(0)}{2}\frac{1}{10^2}+\frac{B_4(0)}{4}\frac{1}{10^4},$$

$$\textstyle\sum_{r=1}^{9}\frac{1}{r}+\frac{1}{20}-\ln 10+\frac{B_2(0)}{2}\frac{1}{10^2}+\frac{B_4(0)}{4}\frac{1}{10^4}+\frac{B_6(0)}{6}\frac{1}{10^6}.$$

Exercise 3.

$$\begin{aligned} B_{2k}(x)-B_{2k}(0) &= 2k\textstyle\int_0^x B_{2k-1}(t)dt \\ &= 2k\textstyle\int_0^x (B_{2k-1}(0)+(2k-1)\int_0^t B_{2k-2}(\xi)d\xi)dt \\ &= 2k(2k-1)\textstyle\int_0^x\int_0^t B_{2k-2}(\xi)d\xi\,dt. \end{aligned}$$

Thus, if $x\in(0,1]$,

$$\begin{aligned} |B_{2k}(x)-B_{2k}(0)| &\le 2k(2k-1)\textstyle\int_0^x\int_0^t |B_{2k-2}(\xi)|d\xi\,dt \\ &\le 2k(2k-1)|B_{2k-2}(0)|\textstyle\int_0^x\int_0^t d\xi\,dt \\ &= 2k(2k-1)|B_{2k-2}(0)|\textstyle\int_0^x t\,dt = 2k(2k-1)|B_{2k-2}(0)|\frac{x^2}{2}. \end{aligned}$$

(recall that $|B_{2k}(x)|\le|B_{2k}(0)|$, $\forall\, x\in[0,1]$). Note that equality holds if $x=0$.

Exercise 4. Let D be a diagonal matrix. Is $D^H=D$? We have $D^H=\overline{D}$, so the question becomes: is $\overline{D}=D$? The answer is: $\overline{D}=D$ iff D is real. Thus, if one of the diagonal entries of D is in $\mathbb{C}\backslash\mathbb{R}$, then $D^H\neq D$; i.e. there exist diagonal matrices which are not hermitian.

If D is diagonal, then DD^H and D^HD are diagonal, and the i,i element of DD^H is equal to $[D]_{ii}\overline{[D]_{ii}}=\overline{[D]_{ii}}[D]_{ii}$, which is the i,i element of D^HD. So, $DD^H=D^HD$, i.e. any diagonal matrix D is normal.

More in general, any matrix $A=QDQ^H$, D diagonal, Q unitary, is normal. In fact,

$$\begin{aligned} (QDQ^H)^H(QDQ^H) &= QD^HQ^HQDQ^H=QD^HDQ^H=QDD^HQ^H \\ &= QDQ^HQD^HQ^H=QDQ^H(QDQ^H)^H. \end{aligned}$$

On the relation between the numbers $B_{2k}(0)$ and $B_{2k}(\frac{1}{2})$

By calculating the first Bernoulli polynomials (they are listed after formula (c) below), we observe that

$$B_2(0)=\tfrac{1}{6},\ B_2(\tfrac{1}{2})=\tfrac{1}{6}-\tfrac{3}{2}\cdot\tfrac{1}{6},$$
$$B_2(0)+B_2(\tfrac{1}{2})=\tfrac{1}{2^2\cdot 3},$$

$$B_4(0)=-\tfrac{1}{30},\ B_4(\tfrac{1}{2})=-\tfrac{1}{30}+\tfrac{15}{8}\cdot\tfrac{1}{30},$$
$$B_4(0)+B_4(\tfrac{1}{2})=-\tfrac{1}{2^4\cdot 3\cdot 5},$$

$$B_6(0)=\tfrac{1}{42},\ B_6(\tfrac{1}{2})=\tfrac{1}{42}-\tfrac{63}{32}\cdot\tfrac{1}{42},$$
$$B_6(0)+B_6(\tfrac{1}{2})=\tfrac{1}{2^6\cdot 3\cdot 7},$$

$$B_8(0) = -\frac{1}{30},\ B_8(\tfrac{1}{2}) = -\frac{1}{30} + \frac{255}{128}\cdot\frac{1}{30},$$
$$B_8(0) + B_8(\tfrac{1}{2}) = -\frac{1}{2^8\cdot 3\cdot 5},$$

$$B_{10}(0) = \frac{5}{66},\ B_{10}(\tfrac{1}{2}) = \frac{5}{66} - \frac{1023}{512}\cdot\frac{5}{66},$$
$$B_{10}(0) + B_{10}(\tfrac{1}{2}) = \frac{5}{2^{10}\cdot 3\cdot 11}.$$

Thus we conjecture that the following identity holds:

$$\begin{aligned} B_{2k}(\tfrac{1}{2}) &= B_{2k}(0) - \frac{2\cdot 2^{2k-1}-1}{2^{2k-1}}\cdot B_{2k}(0) \\ &= -\frac{2^{2k-1}-1}{2^{2k-1}}\ B_{2k}(0) \\ &= -(1-\frac{1}{2^{2k-1}})\ B_{2k}(0). \qquad (c) \end{aligned}$$

Once such conjecture is proved, we will have the inequality $|B_{2k}(\frac{1}{2})| < |B_{2k}(0)|$, $\forall\, k$ (note that $\lim_{k\to+\infty}(B_{2k}(\frac{1}{2})/B_{2k}(0)) = -1$), and, as a consequence, the result $|B_{2k}(x)| \leq |B_{2k}(0)|$, $\forall\, x \in [0,1]$.

The Bernoulli polynomials $B_0, B_1, \ldots, B_{10}$:

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6} = \tfrac{1}{6} + x(x-1),$$
$$B_3(x) = x(x-1)(x-\tfrac{1}{2}), \quad B_4(x) = -\tfrac{1}{30} + x^2(x-1)^2$$

(note that $B_4(x) - B_4(0) = (B_2(x) - B_2(0))^2$),

$$\begin{aligned} B_5(x) &= x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x \\ &= x(x-1)(x-\tfrac{1}{2})(x^2 - x - \tfrac{1}{3}), \end{aligned}$$

$$\begin{aligned} B_6(x) &= x^6 - 3x^5 + \tfrac{5}{2}x^4 - \tfrac{1}{2}x^2 + \tfrac{1}{42} \\ &= \tfrac{1}{42} + x^2(x-1)^2(x^2 - x - \tfrac{1}{2}), \end{aligned}$$

$$\begin{aligned} B_7(x) &= x^7 - \tfrac{7}{2}x^6 + \tfrac{7}{2}x^5 - \tfrac{7}{6}x^3 + \tfrac{1}{6}x \\ &= x(x-\tfrac{1}{2})(x-1)(x^4 - 2x^3 + x + \tfrac{1}{3}), \end{aligned}$$

$$\begin{aligned} B_8(x) &= x^8 - 4x^7 + \tfrac{14}{3}x^6 - \tfrac{7}{3}x^4 + \tfrac{2}{3}x^2 - \tfrac{1}{30} \\ &= -\tfrac{1}{30} + x^2(x-1)^2(x^4 - 2x^3 - \tfrac{1}{3}x^2 + \tfrac{4}{3}x + \tfrac{2}{3}), \end{aligned}$$

$$\begin{aligned} B_9(x) &= x^9 - \tfrac{9}{2}x^8 + 6x^7 - \tfrac{21}{5}x^5 + 2x^3 - \tfrac{3}{10}x \\ &= x(x-\tfrac{1}{2})(x-1)(x^6 - 3x^5 + x^4 + 3x^3 - \tfrac{1}{5}x^2 - \tfrac{9}{5}x - \tfrac{3}{5}), \end{aligned}$$

$$\begin{aligned} B_{10}(x) &= x^{10} - 5x^9 + \tfrac{15}{2}x^8 - 7x^6 + 5x^4 - \tfrac{3}{2}x^2 + \tfrac{5}{66} \\ &= \tfrac{5}{66} + x^2(x-1)^2(x^6 - 3x^5 + \tfrac{1}{2}x^4 + 4x^3 + \tfrac{1}{2}x^2 - 3x - \tfrac{3}{2}). \end{aligned}$$

Exercise. Prove the following assertion

$$\frac{|B_{2k}(0)|}{(2k)!} \to c > 0,\ k \to +\infty.$$

The eigenvalue problem is optimally conditioned (in the spectral norm) for a matrix A iff A is normal

Let M be a non singular $n \times n$ matrix. If $\mu_2(M) = 1$ then cM is unitary for some $c > 0$. As a consequence, any time a matrix A is diagonalized by a matrix with spectral-condition number 1, the same A is also diagonalized by a unitary matrix, that is, A is normal. Thus, we have the following statement:

A is normal iff it is diagonalized by a matrix M with condition number 1.

Assume $\|M\|_2\|M^{-1}\|_2 = 1$. Then

$$\frac{\max_i |\lambda_i(M^H M)|}{\min_i |\lambda_i(M M^H)|} = \rho(M^H M)\rho((M M^H)^{-1}) = \rho(M^H M)\rho((M^{-1})^H(M^{-1})) = 1.$$

But the eigenvalues of MM^H are equal to the eigenvalues of $M^H M$ (AB and BA have the same eigenvalues, even in case both A and B are singular), and the latter are positive ($B^H B$ is positive definite if B is non singular). So, we must have

$$\frac{\max_i \lambda_i(M^H M)}{\min_i \lambda_i(M^H M)} = 1 \text{ i.e. } \exists c > 0 : \max_i \lambda_i(M^H M) = \min_i \lambda_i(M^H M) = c.$$

We also know that there is a matrix Q unitary such that $Q^{-1}M^H MQ$ is diagonal. Thus $M^H M = QcIQ^{-1} = cI$, and the thesis follows.

An AN3 transition matrix

Let P be the 21×21 matrix associated with the 21 students of AN3, whose entries are defined as follows. $P_{ij} = 1/\mu_i$ if student $i \in AN3$ satisfies the following two conditions: 1) has the mobil phone number of student $j \in AN3$; 2) has the mobil phone number of μ_i students of AN3. Otherwise, $P_{ij} = 0$.

For example, student $2 = MC$ has the mobil phone number of students $4 = SC$, $13 = MD$, $16 = DA$ and $21 = II$.

$AC = 1$	$\mu_{AC} = 6,$	$\nu_{AC} = \{CP, MF, DL, GS, JD, \overline{AC}\}$
$MC = 2$	$\mu_{MC} = 4,$	$\nu_{MC} = \{SC, MD, DA, II\}$
$CP = 3$	$\mu_{CP} = 2,$	$\nu_{CP} = \{AC, GL\}$
$SC = 4$	$\mu_{SC} = 3,$	$\nu_{SC} = \{AC, MC, MD\}$
$SM = 5$	$\mu_{SM} = 4,$	$\nu_{SM} = \{AC, GS, JD, \overline{AC}\}$
$RP = 6$	$\mu_{RP} = 4,$	$\nu_{RP} = \{EL, SB, GL, MR\}$
$MF = 7$	$\mu_{MF} = 2,$	$\nu_{MF} = \{AC, CP\}$
$AF = 8$	$\mu_{AF} = 1,$	$\nu_{AF} = \{CM\}$
$FI = 9$	$\mu_{FI} = 2,$	$\nu_{FI} = \{EL, CM\}$
$DL = 10$	$\mu_{DL} = 0,$	$\nu_{DL} = \emptyset$
$EL = 11$	$\mu_{EL} = 4,$	$\nu_{EL} = \{RP, FI, CM, GL\}$
$CM = 12$	$\mu_{CM} = 3,$	$\nu_{CM} = \{AF, FI, EL\}$
$MD = 13$	$\mu_{MD} = 4,$	$\nu_{MD} = \{MC, SC, GL, II\}$
$GS = 14$	$\mu_{GS} = 4,$	$\nu_{GS} = \{AC, SM, JD, \overline{AC}\}$
$SB = 15$	$\mu_{SB} =,$	$\nu_{SB} = \{\}$
$DA = 16$	$\mu_{DA} = 1,$	$\nu_{DA} = \{MC\}$
$JD = 17$	$\mu_{JD} = 6,$	$\nu_{JD} = \{AC, SM, DL, GS, \overline{AC}, II\}$
$\overline{AC} = 18$	$\mu_{\overline{AC}} = 4,$	$\nu_{\overline{AC}} = \{AC, SM, GS, JD\}$
$GL = 19$	$\mu_{GL} = 7,$	$\nu_{GL} = \{CP, SC, RP, EL, MD, MR, II\}$
$MR = 20$	$\mu_{MR} = 3,$	$\nu_{MR} = \{SC, RP, \overline{AC}\}$
$II = 21$	$\mu_{II} = 5,$	$\nu_{II} = \{MC, SC, MD, JD, GL\}$

$1 = AC$ =Andrea Celidonio, $2 = MC$ =Maria Chiara Capuzzo, $3 = CP$ =Claudia Pallotta, $4 = SC$ =Stefano Cipolla, $5 = SM$ =Sara Malacarne, $6 = RP$ =Roberta Piersimoni, $7 = MF$ =Marcello Filosa, $8 = AF$ =Alessandra Fabrizi, $9 = FI$ =Federica Iacovissi, $10 = DL$ =Diego Lopez, $11 = EL$ =Erika Leo, $12 = CM$ =Chiara Minotti $13 = MD$ =Martina De Marchis, $14 = GS$ =Giulia Sambucini, $15 = SB$ =Sofia Basile, $16 = DA$ =Davide Angelocola, $17 = JD$ =Jacopo De Cesaris, $18 = \overline{AC}$ =Alessandra Cataldo, $19 = GL$ =Giorgia Lucci, $20 = MR$ =Maria Grazia Rositano, $21 = II$ =Isabella Iori

	AC	MC	CP	SC	SM	RP	MF	AF	FI	DL	EL	CM	MD	GS	SB	DA	JD	$\overline{AC}$	GL	MR	II
AC			$\frac{1}{6}$				$\frac{1}{6}$			$\frac{1}{6}$				$\frac{1}{6}$			$\frac{1}{6}$	$\frac{1}{6}$			
MC				$\frac{1}{4}$									$\frac{1}{4}$			$\frac{1}{4}$					$\frac{1}{4}$
CP	$\frac{1}{2}$																		$\frac{1}{2}$		
SC	$\frac{1}{3}$	$\frac{1}{3}$											$\frac{1}{3}$								
SM	$\frac{1}{4}$													$\frac{1}{4}$			$\frac{1}{4}$	$\frac{1}{4}$			
RP											$\frac{1}{4}$				$\frac{1}{4}$				$\frac{1}{4}$	$\frac{1}{4}$	
MF	$\frac{1}{2}$		$\frac{1}{2}$																		
AF												1									
FI											$\frac{1}{2}$	$\frac{1}{2}$									
DL																					
EL						$\frac{1}{4}$			$\frac{1}{4}$			$\frac{1}{4}$							$\frac{1}{4}$		
CM								$\frac{1}{3}$	$\frac{1}{3}$		$\frac{1}{3}$										
MD		$\frac{1}{4}$		$\frac{1}{4}$															$\frac{1}{4}$		$\frac{1}{4}$
GS	$\frac{1}{4}$				$\frac{1}{4}$												$\frac{1}{4}$	$\frac{1}{4}$			
SB																					
DA		1																			
JD	$\frac{1}{6}$				$\frac{1}{6}$					$\frac{1}{6}$				$\frac{1}{6}$				$\frac{1}{6}$			$\frac{1}{6}$
$\overline{AC}$	$\frac{1}{4}$				$\frac{1}{4}$									$\frac{1}{4}$			$\frac{1}{4}$				
GL			$\frac{1}{7}$	$\frac{1}{7}$		$\frac{1}{7}$					$\frac{1}{7}$		$\frac{1}{7}$							$\frac{1}{7}$	$\frac{1}{7}$
MR				$\frac{1}{3}$		$\frac{1}{3}$												$\frac{1}{3}$			
II		$\frac{1}{5}$		$\frac{1}{5}$									$\frac{1}{5}$				$\frac{1}{5}$		$\frac{1}{5}$		

$1 = AC$ =Andrea Celidonio, $2 = MC$ =Maria Chiara Capuzzo, $3 = CP$ =Claudia Pallotta,
$4 = SC$ =Stefano Cipolla, $5 = SM$ =Sara Malacarne, $6 = RP$ =Roberta Piersimoni,
$7 = MF$ =Marcello Filosa, $8 = AF$ =Alessandra Fabrizi, $9 = FI$ =Federica Iacovissi,
$10 = DL$ =Diego Lopez, $11 = EL$ =Erika Leo, $12 = CM$ =Chiara Minotti
$13 = MD$ =Martina De Marchis, $14 = GS$ =Giulia Sambucini, $15 = SB$ =Sofia Basile,
$16 = DA$ =Davide Angelocola, $17 = JD$ =Jacopo De Cesaris, $18 = \overline{AC}$ =Alessandra Cataldo,
$19 = GL$ =Giorgia Lucci, $20 = MR$ =Maria Grazia Rositano, $21 = II$ =Isabella Iori

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$|B_{2k}(x)|$ in $[0,1]$ is dominated by $|B_{2k}(0)|$

It is easy to verify that for $j = 0$, for $j = 1$, and for all odd j, $j \geq 3$, the number $B_j(\frac{1}{2})$ satisfies the following identity:

$$B_j(\frac{1}{2}) = \left(\frac{1}{2^{j-1}} - 1\right) B_j(0). \qquad \text{(CP)}$$

As a matter of fact, the same identity holds also for $j = 2k$, $k = 1, 2, \ldots$. Thus, one has the inequality $|B_{2k}(\frac{1}{2})| = (1 - \frac{1}{2^{2k-1}})|B_{2k}(0)| < |B_{2k}(0)|$, and therefore the desired result: $|B_{2k}(x)|$ in $[0, 1]$ is dominated by $|B_{2k}(0)|$.

The proof is by induction: we assume the equality (CP) true for all $j \leq 2k-1$, and we prove it for $j = 2k$. We use the Taylor expansions of B_{2k} centered in 0 and in $\frac{1}{2}$.

First note that since $B_{2k}(x) = B_{2k}(1 - x)$ we have

$$0 = \int_0^1 B_{2k}(x)dx = 2\int_{\frac{1}{2}}^1 B_{2k}(x)dx,$$

and recall the derivation rule $B_n^{(k)}(x) = n(n-1)\cdots(n-k+1)B_{n-k}(x)$.

Thus, by integrating the identity

$$B_{2k}(x) = B_{2k}(\frac{1}{2}) + \sum_{j=1}^{2k} \frac{1}{j!} B_{2k}^{(j)}(\frac{1}{2})(x - \frac{1}{2})^j$$

from $\frac{1}{2}$ to 1, one obtains

$$\begin{aligned} 0 &= B_{2k}(\tfrac{1}{2}) + \textstyle\sum_{j=1}^{2k} \frac{1}{(j+1)!2^j} B_{2k}^{(j)}(\tfrac{1}{2}) \\ &= B_{2k}(\tfrac{1}{2}) + \textstyle\sum_{j=1}^{2k} \frac{2k(2k-1)\cdots(2k-j+1)}{(j+1)!2^j} B_{2k-j}(\tfrac{1}{2}) \end{aligned}$$

($\int_{\frac{1}{2}}^1 (x - \frac{1}{2})^j dx = \frac{1}{(j+1)2^{j+1}}$). Analogously, by integrating the identity

$$B_{2k}(x) = B_{2k}(0) + \sum_{j=1}^{2k} \frac{1}{j!} B_{2k}^{(j)}(0)x^j$$

first from 0 to 1 and then from 0 to $\frac{1}{2}$, we have, respectively,

$$\begin{aligned} 0 &= B_{2k}(0) + \textstyle\sum_{j=1}^{2k} \frac{1}{(j+1)!} B_{2k}^{(j)}(0) \\ &= B_{2k}(0) + \textstyle\sum_{j=1}^{2k} \frac{2k(2k-1)\cdots(2k-j+1)}{(j+1)!} B_{2k-j}(0), \end{aligned}$$

$$\begin{aligned} 0 &= B_{2k}(0) + \textstyle\sum_{j=1}^{2k} \frac{1}{(j+1)!2^j} B_{2k}^{(j)}(0) \\ &= B_{2k}(0) + \textstyle\sum_{j=1}^{2k} \frac{2k(2k-1)\cdots(2k-j+1)}{(j+1)!2^j} B_{2k-j}(0) \end{aligned}$$

($\int_0^1 x^j dx = \frac{1}{j+1}$ and $\int_0^{\frac{1}{2}} x^j dx = \frac{1}{(j+1)2^{j+1}}$).

Now assume (CP) true for $j \leq 2k-1$. Then

$$\begin{aligned} B_{2k}(\tfrac{1}{2}) &= -\textstyle\sum_{j=1}^{2k} \frac{2k(2k-1)\cdots(2k-j+1)}{(j+1)!2^j} B_{2k-j}(\tfrac{1}{2}) \\ &= -\textstyle\sum_{j=1}^{2k} \frac{2k(2k-1)\cdots(2k-j+1)}{(j+1)!2^j} (\frac{1}{2^{2k-j-1}} - 1) B_{2k-j}(0) \\ &= -\textstyle\sum_{j=1}^{2k} \frac{2k(2k-1)\cdots(2k-j+1)}{(j+1)!2^{2k-1}} B_{2k-j}(0) \\ &\quad + \textstyle\sum_{j=1}^{2k} \frac{2k(2k-1)\cdots(2k-j+1)}{(j+1)!2^j} B_{2k-j}(0) \\ &= \tfrac{1}{2^{2k-1}} \Big(-\textstyle\sum_{j=1}^{2k} \frac{2k(2k-1)\cdots(2k-j+1)}{(j+1)!} B_{2k-j}(0) \\ &\quad + \textstyle\sum_{j=1}^{2k} \frac{2k(2k-1)\cdots(2k-j+1)}{(j+1)!2^j} 2^{2k-1} B_{2k-j}(0) \Big) \\ &= \tfrac{1}{2^{2k-1}} (B_{2k}(0) - 2^{2k-1} B_{2k}(0)) = B_{2k}(0) \big(\tfrac{1}{2^{2k-1}} - 1 \big). \end{aligned}$$

That is, (CP) is true also for $j = 2k$.