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In these notes the concepts of circulants, $\tau$ and Toeplitz matrices, Hessenberg algebras and displacement decompositions, spaces of class $\mathbb{V}$ and best least squares fits on such spaces, are introduced and investigated. As a consequence of the results presented, the choice of matrices involved in displacement decompositions, the choice of preconditioners in solving linear systems and the choice of Hessian approximations in quasi-Newton minimization methods, become possible in wider classes of low complexity matrix algebras.

*The Fourier matrix, circulants, and fast discrete transforms*

Consider the following $n \times n$ matrix

$$P_1 = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & 0 \end{bmatrix}.$$

Let $\omega \in \mathbb{C}$. Note that

$$P_1 \begin{bmatrix} 1 \\ 1 \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ \\ 1 \end{bmatrix}, \; P_1 \begin{bmatrix} 1 \\ \omega \\ \\ \omega^{n-1} \end{bmatrix} = \begin{bmatrix} \omega \\ \omega^2 \\ \\ \omega^{n-1} \\ 1 \end{bmatrix} = \omega \begin{bmatrix} 1 \\ \omega \\ \\ \omega^{n-1} \end{bmatrix},$$

where the latter identity holds if $\omega^n = 1$. More in general, if $\omega^n = 1$, we have the following vectorial identities

$$P_1 \begin{bmatrix} 1 \\ \omega^j \\ \\ \omega^{(n-1)j} \end{bmatrix} = \begin{bmatrix} \omega^j \\ \\ \omega^{(n-1)j} \\ 1 \end{bmatrix} = \omega^j \begin{bmatrix} 1 \\ \omega^j \\ \\ \omega^{(n-1)j} \end{bmatrix}, \; j = 0, 1, \ldots, n-1,$$

or, equivalently, the following matrix identity

$$P_1 W = W D_{1\omega^{n-1}},$$

$$D_{1\omega^{n-1}} = \begin{bmatrix} 1 & & \\ & \omega & \\ & & \\ & & \omega^{n-1} \end{bmatrix}, \; W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^j & \omega^{n-1} \\ & & & \\ 1 & \omega^{n-1} & \omega^{(n-1)j} & \omega^{(n-1)(n-1)} \end{bmatrix}.$$

*Proposition.* If $\omega^n = 1$ and if $\omega^j \neq 1$ for $0 < j < n$, then $W^* W = nI$.

proof: since $|\omega| = 1$, $\overline{\omega} = \omega^{-1}$, we have

$$\begin{aligned} [W^*W]_{ij} &= [\overline{W}W]_{ij} = \textstyle\sum_{k=1}^n [\overline{W}]_{ik}[W]_{kj} = \sum_{k=1}^n \overline{\omega}^{(i-1)(k-1)} \omega^{(k-1)(j-1)} \\ &= \textstyle\sum_{k=1}^n \omega^{(k-1)(j-i)} = \sum_{k=1}^n (\omega^{j-i})^{k-1}. \end{aligned}$$

Thus $[W^*W]_{ij} = n$ if $i = j$, and $[W^*W]_{ij} = \frac{1-(\omega^{j-i})^n}{1-\omega^{j-i}} = 0$ if $i \neq j$ (note that the assumption $\omega^j \neq 1$ for $0 < j < n$ is essential in order to make $1 - \omega^{j-i} \neq 0$).

By the result of the above Proposition, we can say that the following (symmetric) Fourier matrix

$$F = \frac{1}{\sqrt{n}} W$$

is unitary, i.e. $F^*F = I$.

*Exercise.* Prove that $F^2 = JP_1$ where $J$ is the permutation matrix $J\mathbf{e}_k = \mathbf{e}_{n+1-k}$, $k = 1, \ldots, n$ ($J$ is usually called anti-identity).

The matrix identity satisfied by $P_1$ and $W$ can be of course rewritten in terms of $F$, $P_1 F = F D_{1\omega^{n-1}}$, thus we obtain the equality

$$P_1 = F D_{1\omega^{n-1}} F^*$$

which states that the Fourier matrix diagonalizes the matrix $P_1$, or, more precisely, that *the columns of the Fourier matrix form a system of $n$ unitarily orthonormal eigenvectors for the matrix $P_1$ with corresponding eigenvalues $1, \omega, \ldots, \omega^{n-1}$.*

But if $F$ diagonalizes $P_1$, then it diagonalizes all polynomials in $P_1$:

$$\begin{aligned}
P_1^{k-1} &= F D_{1\omega^{n-1}}^{k-1} F^*, \\
\textstyle\sum_{k=1}^n a_k P_1^{k-1} &= F \textstyle\sum_{k=1}^n a_k D_{1\omega^{n-1}}^{k-1} F^* \\
&= F \begin{bmatrix} \sum_{k=1}^n a_k & & \\ & \sum_{k=1}^n a_k \omega^{k-1} & \\ & & \ddots \\ & & & \sum_{k=1}^n a_k \omega^{(n-1)(k-1)} \end{bmatrix} F^* \\
&= F d(W\mathbf{a}) F^* = \sqrt{n} F d(F\mathbf{a}) F^*
\end{aligned}$$

where by $d(\mathbf{z})$ we mean the diagonal matrix whose diagonal entries are $z_1, z_2, \ldots, z_n$.

Let us investigate the matrices $P_1^{k-1}$, $k = 1, \ldots, n$, and the matrix $\sum_{k=1}^n a_k P_1^{k-1}$ in the case $n = 4$:

$$P_1^0 = I,\ P_1^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},\ P_1^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ P_1^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$P_1^4 = P_1^3 P_1 = P_1^T P_1 = I = P_1^0,$$

$$\sum_{k=1}^4 a_k P_1^{k-1} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_4 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_1 \end{bmatrix} = \sqrt{4} F d(F\mathbf{a}) F^*,\ F = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix},$$

$$\omega^4 = 1,\ \omega^j \neq 1,\ 0 < j < 4\ (\omega = e^{\pm \mathbf{i} 2\pi/4}).$$

Note that, for $n$ generic, we have the identities $\mathbf{e}_1^T P_1^{k-1} = \mathbf{e}_k^T$, $k = 1, \ldots, n$, and $P_1^n = I$ (prove them!). So, the set $C = \{p(P_1)\}$ of all polynomials in $P_1$

is spanned by the matrices $J_k = P_1^{k-1}$; the particular polynomial $\sum_{k=1}^n a_k J_k$ is simply denoted by $C(\mathbf{a})$. Note that $C(\mathbf{a})$ is the matrix of $C$ with first row $\mathbf{a}^T$:

$$C(\mathbf{a}) = \sum_{k=1}^n a_k J_k = \begin{bmatrix} a_1 & a_2 & & a_{n-1} & a_n \\ a_n & a_1 & & & a_{n-1} \\ & & & & \\ a_3 & & & & a_2 \\ a_2 & a_3 & & a_n & a_1 \end{bmatrix} = Fd(F^T\mathbf{a})d(F^T\mathbf{e}_1)^{-1}F^{-1}.$$

$C$ is known as the space of circulant matrices.

*Exercise.* (i) Repeat all, starting from the $n \times n$ matrix

$$P_{-1} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -1 & & & & 0 \end{bmatrix}$$

and arriving to the $(-1)$-circulant matrix whose first row is $\mathbf{a}^T$, $\mathbf{a} \in \mathbb{C}^n$:

$$C_{-1}(\mathbf{a}) = \begin{bmatrix} a_1 & a_2 & & a_{n-1} & a_n \\ -a_n & a_1 & & & a_{n-1} \\ & & & & \\ -a_3 & & & & a_2 \\ -a_2 & -a_3 & & -a_n & a_1 \end{bmatrix}.$$

(ii) Let $T$ be a Toeplitz $n \times n$ matrix, i.e. $T = (t_{i-j})_{i,j=1}^n$, for some $t_k \in \mathbb{C}$. Show that $T$ can be written as the sum of a circulant and of a $(-1)$-circulant, that is, $T = C(\mathbf{a}) + C_{-1}(\mathbf{b})$, $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$.

Why circulant matrices can be interesting in the applications of linear algebra? The main reason is in the fact that the matrix-vector product $C(\mathbf{a})\mathbf{z}$ can be computed in at most $O(n \log_2 n)$ arithmetic operations (whereas, usually, a matrix-vector product requires $n^2$ multiplications).

*Proposition* FFT. Given $\mathbf{z} \in \mathbb{C}^n$, the complexity of the matrix-vector product $F\mathbf{z}$ is at most $O(n \log_2 n)$. Such operation is called discrete Fourier transform (DFT) of $\mathbf{z}$. As a consequence, the matrix-vector product $C(\mathbf{a})\mathbf{z}$ is computable by two DFTs (after the preprocessing DFT $F\mathbf{a}$).

proof: since $\omega^{(i-1)(k-1)}$ is the $(i,k)$ entry of $W$ and $z_k$ is the $k$ entry of $\mathbf{z} \in \mathbb{C}^n$, we have

$$\begin{aligned} (W\mathbf{z})_i &= \textstyle\sum_{k=1}^n \omega^{(i-1)(k-1)} z_k = \sum_{j=1}^{n/2} \omega^{(i-1)(2j-2)} z_{2j-1} + \sum_{j=1}^{n/2} \omega^{(i-1)(2j-1)} z_{2j} \\ &= \textstyle\sum_{j=1}^{n/2} (\omega^2)^{(i-1)(j-1)} z_{2j-1} + \sum_{j=1}^{n/2} \omega^{(i-1)(2(j-1)+1)} z_{2j} \\ &= \textstyle\sum_{j=1}^{n/2} (\omega^2)^{(i-1)(j-1)} z_{2j-1} + \omega^{i-1} \sum_{j=1}^{n/2} (\omega^2)^{(i-1)(j-1)} z_{2j}. \end{aligned}$$

Note that $\omega$ is in fact a function of $n$, i.e. the right notation for $\omega$ should be $\omega_n$. Then $\omega^2 = \omega_n^2$ is such that $(\omega_n^2)^{n/2} = 1$ and $(\omega_n^2)^i \neq 1$ $0 < i < n/2$; in other words $\omega_n^2 = \omega_{n/2}$. So, we have the identities

$$(W_n\mathbf{z})_i = \sum_{j=1}^{n/2} \omega_{n/2}^{(i-1)(j-1)} z_{2j-1} + \omega_n^{i-1} \sum_{j=1}^{n/2} \omega_{n/2}^{(i-1)(j-1)} z_{2j}, \ i = 1, 2, \ldots, n. \quad (?)$$

It follows that, for $i = 1, \ldots, \frac{n}{2}$,

$$(W_n\mathbf{z})_i = (W_{n/2}\begin{bmatrix} z_1 \\ z_3 \\ \\ z_{n-1} \end{bmatrix})_i + \omega_n^{i-1}(W_{n/2}\begin{bmatrix} z_2 \\ z_4 \\ \\ z_n \end{bmatrix})_i.$$

Moreover, by setting $i = \frac{n}{2} + k$, $k = 1, \ldots, \frac{n}{2}$, in (?), we obtain

$$\begin{aligned}(W_n\mathbf{z})_{\frac{n}{2}+k} &= \textstyle\sum_{j=1}^{n/2} \omega_{n/2}^{\frac{n}{2}(j-1)} \omega_{n/2}^{(k-1)(j-1)} z_{2j-1} + \omega_n^{\frac{n}{2}} \omega_n^{k-1} \sum_{j=1}^{n/2} \omega_{n/2}^{\frac{n}{2}(j-1)} \omega_{n/2}^{(k-1)(j-1)} z_{2j} \\ &= \textstyle\sum_{j=1}^{n/2} \omega_{n/2}^{(k-1)(j-1)} z_{2j-1} - \omega_n^{k-1} \sum_{j=1}^{n/2} \omega_{n/2}^{(k-1)(j-1)} z_{2j} \\ &= (W_{n/2}\begin{bmatrix} z_1 \\ z_3 \\ \\ z_{n-1} \end{bmatrix})_k - \omega_n^{k-1}(W_{n/2}\begin{bmatrix} z_2 \\ z_4 \\ \\ z_n \end{bmatrix})_k, \; k = 1, \ldots, \tfrac{n}{2}.\end{aligned}$$

($\omega_n^{\frac{n}{2}} = -1$; think $\omega = e^{\pm \mathbf{i} 2\pi/n}$). Thus

$$W_n\mathbf{z} = \begin{bmatrix} I & D_{1\omega_n^{\frac{n}{2}-1}} \\ I & -D_{1\omega_n^{\frac{n}{2}-1}} \end{bmatrix} \begin{bmatrix} W_{n/2} & 0 \\ 0 & W_{n/2} \end{bmatrix} Q\mathbf{z},$$

$$D_{1\omega_n^{\frac{n}{2}-1}} = \begin{bmatrix} 1 & & \\ & \omega_n & \\ & & \\ & & \omega_n^{\frac{n}{2}-1} \end{bmatrix}, Q = \begin{bmatrix} 1 & & & & & \\ 0 & 0 & 1 & & & \\ & & & & & \\ & & & & 1 & 0 \\ 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & \\ & & & & & \\ & & & & 0 & 1 \end{bmatrix}. \qquad (??)$$

If $c_n$ denotes the complexity of the matrix-vector product $F_n\mathbf{z}$, then, by the previous formula,

$$c_n \leq 2c_{n/2} + rn, \; r \text{ constant}.$$

But this implies $c_n = O(n \log_2 n)$. The proof of the last assertion is left to the reader.

Of course, any time a $n \times n$ matrix $U$, well defined for all $n$, satisfies for $n$ even an identity of the type

$$U_n = \begin{bmatrix} \text{sparse matrix} \end{bmatrix} \begin{bmatrix} U_{n/2} & 0 \\ 0 & U_{n/2} \end{bmatrix} \begin{bmatrix} \text{permutation matrix} \end{bmatrix},$$

the matrix-vector product $U_n\mathbf{z}$ can be computed in at most $O(n \log_2 n)$ arithmetic operations. The above identity is verified for at least 10 matrices $U$, the Fourier transform and its $(-1)$ version, and the eight Hartley-type transforms. Note, however, that there are also other 16 discrete transforms of complexity $O(n \log_2 n)$, sine-type and the cosine-type transforms. See [],[].

*Exercise* G. Prove that the $n \times n$ matrix $G = G_n$ defined by

$$G_{ij} = \frac{1}{\sqrt{n}}(\cos \frac{(2i+1)(2j+1)\pi}{2n} + \sin \frac{(2i+1)(2j+1)\pi}{2n}), \; i,j = 0, \ldots, n-1,$$

is symmetric, persymmetric, real, unitary, and satisfies the identity:

$$G_n = \frac{1}{\sqrt{2}} \begin{bmatrix} R_+ & R_- \\ -R_-J & R_+J \end{bmatrix} \begin{bmatrix} G_{n/2} & 0 \\ 0 & G_{n/2} \end{bmatrix} Q,\ R_\pm = D_c \pm D_s J,$$

($J$ $\frac{n}{2}\times\frac{n}{2}$ anti-identity) for some suitable $\frac{n}{2}\times\frac{n}{2}$ diagonal matrices $D_c$, $D_s$. Prove, moreover, that each row of $G_n$ has at least a zero entry when $n = 2+4s$ (this is like to say, we will see, that the space $\{Gd(\mathbf{z})G : \mathbf{z} \in \mathbb{C}^n\}$ is not a $h$-space for such values of $n$); and that, instead, for all other $n$, $[G_n]_{1k} \neq 0\ \forall\, k$ (i.e., for all other $n$, the space $\{Gd(\mathbf{z})G : \mathbf{z} \in \mathbb{C}^n\}$ is a 1-space).

*Exercise.* Prove that the space $C_{-1}^S + JC_{-1}^{SK}$, $C_{-1}^S$ symmetric $n\times n$ $(-1)$-circulants, $C_{-1}^{SK}$ skewsymmetric $(-1)$-circulants, is a commutative matrix algebra (a matrix $A$ is skewsymmetric if $A^T = -A$).

*The sine matrix and the (commutative) algebra of $\tau$ matrices*

Consider the $n\times n$ matrix

$$P_0 + P_0^T = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ 0 & 1 & 0 & & \\ & & & & 1 \\ & & & 1 & 0 \end{bmatrix},$$

and set $J_1 = I$, and $J_2 = P_0 + P_0^T$. Note that $\mathbf{e}_1^T J_1 = \mathbf{e}_1^T$, $\mathbf{e}_1^T J_2 = \mathbf{e}_2^T$. Moreover, since

$$(P_0 + P_0^T)^2 = \begin{bmatrix} 1 & 0 & 1 & & & \\ 0 & 2 & 0 & 1 & & \\ 1 & 0 & 2 & & & \\ & 1 & & & & \\ & & & & & 1 \\ & & & & 2 & 0 \\ & & & & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & & & & \\ 0 & 1 & 0 & 1 & & & \\ 1 & 0 & 1 & & & & \\ & 1 & & & & & \\ & & & & & & 1 \\ & & & & & 1 & 0 \\ & & & & 1 & 0 & 0 \end{bmatrix} + I,$$

we have $\mathbf{e}_1^T((P_0+P_0^T)^2 - I) = [0\,0\,1\,0\cdots 0] = \mathbf{e}_3^T$. Set $J_3 = (P_0+P_0^T)^2 - I = J_2(P_0+P_0^T) - J_1$; then $\mathbf{e}_1^T J_3 = \mathbf{e}_3^T$.

More in general, set $J_{i+1} = J_i(P_0+P_0^T) - J_{i-1}$, $i = 2,3,\ldots,n-1$. The matrix $J_{i+1}$ is a polynomial in $P_0+P_0^T$ of degree $i$ with the property $\mathbf{e}_1^T J_{i+1} = \mathbf{e}_{i+1}^T$.

proof: assume $\mathbf{e}_1^T J_j = \mathbf{e}_j^T$, $j = 1,\ldots,i$; then

$$\mathbf{e}_1^T J_{i+1} = \mathbf{e}_1^T(J_i(P_0+P_0^T) - J_{i-1}) = (\mathbf{e}_i^T(P_0+P_0^T)) - \mathbf{e}_{i-1}^T = (\mathbf{e}_{i-1}^T + \mathbf{e}_{i+1}^T) - \mathbf{e}_{i-1}^T = \mathbf{e}_{i+1}^T.$$

Since $J_1, J_2, \ldots, J_n$ are linearly independent, we can say that they span the set $\{p(P_0+P_0^T)\}$ of all polynomials in the matrix $P_0+P_0^T$ (use the Cailey-Hamilton theorem). We call such set $\tau$. Note that the matrices of $\tau$ are determined once their first row is known; with the symbol $\tau(\mathbf{a})$ we denote the matrix of $\tau$ whose first row is $\mathbf{a}^T$, i.e. the matrix $\sum_k a_k J_k$.

Let us find a useful representation of $\tau$ and, in particular, of $\tau(\mathbf{a})$. First observe that the following vectorial equalities hold:

$$\begin{bmatrix} & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & & & \\ & & & & 1 \\ & & & 1 & \end{bmatrix} \begin{bmatrix} \sin\frac{j\pi}{n+1} \\ \sin\frac{2j\pi}{n+1} \\ \\ \sin\frac{nj\pi}{n+1} \end{bmatrix} = 2\cos\frac{j\pi}{n+1} \begin{bmatrix} \sin\frac{j\pi}{n+1} \\ \sin\frac{2j\pi}{n+1} \\ \\ \sin\frac{nj\pi}{n+1} \end{bmatrix}, \ j = 1, \dots, n.$$

Such $n$ equalities can be rewritten as a simple matrix identity $(P_0 + P_0^T)S = SD$ where $S$ is the matrix

$$S_{ij} = \sqrt{\frac{2}{n+1}} \sin\frac{ij\pi}{n+1}, \quad i, j = 1, \dots, n,$$

and $D$ is the diagonal matrix with diagonal entries $D_{jj} = 2\cos\frac{j\pi}{n+1}$. Note that the matrix $S$, called sine matrix, is real, symmetric and unitary (prove it!).

*Remark.* Let $F_{2(n+1)}$ be the Fourier matrix of order $2(n+1)$. Then the sine matrix $S$ satisfies the following relation:

$$\mathbf{i}\big(I - F_{2(n+1)}^2\big)F_{2(n+1)} = \begin{bmatrix} 0 & \mathbf{0}^T & 0 & \mathbf{0}^T \\ \mathbf{0} & S & \mathbf{0} & -SJ \\ 0 & \mathbf{0}^T & 0 & \mathbf{0}^T \\ \mathbf{0} & -JS & \mathbf{0} & JSJ \end{bmatrix}$$

(note that $F_{2(n+1)}^2$ is a permutation matrix). As a consequence, a sine transform can be computed by performing a discrete Fourier transform.

So, the columns of the sine matrix $S$ form a system of unitarily orthonormal eigenvectors for the matrix $P_0 + P_0^T$. In other words, the unitary matrix $S$ diagonalizes $P_0 + P_0^T$ and, of course, diagonalizes any polynomial in $P_0 + P_0^T$, i.e. any $\tau$ matrix:

$$P_0 + P_0^T = SDS, \quad (P_0 + P_0^T)^k = SD^kS,$$
$$\tau = \{p(P_0 + P_0^T)\} = \{\textstyle\sum_{k=1}^n a_k(P_0 + P_0^T)^{k-1} : a_k \in \mathbb{C}\} = \{Sd(\mathbf{z})S : \mathbf{z} \in \mathbb{C}^n\}.$$

In particular, it is clear that the matrix of $\tau$ with first row $\mathbf{a}^T$ is

$$\tau(\mathbf{a}) = \sum_{k=1}^n a_k J_k = Sd(S^T\mathbf{a})d(S^T\mathbf{e}_1)^{-1}S^{-1}.$$

The latter formula states that matrix-vector products involving $\tau$ matrices have complexity at most $O(n \log_2 n)$.

*Proposition* PC. Given $X \in \mathbb{C}^{n\times n}$ generic, we have

$$\{p(X)\} \subset \{A \in \mathbb{C}^{n\times n} : AX = XA\},$$
$$\dim(\{p(X)\}) \le n \le \dim\{A \in \mathbb{C}^{n\times n} : AX = XA\}$$

and if one equality holds then the other equality holds too. So, $X$ is non derogatory if and only if $\dim(\{p(X)\}) = n = \dim\{A \in \mathbb{C}^{n\times n} : AX = XA\}$.

The above Proposition suggests the following further representation of the space $\tau$:

$$\tau = \{A \in \mathbb{C}^{n\times n} : A(P_0 + P_0^T) = (P_0 + P_0^T)A\}.$$

The fact that any matrix of $\tau$ must commute with $P_0 + P_0^T$ is equivalent to require that the following $n^2$ *cross-sum* conditions hold:

$$a_{i,j-1} + a_{i,j+1} = a_{i-1,j} + a_{i+1,j}, \ \ i,j = 1,\dots,n$$

where we have set $a_{0,j} = a_{n+1,j} = a_{i,0} = a_{i,n+1} = 0$, $i,j = 1,\dots,n$. We can use such conditions in order to write down the generic $\tau$ matrix whose first row is $[a_1\, a_2\, \cdots\, a_n]$. For example, for $n = 4$, we can say that

$$\tau(\mathbf{a}) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1+a_3 & a_2+a_4 & a_3 \\ a_3 & a_2+a_4 & a_1+a_3 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix}, \ J_1 = I, \ J_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$J_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ J_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ J_2 - J_4 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix},$$

and so on.

*Exercise*. Prove that for $n$ even the matrix $J_2$ is invertible, and, if possible, compute the inverse.

solution: We know that if $J_2$ is invertible, then $J_2^{-1} \in \tau$ (from the fact that $J_2$ commutes with $P_0 + P_0^T$ it follows that also $J_2^{-1}$ commutes with $P_0 + P_0^T$!). Thus $J_2^{-1} = \tau(\mathbf{z})$ for some $\mathbf{z} \in \mathbb{C}^n$. Note that the matrix identity $\tau(\mathbf{z})J_2 = I$ is equivalent to the vectorial identity $\mathbf{z}^T J_2 = \mathbf{e}_1^T$. So, for example, for $n = 4$ we have the condition

$$[z_1\ z_2\ z_3\ z_4] \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [1\ 0\ 0\ 0],$$

which yields $z_1 = 0$, $z_2 = 1$, $z_3 = 0$, $z_4 = -1$, and thus

$$J_2^{-1} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} = J_3 - J_4.$$

The proof for $n = 6, 8, \dots$ is left to the reader.

*Exercise* Ttau. Let $T$ be a symmetric Toeplitz $n \times n$ matrix, i.e. $T = (t_{|i-j|})_{i,j=1}^n$, for some $t_k \in \mathbb{C}$. Show that $T = A + B$ where $A$ is a $\tau$ matrix of order $n$ and

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ R \in \tau, \ R \in \mathbb{C}^{(n-2)\times(n-2)}.$$

*Exercise*. Write down rank one $\tau$ matrices (try first $n = 2$, $n = 3$, $n = 4$, $n = 5$, $n = 6$, $\dots$)

*Hessenberg algebras*

Let $X$ be a lower Hessenberg $3 \times 3$ matrix

$$X = \begin{bmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

We now show that the space of all polynomials in $X$ is spanned by three matrices $J_1, J_2, J_3$ such that $\mathbf{e}_1^T J_k = \mathbf{e}_k^T$, $k = 1, 2, 3$, provided that $X_{ii+1} \neq 0$, $i = 1, 2$.

As $J_1$ we take the identity, $J_1 = X^0 = I$. Let us define $J_2$:

$$X - a_{11}I = \begin{bmatrix} 0 & b_1 & 0 \\ a_{21} & a_{22} - a_{11} & b_2 \\ a_{31} & a_{32} & a_{33} - a_{11} \end{bmatrix},$$

$b_1 \neq 0 \Rightarrow$

$$J_2 = \frac{1}{b_1}(X - a_{11}I) = \begin{bmatrix} 0 & 1 & 0 \\ a_{21}/b_1 & (a_{22} - a_{11})/b_1 & b_2/b_1 \\ a_{31}/b_1 & a_{32}/b_1 & (a_{33} - a_{11})/b_1 \end{bmatrix}.$$

Note that $\mathbf{e}_1^T J_2 = \mathbf{e}_2^T$. Then, let us define $J_3$:

$$J_2^2 = \begin{bmatrix} a_{22}/b_1 & (a_{22} - a_{11})/b_1 & b_2/b_1 \\ & & \\ & & \end{bmatrix},$$

$b_2 \neq 0 \Rightarrow$

$$J_3 = \frac{b_1}{b_2}(J_2^2 - \frac{a_{22}}{b_1}I - \frac{a_{22} - a_{11}}{b_1}J_2).$$

Note that $\mathbf{e}_1^T J_3 = \mathbf{e}_3^T$. Finally, Cailey-Hamilton theorem yields the thesis, $\{p(X)\} = \text{Span}\,\{J_1, J_2, J_3\}$.

The following Proposition generalizes to a generic $n$ the above remarks. For a detailed proof see [].

*Proposition.* Let $X$ be a lower Hessenberg $n \times n$ matrix. Then the space $H_X$ of all polynomials in $X$

$$H_X = \{\sum_{k=1}^{n} \alpha_k X^{k-1} : \alpha_k \in \mathbb{C}\}$$

is spanned by $n$ matrices $J_1, \ldots, J_n$ such that $\mathbf{e}_1^T J_k = \mathbf{e}_k^T$, $k = 1, \ldots, n$, provided that $X_{ii+1} \neq 0$, $i = 1, \ldots, n-1$; in such case $H_X$ is called Hessenberg algebra, and for any $\mathbf{a} \in \mathbb{C}^n$ there is a unique matrix in $H_X$ with first row $\mathbf{a}^T$ which is denoted by $H_X(\mathbf{a})$, i.e. $H_X(\mathbf{a}) = \sum_k a_k J_k$.

Of course, by Proposition PC, any Hessenberg algebra $H_X$ admits also the following representation

$$H_X = \{A \in \mathbb{C}^{n \times n} : AX = XA\}.$$

Until now we have seen two examples of Hessenberg algebras, $\xi$-circulants $\xi \neq 0$ ($X = P_\xi$) and tau matrices ($X = P_0 + P_0^T$). Both can be simultaneously diagonalized by a suitable matrix. An example of Hessenberg algebra whose matrices cannot be simultaneously diagonalized is $H_{P_0}$, the space of all upper triangular Toeplitz matrices. The matrix $H_{P_0}(\mathbf{a})$ is displayed here below

$$H_{P_0}(\mathbf{a}) = \begin{bmatrix} a_1 & a_2 & & & a_n \\ & a_1 & a_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & a_2 \\ & & & & a_1 \end{bmatrix}.$$

Even the matrix $P_0$, i.e. the matrix generating the space, is not diagonalizable. (We shall see, however, that $H_{P_0}$ can be embedded in the space of $2n \times 2n$ circulants, which are diagonalizable).

Note that when $X = P_0$ or, more in general, when $X = P_\xi$, the matrices $J_k$ are simply the powers of $X$, i.e. $J_k = X^{k-1}$. For example, for $n = 3$

$$J_1 = I,\ J_2 = X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \xi & 0 & 0 \end{bmatrix},\ J_3 = X^2 = \begin{bmatrix} 0 & 0 & 1 \\ \xi & 0 & 0 \\ 0 & \xi & 0 \end{bmatrix}.$$

Hessenberg algebras make up a subclass of commutative matrix algebras of the class of 1-spaces defined here below

*Definition.* $\mathcal{L} \subset \mathbb{C}^{n\times n}$ is said to be a 1-space if $\mathcal{L} = \text{Span}\,\{J_1, \ldots, n\}$ with $J_k$ such that $\mathbf{e}_1^T J_k = \mathbf{e}_k^T$, $k = 1, \ldots, n$.

An example of 1-space $\mathcal{L}$ which is not a Hessenberg algebra is the following

$$\mathcal{L} = \{\begin{bmatrix} A & JB \\ JB & A \end{bmatrix} :\ A, B\ \frac{n}{2} \times \frac{n}{2} \text{ circulants}\}.$$

One can easily prove that $\mathcal{L}$ is a non commutative matrix algebra. An example of 1-space which is not a matrix algebra is the set of all $n \times n$ symmetric Toeplitz matrices (see the next section).

*Toeplitz linear systems and displacement decompositions*

A $n \times n$ Toeplitz matrix is a matrix of the form $T = (t_{i-j})_{i,j=1}^n$. In applications often one has to solve Toeplitz linear systems $T\mathbf{x} = \mathbf{b}$.

For example, here below is a $3 \times 3$ Toeplitz matrix:

$$T = \begin{bmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{bmatrix}.$$

An example of Toeplitz matrix $T$ is the coefficient matrix of the linear system arising when solving, by finite differences or by finite elements, the boundary value differential problem $-u" = f$, $u(a) = \alpha$, $u(b) = \beta$. In such case $T$ is symmetric and its first row is $[2\ -1\ 0 \cdots 0]$. Another example, important is applied probability, is $T = (t^{|i-j|})_{i,j=1}^n$ with $|t|$ ($t \in \mathbb{C}$) less than 1.

*Exercise.* The vector space of all $n \times n$ *symmetric* Toeplitz matrices is a 1-space, being $(t_{|i-j|})_{i,j=1}^n$ equal to the sum $\sum_{k=1}^n t_{k-1} J_k$ where the $J_k$ are the symmetric

Toeplitz matrices with first row $\mathbf{e}_k^T$. Prove that such 1-space is not a matrix algebra.

In the framework of displacement theory it is possible to obtain some decompositions of $T^{-1}$ involving matrices from Hessenberg algebras (or from more general commutative $h$-spaces) of the type

$$T^{-1} = \sum_{k=1}^{\alpha} M_k N_k, \ \ 2 \le \alpha \le 4.$$

Usually, the matrices $M_k$ and $N_k$, appearing in such formulas, can be multiplied by a vector in $O(n \log_2 n)$ arithmetic operations; thus fast direct solvers of Toeplitz linear system naturally arise. Here below there is one example of such formulas:

$$T^{-1} = L_1 U_1 + L_2 U_2. \qquad (GS)$$

The $L_j$ and $U_j$ are suitable lower and upper triangular Toeplitz matrices, i.e. elements or transposed of elements from the Hessenberg algebra $H_{P_0}$.

*Remark.* By the Gohberg-Semencul formula (GS), if the $L_j$ and $U_j$ are known (a way to obtain them is indicated in []), then the matrix-vector product $T^{-1}\mathbf{b}$ can be computed in at most $O(n \log_2 n)$ arithmetic operations. That is, assuming preprocessing on $T$, the complexity of the problem of solving any Toeplitz system $T\mathbf{x} = \mathbf{b}$ is at most $O(n \log_2 n)$.

proof: it is enough to prove that any Toeplitz matrix (in particular the triangular ones) can be multiplied by a vector by means of a finite number of discrete Fourier transforms. The latter result is immediate if we observe that any $n \times n$ Toeplitx matrix can be embedded into a $(2n+k) \times (2n+k)$, $k \ge 0$, circulant matrix; for example, if $n = 3$ we have

$$T = \begin{bmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{bmatrix}, \ C = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & & t_2 & t_1 \\ t_1 & t_0 & t_{-1} & t_{-2} & & t_2 \\ t_2 & t_1 & t_0 & t_{-1} & t_{-2} & \\ & t_2 & t_1 & t_0 & t_{-1} & t_{-2} \\ t_{-2} & & t_2 & t_1 & t_0 & t_{-1} \\ t_{-1} & t_{-2} & & t_2 & t_1 & t_0 \end{bmatrix}$$

$(k = 0)$. It is clear that the vector $T \cdot \mathbf{z}$ is the first part of the vector $C \begin{bmatrix} \mathbf{z} \\ \mathbf{0} \end{bmatrix}$.

The above representation (GS) for the inverse of a Toeplitz matrix, which can be very useful in order to solve Toeplitz linear systems efficiently [], follows from the *displacement decomposition* formula stated in the following theorem

*Theorem* DD. Let $X$ be a lower Hessenberg $n \times n$ matrix. Assume that $X_{ij} = X_{i+1,j+1}$, $i, j = 1, \ldots, n-1$ ($X$ has Toeplitz structure), and that $b = X_{ii+1} \ne 0$, and consider the (commutative) Hessenberg algebra $H_X$ generated by $X$ (note that $H_X$ is a 1-space).

Assume that $A \in \mathbb{C}^{n \times n}$ is such that $AX - XA = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$. Then

$$bA = -\sum_{m=1}^{\alpha} H_{P_0}(\tilde{\mathbf{x}}_m)^T H_X(\mathbf{y}_m) + bH_X(A^T \mathbf{e}_1) \qquad (DD)$$

where, for $\mathbf{z} \in \mathbb{C}^n$, $H_{P_0}(\mathbf{z})$ is the upper triangular Toeplitz matrix with first row $\mathbf{z}^T$, $H_X(\mathbf{z})$ is the matrix of $H_X$ with first row $\mathbf{z}^T$, and $\tilde{\mathbf{z}}$ is the vector $[0\, z_1 \cdots z_{n-1}]^T$.

Note: Besides DD several other displacement decompositions hold, which can be general like DD, i.e. representing generic matrices $A$, or specialized for centrosymmetric $A$ (see []). Such decompositions yield formulas for the inverses of Toeplitz, Toeplitz plus Hankel, and Toeplitz plus Hankel-like matrices useful in order to solve Toeplitz plus Hankel-like linear systems. Recall that a Hankel matrix is nothing else a matrix of the form $JT$ where $T$ is Toeplitz (the well known Hilbert matrix is an example of Hankel matrix).

In order to prove Theorem DD, the following Lemma is fundamental.

*Lemma* []. Let $\mathcal{L}$ be a commutative 1-space of $n \times n$ matrices, i.e. $\mathcal{L} = \{\sum_k \alpha_k J_k : \alpha_k \in \mathbb{C}\}$, with $J_k \in \mathbb{C}^{n\times n}$ such that $\mathbf{e}_1^T J_k = \mathbf{e}_k^T$, $J_k J_s = J_s J_k$. Let $X$ be an element of $\mathcal{L}$, and assume that $A \in \mathbb{C}^{n\times n}$ is such that $AX - XA = \mathbf{x}\mathbf{y}^T$. Then $\mathbf{x}^T \mathcal{L}(\mathbf{y})^T = \mathbf{0}^T$.

proof: note that the equality $J_k J_s = J_s J_k$ implies $\mathbf{e}_1^T J_k J_s = \mathbf{e}_1^T J_s J_k$, $\mathbf{e}_k^T J_s = \mathbf{e}_s^T J_k$ $\forall\, s, k$, thus

$$\begin{aligned}
\mathbf{x}^T \mathcal{L}(\mathbf{y})^T \mathbf{e}_r &= \mathbf{x}^T (\textstyle\sum_k y_k J_k)^T \mathbf{e}_r = \mathbf{x}^T \sum_k y_k J_k^T \mathbf{e}_r \\
&= \mathbf{x}^T \textstyle\sum_k y_k J_r^T \mathbf{e}_k = \mathbf{x}^T J_r^T \sum_k y_k \mathbf{e}_k \\
&= \mathbf{x}^T J_r^T \mathbf{y} = \textstyle\sum_{i,j} x_i y_j [J_r^T]_{ij} \\
&= \textstyle\sum_{i,j} x_i y_j [J_r]_{ji} = \sum_{i,j} [AX - XA]_{ij} [J_r]_{ji} \\
&= \textstyle\sum_i [(AX - XA) J_r]_{ii} = \sum_i [(AJ_r)X - X(AJ_r)]_{ii} \\
&= \operatorname{tr}((AJ_r)X) - \operatorname{tr}(X(AJ_r)) = 0
\end{aligned}$$

(recall that the two matrices $MN$ and $NM$, $M, N \in \mathbb{C}^{n\times n}$, have the same characteristic polynomial, even if (in case $\det(M) = \det(N) = 0$) $MN$ and $NM$ might be not similar each other).

We now report a draft of the proof of Theorem DD (for a more detailed proof see []). In order to obtain the equality (DD), which is of the type

$$bA = E + bH_X(A^T \mathbf{e}_1),$$

it is enough to prove that

$$EX - XE = (bA)X - X(bA), \qquad (***)$$

and to observe that the first row of $E$ is null. In fact, the above equality implies $(bA - E)X - X(bA - E) = 0$, and thus $bA - E \in H_X$. The Lemma, applied for $\mathcal{L} = H_X$, is fundamental in proving (***).

*A matrix algebra which is not a* 1*-space: the class of spaces in* $\mathbb{V}$

Remember that a matrix $A$ is said symmetric if $A^T = A$ ($a_{ji} = a_{ij}$), skewsymmetric if $A^T = -A$ ($a_{ji} = -a_{ij}$) and persymmetric if $A^T = JAJ$ ($a_{ji} = a_{n+1-i,n+1-j}$).

Consider a $6 \times 6$ symmetric $(-1)$-circulant matrix $A \in C_{-1}^{S}$,

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & 0 & -a_2 & -a_1 \\ a_1 & a_0 & a_1 & a_2 & 0 & -a_2 \\ a_2 & a_1 & a_0 & a_1 & a_2 & 0 \\ 0 & a_2 & a_1 & a_0 & a_1 & a_2 \\ -a_2 & 0 & a_2 & a_1 & a_0 & a_1 \\ -a_1 & -a_2 & 0 & a_2 & a_1 & a_0 \end{bmatrix},$$

a $6 \times 6$ skewsymmetric $(-1)$-circulant matrix $B \in C_{-1}^{SK}$, and the matrix $JB$,

$$B = \begin{bmatrix} 0 & b_1 & b_2 & b_3 & b_2 & b_1 \\ -b_1 & 0 & b_1 & b_2 & b_3 & b_2 \\ -b_2 & -b_1 & 0 & b_1 & b_2 & b_3 \\ -b_3 & -b_2 & -b_1 & 0 & b_1 & b_2 \\ -b_2 & -b_3 & -b_2 & -b_1 & 0 & b_1 \\ -b_1 & -b_2 & -b_3 & -b_2 & -b_1 & 0 \end{bmatrix}, \; JB = \begin{bmatrix} -b_1 & -b_2 & -b_3 & -b_2 & -b_1 & 0 \\ -b_2 & -b_3 & -b_2 & -b_1 & 0 & b_1 \\ -b_3 & -b_2 & -b_1 & 0 & b_1 & b_2 \\ -b_2 & -b_1 & 0 & b_1 & b_2 & b_3 \\ -b_1 & 0 & b_1 & b_2 & b_3 & b_2 \\ 0 & b_1 & b_2 & b_3 & b_2 & b_1 \end{bmatrix}.$$

The vector space $\gamma$ of all matrices of the type $A + JB$ has dimension equal to 6 (recall that $\dim(A + JB) = \dim(A) + \dim(JB) - \dim(A \cap JB)$).

We now show that there is not a basis $\{J_k\}$ for $\gamma$ such that $\mathbf{e}_1^T J_k = \mathbf{e}_k^T$, $k = 1, 2, 3, 4, 5, 6$, i.e. $\gamma$ is not a 1-space. Note that

$$\mathbf{e}_1^T (A + JB) = [(a_0 - b_1)(a_1 - b_2)(a_2 - b_3)(-b_2)(-a_2 - b_1)(-a_1)],$$

so, the equality $\mathbf{e}_1^T (A + JB) = \mathbf{e}_2^T$ is satisfied if and only if

$$a_0 - b_1 = 0,\; a_1 - b_2 = 1,\; a_2 - b_3 = 0,\; -b_2 = 0,\; -a_2 - b_1 = 0,\; -a_1 = 0.$$

Since we have both the conditions $b_2 = 0$ and $b_2 = -1$, a matrix $J_2 \in \gamma$ such that $\mathbf{e}_1^T J_2 = \mathbf{e}_2^T$ cannot exist.

However, there exists a basis $\{J_k\}$ of $\gamma$ satisfying the equalities $(\mathbf{e}_1 + \mathbf{e}_6)^T J_k = \mathbf{e}_k^T$, $k = 1, 2, 3, 4, 5, 6$. For example, a matrix $J_2 \in \gamma$ with the property $(\mathbf{e}_1 + \mathbf{e}_6)^T J_2 = \mathbf{e}_2^T$ is obtained as follows. Note that

$$\mathbf{e}_6^T (A + JB) = [(-a_1)(-a_2 + b_1)(b_2)(a_2 + b_3)(a_1 + b_2)(a_0 + b_1)],$$

so, for the sum of the first and of the sixth rows of $A + JB$, we obtain the formula

$$\begin{aligned} &(\mathbf{e}_1 + \mathbf{e}_6)^T (A + JB) \\ &= [(a_0 - b_1 - a_1)(a_1 - b_2 - a_2 + b_1)(a_2 - b_3 + b_2)(-b_2 + a_2 + b_3)(-a_2 - b_1 + a_1 + b_2)(-a_1 + a_0 + b_1)]. \end{aligned}$$

Thus the condition $(\mathbf{e}_1 + \mathbf{e}_6)^T (A + JB) = \mathbf{e}_2^T$ is satisfied if and only if the following system of equations has solution

$$\begin{aligned} &a_0 - b_1 - a_1 = 0, \quad a_1 - b_2 - a_2 + b_1 = 1, \\ &a_2 - b_3 + b_2 = 0, \quad -b_2 + a_2 + b_3 = 0, \\ &-a_2 - b_1 + a_1 + b_2 = 0, \quad -a_1 + a_0 + b_1 = 0, \end{aligned}$$

and such system has the unique solution $a_0 = a_1 = \frac{1}{2}$, $a_2 = 0$, $b_2 = b_3 = -\frac{1}{2}$,

$b_1 = 0$. The matrix $J_2 \in \gamma$ such that $(\mathbf{e}_1 + \mathbf{e}_6)^T J_2 = \mathbf{e}_2^T$ is displayed here below:

$$J_2 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Analogously, one obtains the other $J_k$ such that $(\mathbf{e}_1 + \mathbf{e}_6)^T J_k = \mathbf{e}_k^T$:

$$J_1 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

($a_1 = a_2 = 0$, $a_0 = \frac{1}{2}$, $b_2 = b_3 = -\frac{1}{2}$, $b_1 = -\frac{1}{2}$), ($a_0 = a_1 = a_2 = \frac{1}{2}$, $b_1 = b_2 = 0$, $b_3 = -\frac{1}{2}$),

$$J_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad J_5 = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix},$$

($a_0 = a_1 = a_2 = \frac{1}{2}$, $b_1 = b_2 = 0$, $b_3 = \frac{1}{2}$), ($a_0 = a_1 = \frac{1}{2}$, $a_2 = 0$, $b_2 = b_3 = \frac{1}{2}$, $b_1 = 0$),

$$J_6 = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

($a_1 = a_2 = 0$, $a_0 = \frac{1}{2}$, $b_1 = b_2 = b_3 = \frac{1}{2}$).

Of course $\gamma = \text{Span}\,\{J_1, J_2, J_3, J_4, J_5, J_6\}$ (the $J_k$ are linearly independent!). The matrix $\sum_k a_k J_k$ is denoted by $\gamma(\mathbf{a})$. Note that $(\mathbf{e}_1 + \mathbf{e}_6)^T \gamma(\mathbf{a}) = \mathbf{a}^T$, so $\gamma(\mathbf{a})$ is called the matrix of $\gamma$ whose $(\mathbf{e}_1 + \mathbf{e}_6)$-row is $\mathbf{a}^T$.

More in general, one can easily prove that the set $\gamma = C^S_{-1} + JC^{SK}_{-1}$, $C^S_{-1}$ $n \times n$ symmetric $(-1)$-circulants, $C^{SK}_{-1}$ $n \times n$ skewsymmetric $(-1)$-circulants, is a vector space of dimension $n$, and is a commutative matrix algebra. Moreover, it is a 1-space if and only if $n$ is one of the integers $\{3, 4, 5, 7, 8, 9, 11, 12, 13, 15, \ldots\}$; for the remaining values of $n$, i.e. for $n = 2 + 4s$ $s \in \mathbb{Z}$, no row of a matrix $A + JB$ of $\gamma$ determines $A + JB$, that is, there is no index $h$ for which there exists a basis $\{J_k\}$ of $\gamma$ with the property $\mathbf{e}_h^T J_k = \mathbf{e}_k^T$. Instead, for all $n$ the sum of the first and of the $n$th row of $A + JB$ determines $A + JB$, i.e. there exists a basis $\{J_k\}$ of $\gamma$ with the property $(\mathbf{e}_1 + \mathbf{e}_n)^T J_k = \mathbf{e}_k^T$.

The matrices of $\gamma$ can be simultaneously diagonalized by a fast discrete transform. More precisely, for any value of $n$ the following equality holds:

$$\gamma = \{Gd(\mathbf{z})G^{-1} : \mathbf{z} \in \mathbb{C}^n\},$$
$$G_{ij} = \frac{1}{\sqrt{n}}\Big(\cos\frac{(2i-1)(2j-1)\pi}{2n} + \sin\frac{(2i-1)(2j-1)\pi}{2n}\Big),\ i,j = 1,\dots,n$$

(see []). Note that the matrix $G$ is real, symmetric, persymmetric and unitary. The fact that the matrix-vector product $G\mathbf{z}$ can be computed in at most $O(n\log_2 n)$ arithmetic operations follows from the representation of $G_n$, $G_n := G$, stated in Exercise G.

Note that for $n = 6$ the matrix $G$ has ten zeros among its entries, and that these zeros are positioned as follows:

$$G = \begin{bmatrix} & & & & 0 & \\ & 0 & & 0 & & 0 \\ & & & & 0 & \\ & 0 & & & & \\ 0 & & 0 & & 0 & \\ & 0 & & & & \end{bmatrix}.$$

Thus each of the vectors $G^T\mathbf{e}_k$, $k = 1,\dots,6$, has at least one zero entry, i.e. the matrix $d(G^T\mathbf{e}_k)^{-1}$ is never well defined. The latter assertion is yet true whenever $n = 2 + 4s$ $s \in \mathbb{Z}$. In other words, $\gamma$ can be represented as

$$\gamma = \{Gd(G^T\mathbf{z})d(G^T\mathbf{e}_k)^{-1}G^{-1} : \mathbf{z} \in \mathbb{C}^n\}.$$

if and only $n \neq 2 + 4s$ $s \in \mathbb{Z}$ (for such $n$ one can choose $k = 1$).

However, as one may guess from the above discussion (detailed, for $n = 6$), it can be easily shown that $[G^T(\mathbf{e}_1 + \mathbf{e}_n)]_k \neq 0$ $\forall\, k$ and $\forall\, n$. So, we have the following representation for $\gamma$

$$\gamma = \{Gd(G^T\mathbf{z})d(G^T(\mathbf{e}_1 + \mathbf{e}_n))^{-1}G^{-1} : \mathbf{z} \in \mathbb{C}^n\}$$

valid for all $n$ (also for $n = 2+4s$ where $d(G^T\mathbf{e}_k)$ are not invertible). The latter formula confirms the fact that the matrices of $\gamma$ are uniquely defined by the sum of their 1st and $n$th rows; in particular, since the sum of the first and of the $n$th row of the matrix $Gd(G^T\mathbf{a})d(G^T(\mathbf{e}_1 + \mathbf{e}_n))^{-1}G^{-1}$ is equal to $\mathbf{a}^T$, we can say that such matrix is exactly the matrix $\gamma(\mathbf{a})$ (already defined above for $n = 6$), i.e. the matrix of $\gamma$ whose $(\mathbf{e}_1 + \mathbf{e}_n)$-row is $\mathbf{a}^T$.

It is now natural to introduce a class of spaces which include (besides the 1-spaces, like the Hessenberg algebras) also spaces like the algebra $\gamma$.

*Definition.* A subset $\mathcal{L}$ of $\mathbb{C}^{n\times n}$ is said to be a space in $\mathbb{V}$, if $\mathcal{L} = \text{Span}\,\{J_1,\dots,J_n\}$ with $\mathbf{v}^TJ_k = \mathbf{e}_k^T$ for some $\mathbf{v} \in \mathbb{C}^n$. Given $\mathbf{z} \in \mathbb{C}^n$, the matrix $\sum_k z_kJ_k \in \mathcal{L}$ is denoted by $\mathcal{L}(\mathbf{z})$. Since $\mathbf{v}^T\mathcal{L}(\mathbf{z}) = \mathbf{z}^T$, $\mathcal{L}(\mathbf{z})$ is called the matrix of $\mathcal{L}$ whose $\mathbf{v}$-row is $\mathbf{z}^T$.

*Example.* $\mathcal{L} = sd\,M = \{Md(\mathbf{z})M^{-1} : \mathbf{z} \in \mathbb{C}^n\}$ is in $\mathbb{V}$ since

$$\mathcal{L} = \text{Span}\,\{J_1,\dots,J_n\},\ J_k = Md(M^T\mathbf{e}_k)d(M^T\mathbf{v})^{-1}M^{-1},$$

for any vector $\mathbf{v}$ such that $[M^T\mathbf{v}]_i \neq 0$ $\forall\, i$, and

$$\mathbf{v}^TJ_k = \mathbf{v}^TMd(M^T\mathbf{e}_k)d(M^T\mathbf{v})^{-1}M^{-1} = \mathbf{e}_k^TMd(M^T\mathbf{v})d(M^T\mathbf{v})^{-1}M^{-1} = \mathbf{e}_k^T.$$

Note that $\mathcal{L}(\mathbf{z}) = Md(M^T\mathbf{z})d(M^T\mathbf{v})^{-1}M^{-1}$.

A matrix $X \in \mathbb{C}^{n\times n}$ is said to be non derogatory if the condition $p(X) = 0$, $p$ polynomial, implies $\partial p \geq n$. Note that by the Cailey-Hamilton theorem the characteristic polynomial of $X$ is null in $X$. So, $X$ is non derogatory if and only if the set $\{p(X)\}$ of all polynomials in $X$ has dimension $n$. In [] it is stated the following result, which proves that $\mathbb{V}$ is a wide class of spaces of matrices.

*Theorem* ND. Let $X$ be a $n \times n$ matrix with complex entries. Then $X$ is non derogatory if and only if $\{p(X)\} := \{p(X) : p \text{ polynomials}\}$ is in $\mathbb{V}$.

The Proposition here below collects several properties of the spaces in $\mathbb{V}$. They will be used (in the next section) in order to prove important properties of the best least squares fit in $\mathcal{L}$ of a matrix $A$, holding for all spaces $\mathcal{L}$ of a particular subclass of $\mathbb{V}$.

*Proposition* $\mathbb{V}$ *(properties of spaces in* $\mathbb{V}$*).* Let $\mathcal{L}$ be a space in $\mathbb{V}$, i.e. $\mathcal{L} = \text{Span}\,\{J_1, \dots, J_n\}$ with $\mathbf{v}^T J_k = \mathbf{e}_k^T$ for some $\mathbf{v} \in \mathbb{C}^n$.

(1) If $X \in \mathcal{L}$ and $\mathbf{v}^T X = \mathbf{0}^T$, then $X = 0$, thus $\mathbf{v}^T X = \mathbf{v}^T Y$, $X, Y \in \mathcal{L}$, implies $X = Y$

proof: $\mathbf{0}^T = \mathbf{v}^T X = \mathbf{v}^T \sum_k \alpha_k J_k = \sum_k \alpha_k \mathbf{e}_k^T = [\alpha_1 \ \cdots \ \alpha_n] \Rightarrow \alpha_k = 0 \ \forall\, k$.

(2) If $J_i X \in \mathcal{L}$, $X \in \mathbb{C}^{n\times n}$, then $J_i X = \sum_k [X]_{ik} J_k$

proof: there exist $\alpha_k$ such that $J_i X = \sum_k \alpha_k J_k$; multiplying the latter identity by $\mathbf{v}^T$ we have

$$\mathbf{e}_i^T X = \mathbf{v}^T J_i X = \sum_k \alpha_k \mathbf{e}_k^T = [\alpha_1 \ \cdots \ \alpha_n]$$

which implies $\alpha_k = [X]_{ik}$.

(3) Let $P_k \in \mathbb{C}^n$ be defined by $\mathbf{e}_s^T P_k = \mathbf{e}_k^T J_s$ (note that $\mathbf{e}_k^T = \mathbf{v}^T J_k = \sum_i v_i \mathbf{e}_i^T J_k = \sum_i v_i \mathbf{e}_k^T P_i = \mathbf{e}_k^T \sum_i v_i P_i$, and thus $\sum v_k P_k = I$). Then the following assertions are equivalent:

(i) $\mathcal{L}$ is closed under matrix multiplication

(ii) $J_i J_j = \sum_k [J_j]_{ik} J_k \ \forall\, i, j$

(iii) $P_r J_j = J_j P_r \ \forall\, r, j$

(iv) $P_k P_r = \sum_i [P_r]_{ki} P_i$

proof: The implication (i) $\Rightarrow$ (ii) follows from (2) for $X = J_j$. The opposite implication is obvious. The fact that conditions (ii) and (iii) are equivalent follows by taking the $(r, s)$ entry of the equality in (ii):

$$\mathbf{e}_i^T P_r J_j \mathbf{e}_s = \mathbf{e}_r^T J_i J_j \mathbf{e}_s = \sum_k [J_j]_{ik} [J_k]_{rs} = \sum_k [J_j]_{ik} [P_r]_{ks} = [J_j P_r]_{is}.$$

The fact that conditions (iii) and (iv) are equivalent follows from the identities:

$$[P_k P_r]_{ms} = [J_m P_r]_{ks} = [P_r J_m]_{ks} = [J_k J_m]_{rs} = \sum_i [J_k]_{ri} [J_m]_{is} = \sum_i [P_r]_{ki} [P_i]_{ms}.$$

(3.5) If $I \in \mathcal{L}$, then $\sum_i v_i J_i = I$ and $\mathbf{v}^T P_k = \mathbf{e}_k^T$, i.e. also the space $\text{Span}\,\{P_1, \dots, P_n\}$ is in $\mathbb{V}$ (with the same $\mathbf{v}$)

proof: both $I$ and $\sum_i v_i J_i$ have $\mathbf{v}^T$ as $\mathbf{v}$-row, and both, by assumption, are in $\mathcal{L}$, so they must be equal; moreover, we have

$$\mathbf{v}^T P_k = \sum_i v_i \mathbf{e}_i^T P_k = \sum_i v_i \mathbf{e}_k^T J_i = \mathbf{e}_k^T \sum_i v_i J_i = \mathbf{e}_k^T.$$

(4) If $\mathcal{L}$ is closed under matrix multiplication, then

$$\mathcal{L}(\mathcal{L}(\mathbf{z})^T \mathbf{z}') = \mathcal{L}(\mathbf{z}')\mathcal{L}(\mathbf{z}),\ \forall\, \mathbf{z}, \mathbf{z}' \in \mathbb{C}^n$$

proof: since $\mathcal{L}$ is closed, the matrix $\mathcal{L}(\mathbf{z}')\mathcal{L}(\mathbf{z})$ is in $\mathcal{L}$; moreover, its $\mathbf{v}$-row is $\mathbf{z}'^T \mathcal{L}(\mathbf{z})$; the thesis follows from the fact that also $\mathcal{L}(\mathcal{L}(\mathbf{z})^T \mathbf{z}')$ is the matrix of $\mathcal{L}$ whose $\mathbf{v}$-row is $\mathbf{z}'^T \mathcal{L}(\mathbf{z})$.

(5) Assume $I \in \mathcal{L}$ and $\mathcal{L}$ closed under matrix multiplication. Then $X \in \mathcal{L}$ is non singular if and only if $\exists\, \mathbf{z} \in \mathbb{C}^n$ such that $\mathbf{z}^T X = \mathbf{v}^T$; in this case $X^{-1} = \mathcal{L}(\mathbf{z})$

proof: by inspecting the $k$th row of the matrix $\mathcal{L}(\mathbf{z})X$, and applying properties (3) and (3.5), we obtain the identities

$$\mathbf{e}_k^T \mathcal{L}(\mathbf{z})X = \mathbf{e}_k^T \sum_s z_s J_s X = \sum_s z_s \mathbf{e}_s^T P_k X = \mathbf{z}^T P_k X = \mathbf{z}^T X P_k = \mathbf{v}^T P_k = \mathbf{e}_k^T,$$

or, equivalently, the equality $\mathcal{L}(\mathbf{z})X = I$, which implies that $X$ is non singular and $X^{-1} = \mathcal{L}(\mathbf{z})$.

*The best least squares fit to $A$ in $\mathcal{L} \subset \mathbb{C}^{n\times n}$*

Given a subspace $\mathcal{L} \subset \mathbb{C}^{n\times n}$ and a $n \times n$ matrix $A$, it is well defined $\mathcal{L}_A$, the projection on $\mathcal{L}$ of $A$. In the following theorem we state some assumptions on $\mathcal{L}$ (in particular we consider $n$-dimensional subspaces of $\mathbb{C}^{n\times n}$) which assure that $\mathcal{L}_A$ is hermitian whenever $A$ is, and that the eigenvalues of $\mathcal{L}_A$ are bounded by those of $A$. Such assumptions imply that $\mathcal{L} \in \mathbb{V}$.

*Theorem $\mathcal{L}_A$*. Assumptions:

$\mathcal{L} \subset \mathbb{C}^{n\times n}$, $I \in \mathcal{L}$, $\mathcal{L} =$ Span$\{J_1, \dots, J_n\}$ with $J_k$ such that

$$J_i^H J_j = \sum_{k=1}^{n} \overline{[J_k]_{ij}} J_k,\ i, j = 1, \dots, n. \tag{$*$}$$

$A \in \mathbb{C}^{n\times n}$.

$\mathcal{L}_A \in \mathcal{L}$, $\|A - \mathcal{L}_A\|_F \le \|A - X\|_F$, $\forall\, X \in \mathcal{L}$ (such matrix $\mathcal{L}_A$ is well defined since $\mathbb{C}^{n\times n}$ is a Hilbert space with respect the norm $\|\cdot\|_F$ induced by the inner product $(A, B)_F = \sum_{ij} \overline{a_{ij}} b_{ij}$ and $\mathcal{L}$ is a subspace of $\mathbb{C}^{n\times n}$).

Thesis: If $A = A^H$, then $\mathcal{L}_A = \mathcal{L}_A^H$ and $\min \lambda(A) \le \lambda(\mathcal{L}_A) \le \max \lambda(A)$.

Note: if $A$ is real symmetric, then $\mathcal{L}_A$ is in general hermitian; it is real symmetric under the further condition that $\mathcal{L}$ is spanned by real matrices (prove it!).

Note: we shall see that the hypotheses of Theorem $\mathcal{L}_A$ are satisfied by spaces of the type $\{M d(\mathbf{z}) M^{-1} : \mathbf{z} \in \mathbb{C}^n\}$ if $M^H M$ is diagonal and its diagonal entries are positive; however, the same hypotheses can be satisfied also by non commutative spaces (we shall see an example, for others see []), so also in the latter cases we can say that the conclusions of Theorem $\mathcal{L}_A$ hold.

*Applications of Theorem $\mathcal{L}_A$.* If the conditions of Theorem $\mathcal{L}_A$ are satisfied, then $\mathcal{L}_A$ is positive definite (i.e. $\mathcal{L}_A = \mathcal{L}_A^H$ and $\mathbf{z} \in \mathbb{C}^n$ $\mathbf{z} \neq \mathbf{0} \Rightarrow \mathbf{z}^H \mathcal{L}_A \mathbf{z}$ positive) whenever $A$ is positive definite. So, in order to solve the linear system

$$A\mathbf{x} = \mathbf{b}, \; A \text{ positive definite}$$

we can solve the equivalent system

$$\mathcal{L}_A^{-1} A\mathbf{x} = \mathcal{L}_A^{-1}\mathbf{b}$$

whose coefficient matrix has real and positive eigenvalues, often better distributed than those of $A$ (this results, for example when solving $A\mathbf{x} = \mathbf{b}$ iteratively, in less iterations).

Moreover, if the conditions of Theorem $\mathcal{L}_A$ are satisfied and $\mathcal{L}$ is spanned by real matrices, then then $\mathcal{L}_A$ is real positive definite (i.e. $\mathcal{L}_A = \mathcal{L}_A^T$, $\mathcal{L}_A \in \mathbb{R}^{n\times n}$, and $\mathbf{z} \in \mathbb{C}^n$ $\mathbf{z} \neq \mathbf{0} \Rightarrow \mathbf{z}^H \mathcal{L}_A \mathbf{z}$ positive) whenever $A$ is real positive definite. That is, the following implications hold

$$\begin{array}{l} B_k \text{ real positive definite } \Rightarrow \\ \mathcal{L}_{B_k} \text{ real positive definite } \Rightarrow \\ \varphi(\mathcal{L}_{B_k}, \mathbf{s}_k, \mathbf{y}_k) \text{ real positive definite} \Rightarrow \\ \mathcal{L}_{\varphi(\mathcal{L}_{B_k}, \mathbf{s}_k, \mathbf{y}_k)} \text{ real positive definite} \end{array}$$

(provided $\mathbf{s}_k^T \mathbf{y}_k$ is positive), and thus both the S and NS LQN search directions,

$$\mathbf{d}_{k+1} = -\varphi(\mathcal{L}_{B_k}, \mathbf{s}_k, \mathbf{y}_k)^{-1} \nabla f(\mathbf{x}_{k+1}) \;\text{ and }\; \mathbf{d}_{k+1} = -\mathcal{L}^{-1}_{\varphi(\mathcal{L}_{B_k}, \mathbf{s}_k, \mathbf{y}_k)} \nabla f(\mathbf{x}_{k+1}),$$

are well defined descent directions in $\mathbf{x}_{k+1}$ for the function $f : \mathbb{R}^n \to \mathbb{R}$ (see [] for the definitions of $B_k$, $\mathbf{s}_k$, $\mathbf{y}_k$, $\varphi$, S and NS LQN).

proof (of Theorem $\mathcal{L}_A$): The matrix $\mathcal{L}_A = \sum_s \alpha_s J_s$ is uniquely defined by the following condition

$$(X, A - \mathcal{L}_A)_F = 0, \; X \in \mathcal{L}$$

or, equivalently, by the $n$ conditions $(J_k, A - \sum_s \alpha_s J_s)_F = 0$, $k = 1, \dots, n$, which can be rewritten as follows

$$\sum_{s=1}^{n} (J_k, J_s)_F \alpha_s = (J_k, A)_F, \; k = 1, \dots, n.$$

In other words we have the formula

$$\mathcal{L}_A = \sum_s [B^{-1}\mathbf{c}]_s J_s, \; B_{ks} = (J_k, J_s)_F, \; c_k = (J_k, A)_F, \; k, s = 1, \dots, n.$$

*Remark.* $B$ is positive definite, i.e. $B = B^*$ and $\mathbf{z}^H B \mathbf{z} > 0 \; \forall \mathbf{z} \in \mathbb{C}^n \; \mathbf{z} \neq \mathbf{0}$.

proof: $\overline{B_{ks}} = \overline{(J_k, J_s)_F} = (J_s, J_k)_F = B_{sk}$, that is, $B$ is a hermitian matrix. Moreover, since $0 < (\sum_s z_s J_s, \sum_s z_s J_s)_F = \sum_{k,s} \overline{z_k} z_s (J_k, J_s)_F = \mathbf{z}^H B \mathbf{z}$ whenever $\mathbf{z} \neq \mathbf{0}$, the matrix $B$ is also positive definite.

*Remark.* Let $v_k \in \mathbb{C}$ be such that $I = \sum_k v_k J_k$ (such $v_k$ exist because $I \in \mathcal{L}$). Then the vector $\mathbf{v}$ whose entries are the $v_k$ satisfies the equalities $\mathbf{v}^T J_k = \mathbf{e}_k^T$, thus $\mathcal{L} \in \mathbb{V}$ and all results stated for spaces in $\mathbb{V}$ hold for our space $\mathcal{L}$.

proof: Multiply (*) by $\overline{v_i}$ and sum on $i$:

$$\overline{v_i} J_i^H J_j = \sum_k \overline{v_i} \overline{[J_k]_{ij}} J_k, \;\; J_j = \sum_k (\sum_i \overline{v_i} \overline{[J_k]_{ij}}) J_k = \sum_k (\mathbf{v}^H \overline{J_k} \mathbf{e}_j) J_k.$$

This implies $\mathbf{v}^H \overline{J_k} \mathbf{e}_j = 0$ if $k \neq j$ and $\mathbf{v}^H \overline{J_k} \mathbf{e}_j = 1$ if $k = j$, i.e. $\mathbf{v}^T J_k = \mathbf{e}_k^T$.

As an immediate consequence of the above two Remarks, we have that $\mathcal{L}_A$ is the matrix of $\mathcal{L}$ whose $\mathbf{v}$-row is $(B^{-1}\mathbf{c})^T$,

$$\mathcal{L}_A = \sum_s [B^{-1}\mathbf{c}]_s J_s = \mathcal{L}(B^{-1}\mathbf{c}),$$

and, moreover,

$$\mathcal{L}_A = \mathcal{L}(B^{-1}\mathbf{c}) = \mathcal{L}((B^H)^{-1}\mathbf{c}) = \mathcal{L}((\overline{B}^{-1})^T \mathbf{c}).$$

*Remark.* $\mathcal{L}$ is closed under conjugate transposition.
proof: multiply (*) by $v_j$ and sum on $j$:

$$J_i^H v_j J_j = \sum_k v_j \overline{[J_k]_{ij}} J_k, \;\; J_i^H = \sum_k (\sum_j v_j \overline{[J_k]_{ij}}) J_k \;\; \Rightarrow \;\; J_i^H \in \mathcal{L}.$$

The latter Remark yields part of the thesis of Theorem $\mathcal{L}_A$, because it implies that $\mathcal{L}_A^H \in \mathcal{L}$, and this fact together with the equalities

$$\|A - \mathcal{L}_A\|_F = \|A^H - \mathcal{L}_A^H\|_F = \|A - \mathcal{L}_A^H\|_F$$

(remember that our $A$ is hermitian!) and the unicity of the best approximation of $A$, yield the identity $\mathcal{L}_A = \mathcal{L}_A^H$. In other words, under our conditions on $\mathcal{L}$ the projection on $\mathcal{L}$ of a hermitian matrix is hermitian too.
*Remark.* $\mathcal{L}$ is closed under matrix multiplication ($\mathcal{L}$ is a matrix algebra).
proof: the set $\{J_i^H\}$ forms an alternative basis for $\mathcal{L}$ (prove it!), thus there exist $z_i^{(s)} \in \mathbb{C}$ such that $J_s = \sum_i z_i^{(s)} J_i^H$. Multiply (*) by $z_i^{(s)}$ and sum on $i$,

$$z_i^{(s)} J_i^H J_j = \sum_k z_i^{(s)} \overline{[J_k]_{ij}} J_k, \;\; J_s J_j = \sum_k (\sum_i z_i^{(s)} \overline{[J_k]_{ij}}) J_k,$$

to observe that $J_s J_j \in \mathcal{L}$.
*Remark.* $\overline{B} = \sum_k \overline{\mathrm{tr}\,(J_k)} J_k$, thus $\overline{B} \in \mathcal{L}$, and, since $\overline{B}$ is non singular (it is positive definite!), by the result $\mathbb{V}$ (5) also the matrix $\overline{B}^{-1}$ is in $\mathcal{L}$.
proof: by equality (*) we have:

$$\begin{aligned} B_{ij} &= (J_i, J_j)_F = \textstyle\sum_{r,t} \overline{[J_i]_{rt}} [J_j]_{rt} = \sum_{r,t} [J_i^H]_{tr} [J_j]_{rt} \\ &= \textstyle\sum_t [J_i^H J_j]_{tt} = \sum_t \sum_k \overline{[J_k]_{ij}} [J_k]_{tt} = \sum_k \mathrm{tr}\,(J_k) \overline{[J_k]_{ij}}. \end{aligned}$$

The latter two Remarks, together with $\mathbb{V}$ (4), let us rewrite again $\mathcal{L}_A$ as follows

$$\mathcal{L}_A = \ldots = \mathcal{L}((\overline{B}^{-1})^T \mathbf{c}) = \mathcal{L}(\mathbf{c}) \overline{B}^{-1}.$$

Now note that there exists a hermitian matrix $M$ such that $M^2 = \overline{B}^{-1}$, and that the matrices $\mathcal{L}_A$ and $M\mathcal{L}(\mathbf{c})M$ have the same eigenvalues (by the last representation of $\mathcal{L}_A$ they are similar!). So, if $\lambda(\mathcal{L}_A)$ is the generic eigenvalue of $\mathcal{L}_A$, then there exists $\mathbf{x} \in \mathbb{C}^n$ $\|\mathbf{x}\|_2 = 1$ such that

$$\lambda(\mathcal{L}_A) = \mathbf{x}^H M \mathcal{L}(\mathbf{c}) M\mathbf{x} = (M\mathbf{x})^H \mathcal{L}(\mathbf{c})(M\mathbf{x}).$$

*Remark.* If $\mathbf{z} \in \mathbb{C}^n$, then $\mathbf{z}^H \mathcal{L}(\mathbf{c})\mathbf{z} = \sum_k (P_k^H \mathbf{z})^H A (P_k^H \mathbf{z})$.
proof: here again equality (*) is fundamental:

$$\begin{aligned}
\mathbf{z}^H \mathcal{L}(\mathbf{c})\mathbf{z} &= \mathbf{z}^H (\textstyle\sum_k (J_k, A)_F J_k)\mathbf{z} \\
&= \textstyle\sum_{i,j=1}^n \overline{z_i} z_j \sum_{r,t=1}^n a_{rt} \overline{\sum_k [J_k]_{ij} \overline{[J_k]_{rt}}} \\
&= \textstyle\sum_{i,j=1}^n \overline{z_i} z_j \sum_{r,t=1}^n a_{rt} \overline{[J_i^H J_j]_{rt}} \\
&= \textstyle\sum_{r,t=1}^n a_{rt} \sum_{i,j=1}^n \overline{z_i} z_j \sum_k \overline{[J_i^H]_{rk} [J_j]_{kt}} \\
&= \textstyle\sum_k \sum_{r,t} a_{rt} \overline{(\sum_i z_i [P_k^H]_{ri})} (\sum_j z_j [P_k^H]_{tj}) \\
&= \textstyle\sum_k \sum_{r,t} a_{rt} \overline{(P_k^H \mathbf{z})_r} (P_k^H \mathbf{z})_t.
\end{aligned}$$

By the above Remark we have:

$$\begin{aligned}
\lambda(\mathcal{L}_A) &= \textstyle\sum_k (P_k^H M\mathbf{x})^H A (P_k^H M\mathbf{x}) \le \max \lambda(A) \sum_k (P_k^H M\mathbf{x})^H (P_k^H M\mathbf{x}) \\
&= \max \lambda(A) \mathbf{x}^H M (\textstyle\sum_k P_k P_k^H) M\mathbf{x}.
\end{aligned}$$

But the matrix in the brackets is nothing else the matrix $\overline{B}$:

*Remark.* $\overline{B} = \sum_k P_k P_k^H$
proof:

$$B_{ij} = (J_i, J_j)_F = \sum_{r,s} \overline{[J_i]_{rs}} [J_j]_{rs} = \sum_{r,s} \overline{[P_r]_{is}} [P_r]_{js} = \sum_{r,s} \overline{[P_r]_{is}} [P_r^T]_{sj} = \sum_r [\overline{P_r} P_r^T]_{ij}.$$

We can now conclude one of the inequalities (stated in Theorem $\mathcal{L}_A$) satisfied by the eigenvalues of $A$ and $\mathcal{L}_A$:

$$\lambda(\mathcal{L}_A) \le \max \lambda(A) \mathbf{x}^H M M^{-2} M\mathbf{x} = \max \lambda(A) \mathbf{x}^H \mathbf{x} = \max \lambda(A).$$

Analogously, one can prove that $\lambda(\mathcal{L}_A) \ge \min \lambda(A)$. $\square$

*Exercise* DG. Prove that the *dihedral group* space

$$\mathcal{L} = \{ \begin{bmatrix} X & JY \\ JY & X \end{bmatrix} : X, Y \ \frac{n}{2} \times \frac{n}{2} \text{ circulants} \}$$

satisfies the hypothesis of Theorem $\mathcal{L}_A$, i.e. $I \in \mathcal{L}$, $\mathcal{L} = \text{Span}\{J_1, \ldots, J_n\}$ with $J_k$ linearly independent such that

$$J_i^H J_j = \sum_{k=1}^n \overline{[J_k]_{ij}} J_k, \ i, j = 1, \ldots, n.$$

Thus, the projection $\mathcal{L}_A$ on $\mathcal{L}$ is hermitian and such that $\min \lambda(A) \le \lambda(\mathcal{L}_A) \le \max \lambda(A)$ whenever $A$ is hermitian. Note that $\mathcal{L}$ is not commutative.

*Proposition* c$\mathbb{V}$ *(properties of commutative spaces in* $\mathbb{V}$*)* [mitia]. Let $\mathcal{L}$ be a space in $\mathbb{V}$, i.e. $\mathcal{L} = \text{Span}\{J_1, \ldots, J_n\}$ with $\mathbf{v}^T J_k = \mathbf{e}_k^T$ for some $\mathbf{v} \in \mathbb{C}^n$. Assume that $\mathcal{L}$ is commutative. Then

(1) $\mathbf{e}_i^T J_j = \mathbf{e}_j^T J_i$, $\forall\, i,j$, and thus $J_k = P_k$

proof: $J_i J_j = J_j J_i \Rightarrow \mathbf{v}^T J_i J_j = \mathbf{v}^T J_j J_i$, and the definition $\mathbf{v}^T J_k = \mathbf{e}_k^T$ yields the thesis.

(2) $\mathbf{z}^T \mathcal{L}(\mathbf{z}') = \mathbf{z}'^T \mathcal{L}(\mathbf{z})$

proof: by (1) we have

$$\mathbf{z}^T \mathcal{L}(\mathbf{z}') = \sum_i z_i \mathbf{e}_i^T \sum_k z'_k J_k = \sum_i z_i \sum_k z'_k \mathbf{e}_k^T J_i = \sum_k z'_k \mathbf{e}_k^T \sum_i z_i J_i.$$

(3) $I = \mathcal{L}(\mathbf{v}) \in \mathcal{L}$

proof: note that $\mathcal{L}(\mathbf{v}) = \sum v_i J_i \in \mathcal{L}$ and $\mathbf{e}_k^T \mathcal{L}(\mathbf{v}) = \sum_i v_i \mathbf{e}_i^T J_k = \mathbf{v}^T J_k = \mathbf{e}_k^T$, so $I = \mathcal{L}(\mathbf{v}) \in \mathcal{L}$.

(4) $\mathcal{L}$ is closed under matrix multiplication

proof: from (1) we have that $J_k = P_k$, thus $P_k J_s = J_k J_s = J_s J_k = J_s P_k$, which is one of the necessary and sufficient conditions for the multiplicative closure.

*Example of commutative* $\mathcal{L} \in \mathbb{V}$. Let $M$ be a non singular $n \times n$ matrix with complex entries, and set $\mathcal{L} = sd\,M = \{M d(\mathbf{z}) M^{-1} : \mathbf{z} \in \mathbb{C}^n\}$. Note that $\mathcal{L} \in \mathbb{V}$, in fact if $\mathbf{v}$ is any vector such that $[M^T \mathbf{v}]_k \neq 0$, $\forall\, k$, then the matrices $J_k = M d(M^T \mathbf{e}_k) d(M^T \mathbf{v})^{-1} M^{-1}$ satisfy the identities $\mathbf{v}^T J_k = \mathbf{e}_k^T$ and span $\mathcal{L}$. We have, for $\mathcal{L}$, the following alternative representation:

$$\mathcal{L} = \{M d(M^T \mathbf{z}) d(M^T \mathbf{v})^{-1} M^{-1} : \mathbf{z} \in \mathbb{C}^n\}.$$

It is clear that the matrix of $\mathcal{L}$ whose $\mathbf{v}$-row is $\mathbf{z}^T$ is

$$\mathcal{L}(\mathbf{z}) = M d(M^T \mathbf{z}) d(M^T \mathbf{v})^{-1} M^{-1}.$$

Obviously, $\mathcal{L}$ is commutative.

*Proposition* ch$\mathbb{V}$ *(properties of commutative, closed under conjugate transposition spaces in* $\mathbb{V}$*)* [stefano]. Let $\mathcal{L}$ be a space in $\mathbb{V}$, i.e. $\mathcal{L} = \text{Span}\,\{J_1, \ldots, J_n\}$ with $\mathbf{v}^T J_k = \mathbf{e}_k^T$ for some $\mathbf{v} \in \mathbb{C}^n$. Assume that $\mathcal{L}$ is commutative and closed under conjugate transposition. Then, besides the above c$\mathbb{V}$ (1),(2),(3),(4), we have

$$J_i^H J_j = \sum_k \overline{[J_k]_{ij}} J_k, \;\; i,j = 1, \ldots, n$$

proof: since $J_i^H \in \mathcal{L}$ and $\mathcal{L}$ is commutative, one has $J_i^H J_j = J_j J_i^H$; since $\mathcal{L}$ is closed under matrix multiplication and $J_i^H \in \mathcal{L}$, one has that $J_j J_i^H \in \mathcal{L}$; by $\mathbb{V}$ (2), it follows that

$$J_i^H J_j = J_j J_i^H = \sum_k [J_i^H]_{jk} J_k = \sum_k \overline{[J_i]_{kj}} J_k = \sum_k \overline{[J_k]_{ij}} J_k,$$

where in the latter identity we have used property c$\mathbb{V}$(1).

*Example of commutative, closed under conjugate transposition* $\mathcal{L} \in \mathbb{V}$. Let $M$ be a non singular $n \times n$ matrix with complex entries, and set $\mathcal{L} = sd\,M = \{M d(\mathbf{z}) M^{-1} : \mathbf{z} \in \mathbb{C}^n\}$. We already know that $\mathcal{L}$ is a space in $\mathbb{V}$ which is commutative. We want to prove that $\mathcal{L}$ *is closed under conjugate transposition if and only if* $M^H M$ *is diagonal with positive diagonal entries.* Moreover, in such case there is a diagonal matrix $d(\mathbf{w})$ $\mathbf{w} \in \mathbb{C}^n$ such that $\tilde{M} = Md(\mathbf{w})$ is unitary and $\mathcal{L} = \{\tilde{M}d(\mathbf{z})\tilde{M}^{-1} : \mathbf{z} \in \mathbb{C}^n\}$.

One implication is easy: assume $M^H M = D$, $D_{ii}$ positive $\forall\, i$, $D_{ij} = 0$ $i \neq j$; then

$$(Md(\mathbf{z})M^{-1})^H = (M^H)^{-1}d(\overline{\mathbf{z}})M^H = MD^{-1}d(\overline{\mathbf{z}})DM^{-1} = Md(\overline{\mathbf{z}})M^{-1} \ \in \ \mathcal{L}.$$

Now assume that $\mathcal{L}$ is closed under conjugate transposition. Thus, for any $\mathbf{z} \in \mathbb{C}^n$ there exists $\mathbf{w} \in \mathbb{C}^n$ such that $(Md(\mathbf{z})M^{-1})^H = Md(\mathbf{w})M^{-1}$. But this implies

$$d(\overline{\mathbf{z}})C = Cd(\mathbf{w}),\ C = M^H M,$$

or, equivalently, $c_{ij}\overline{z}_i = c_{ij}w_j$. Assume the $z_i$ distinct. Then the identities $c_{i1}\overline{z}_i = c_{i1}w_1$, $i = 1, \ldots, n$, imply $c_{i1} = 0$ for all $i$ except one of them, say $i_1$ (otherwise, $c_{i1} \neq 0$ and $c_{k1} \neq 0$, $i \neq k$, would imply $w_1 = \overline{z}_i = \overline{z}_k$!). Analogously, the identities $c_{i2}\overline{z}_i = c_{i2}w_2$, $i = 1, \ldots, n$, imply $c_{i2} = 0$ for all $i$ except one of them, say $i_2$, and such $i_2$ must be different from $i_1$ otherwise $C$ would be singular. Proceeding in this way one concludes that

$$C = DR,\ D \text{ diagonal},\ R \text{ permutation}.$$

The fact that $C$ is hermitian implies that the permutation matrix $R$ must be symmetric. The fact that the diagonal entries of $C$ cannot be zero implies that $R$ is the identity. Finally, the fact that $C$ is positive definite implies that $D_{ii}$ are real and positive. Let us prove the last assertion. For $\tilde{M} = Md(\mathbf{w})$ we have

$$\tilde{M}^H\tilde{M} = d(\overline{\mathbf{w}})M^H Md(\mathbf{w}) = d(\overline{\mathbf{w}})Dd(\mathbf{w}).$$

Choose $|w_i| = 1/\sqrt{D_{ii}}$; then $\tilde{M}^H\tilde{M} = I$ and, for any $\mathbf{z} \in \mathbb{C}^n$, $Md(\mathbf{z})M^{-1} = Md(\mathbf{w})d(\mathbf{z})d(\mathbf{w})^{-1}M^{-1} = \tilde{M}d(\mathbf{z})\tilde{M}^{-1} = \tilde{M}d(\mathbf{z})\tilde{M}^H$.

*Question.* Find an example of $\mathcal{L}$ satisfying the assumptions of Proposition ch$\mathbb{V}$ which is not of the type $\{Md(\mathbf{z})M^{-1} : \mathbf{z} \in \mathbb{C}^n\}$, $M^H M$ diagonal with positive diagonal entries.

*Remark.* If $\mathcal{L} = \{Ud(\mathbf{z})U^* : \mathbf{z} \in \mathbb{C}^n\}$ where $U$ is a unitary matrix, then the thesis of Theorem $\mathcal{L}_A$ can be proved very simply as follows. Since $\forall\, \mathbf{z} \in \mathbb{C}^n$

$$\|A - Ud(\mathbf{z})U^*\|_F = \|U^*AU - d(\mathbf{z})\|_F,$$

it is clear that $\|A - Ud(\mathbf{z})U^*\|_F$ is minimum for $z_i = (U^*AU)_{ii}$. So, the following formula for $\mathcal{L}_A$ holds

$$\mathcal{L}_A = U\,\mathrm{diag}\,((U^*AU)_{ii})U^*$$

from which it immediately follows the assertion: $A = A^* \Rightarrow \mathcal{L}_A$ hermitian and $\min \lambda(A) \leq \lambda(\mathcal{L}_A) \leq \max \lambda(A)$. As an application compute $\mathcal{L}_{\mathbf{x}\mathbf{y}^T}$, $\mathbf{x}$, $\mathbf{y} \in \mathbb{C}^n$.

*Other remarks/exercises on spaces in* $\mathbb{V}$

*Exercise.* Prove that there are (besides $I$) infinite matrices in the algebra $\gamma$ whose first row is $\mathbf{e}_1^T$.

*Exercise.* Consider the matrix in Exercise G. Prove that the sum of its first and last rows is a vector with all entries nonzero (i.e. $\gamma$ is a space in $\mathbb{V}$ with $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_n$)

proof: for $j = 1, \dots, n$ we have

$$\begin{aligned}
[G]_{1j} + [G]_{nj} &= (\cos\tfrac{(2j-1)\pi}{2n} + \sin\tfrac{(2j-1)\pi}{2n}) \\
&\quad +(\cos\tfrac{(2n-1)(2j-1)\pi}{2n} + \sin\tfrac{(2n-1)(2j-1)\pi}{2n}) \\
&= (\cos\tfrac{(2j-1)\pi}{2n} + \sin\tfrac{(2j-1)\pi}{2n}) \\
&\quad +(\cos((2j-1)\pi - \tfrac{(2j-1)\pi}{2n}) + \sin((2j-1)\pi - \tfrac{(2j-1)\pi}{2n})) \\
&= \cos\tfrac{(2j-1)\pi}{2n} + \sin\tfrac{(2j-1)\pi}{2n} \\
&\quad - \cos\tfrac{(2j-1)\pi}{2n} + \sin\tfrac{(2j-1)\pi}{2n} = 2\sin\tfrac{(2j-1)\pi}{2n} \neq 0.
\end{aligned}$$

*Exercise.* Prove that any $n$-dimensional space $\mathcal{L} \subset \mathbb{C}^{n\times n}$ with the property $A\mathbf{e}_j = \mathbf{0}\ \forall A \in \mathcal{L}$ cannot be in $\mathbb{V}$. (Suggestion: show that $J_j \in \mathcal{L}$ such that $\mathbf{v}^T J_j = \mathbf{e}_j^T$ does not exist).

*Exercise.* Let $\mathcal{L}$ be in $\mathbb{V}$ and closed under matrix multiplication. Prove that
(i) If the matrices in $\mathcal{L}$ are symmetric, then $\mathcal{L}$ is commutative
(ii) If the matrices in $\mathcal{L}$ are persymmetric, then $\mathcal{L}$ is commutative

proof: by the assumptions on $\mathcal{L}$, we have, in the symmetric case,

$$J_s J_k = J_s^T J_k^T = (J_k J_s)^T = J_k J_s,$$

and, in the persymmetric case,

$$J_s J_k = J_s J J J_k = J J_s^T J_k^T J = J((J_k J_s)^T)J = J(J J_k J_s J)J = J_k J_s.$$

*Question.* And if we start from the more general definition $\mathcal{L} \subset \mathbb{C}^{n\times n}$, $\mathcal{L} = \text{Span}\,\{J_1, \dots, J_n\}$ with $\mathbf{v}^T J_k = \mathbf{u}_k$ for some $\mathbf{v}, \mathbf{u}_k \in \mathbb{C}^n$ such that $\mathbf{u}_k^H \mathbf{u}_s = 0$ $k \neq s$ ?

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EXERCISE.

$$P_\xi = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ \xi & & & & 0 \end{bmatrix},\ \xi \neq 0$$

(1) If $\rho^n = \xi$ and $\omega^n = 1$, then

$$P_\xi \begin{bmatrix} 1 \\ \rho\omega^j \\ \\ \rho^{n-1}\omega^{(n-1)j} \end{bmatrix} = \rho\omega^j \begin{bmatrix} 1 \\ \rho\omega^j \\ \\ \rho^{n-1}\omega^{(n-1)j} \end{bmatrix},\ j = 0, 1, \dots, n-1.$$

Thus, if $W = (\omega^{ij})_{i,j=0}^{n-1}$ and $D_{1\rho^{n-1}} = \text{diag}\,(\rho^i,\, i = 0, \dots, n-1)$, then

$$P_\xi(D_{1\rho^{n-1}}W) = (D_{1\rho^{n-1}}W)\rho D_{1\omega^{n-1}},\ D_{1\omega^{n-1}} = \text{diag}\,(\omega^j,\, j = 0, \dots, n-1).$$

(2) If, moreover, $|\xi| = 1$ and $\omega^i \neq 1$ $0 < i < n$, then $U = \frac{1}{\sqrt{n}} D_{1\rho^{n-1}} W$ is unitary and

$$\begin{aligned}
C_\xi &:= H_{P_\xi} = \{\textstyle\sum_{k=1}^n z_k P_\xi^{k-1} : z_k \in \mathbb{C}\} \\
&= \{U d(\mathbf{z}) U^* : \mathbf{z} \in \mathbb{C}^n\} = \{U d(U^T\mathbf{z}) d(U^T\mathbf{e}_1)^{-1} U^{-1} : \mathbf{z} \in \mathbb{C}^n\}.
\end{aligned}$$

The matrix $C_\xi(\mathbf{a}) := \sum_{k=1}^n a_k P_\xi^{k-1} = Ud(U^T\mathbf{a})d(U^T\mathbf{e}_1)^{-1}U^{-1}$ is the $\xi$-circulant matrix whose first row is $\mathbf{a}^T$:

$$C_\xi(\mathbf{a}) = \begin{bmatrix} a_1 & a_2 & & a_{n-1} & a_n \\ \xi a_n & a_1 & & & a_{n-1} \\ & & & & \\ \xi a_3 & & & & a_2 \\ \xi a_2 & \xi a_3 & & \xi a_n & a_1 \end{bmatrix}.$$

EXERCISE.

Set $X = P_1 + P_1^T$.

1) Find a convenient representation for the set $C^S$ of all polynomials in $X$, and deduce the dimension of $C^S$.

2) Prove that any matrix of the form $C_1 + JC_2$, $C_1, C_2$ circulants, $J$ anti-identity, belongs to the space $\{A \in \mathbb{C}^{n\times n} : AX = XA\}$. Deduce a lower bound for the dimension of $\{A \in \mathbb{C}^{n\times n} : AX = XA\}$.

Repeat the exercise for $X = P_{-1} + P_{-1}^T$.

EXERCISE

1) Write precisely an algorithm that computes the matrix-vector product $T \cdot \mathbf{z}$, $T = (t_{i-j})_{i,j=1}^n$, in $O(n \log_2 n)$ arithmetic operations.

2) Write an algorithm that computes the matrix-vector product $(T^{-1}) \cdot \mathbf{z}$ in $O(n \log_2 n)$ arithmetic operations (after preprocessing on $T$).

EXERCISE

Do exercise DG

EXERCISE Prove Theorem DD

EXERCISE Let $T$ be a $n \times n$ Toeplitz matrix

1) Prove that $TP_0 - P_0T$ has rank at most 2, and write the Gohberg-Semencul formula for $T^{-1}$

2) Let $H$ be a Hankel matrix, i.e. $H = (h_{i+j-2})_{i,j=1}^n$, and show that $(T + H)(P_0 + P_0^T) - (P_0 + P_0^T)(T + H)$ has rank at most 4

3) Write the matrix of the algebra $\tau$ whose first row is $[2\ \ -1\ 0\ 0\ \cdots\ 0]$, and observe that it is a Toeplitz matrix. Call $A$ such matrix and compute $A^{-1}$ explicitly by using the fact that $A^{-1} \in \tau$ (so, it is sufficient to compute its first row). Is $A^{-1}$ a Toeplitz matrix ?

4) Do exercise Ttau

EXERCISE Under the assumptions of Theorem $\mathcal{L}_A$

1) prove the last of the following identities

$$\mathcal{L}_A = \mathcal{L}(B^{-1}\mathbf{c}) = \mathcal{L}(\mathbf{c})\overline{B}^{-1} = \overline{B}^{-1}\mathcal{L}(\mathbf{c})$$

(hint: find an expression of $\overline{B}$ in terms of the matrices $P_k$, $\mathbf{e}_s^T P_k = \mathbf{e}_k^T J_s\ \forall\, s, k$)

2) prove that $B$ is positive definite (besides hermitian)

EXERCISE (0) Let $\mathcal{L} \in \mathbb{C}^{n\times n}$, $\mathcal{L} = \mathrm{Span}\,\{J_1, \ldots, J_n\}$ with $\mathbf{v}^T J_k = \mathbf{e}_k^T$ for some vector $\mathbf{v}$ (that is, $\mathcal{L} \in \mathbb{V}$). Assume that $\mathcal{L}$ is closed under matrix multiplication, that $I \in \mathcal{L}$, and that $J_i^H = \alpha_i J_{t_i}$, $|\alpha_i| = 1\ \forall\, i$, for some $t_i \in \{1, 2, \ldots, n\}$.

(1) Prove that $J_i^H J_j = \sum_k \overline{[J_k]_{ij}} J_k$, $\forall\, i, j$.

(2) Assume that $\{1, 2, \ldots, n\}$ is a group with identity element 1. Prove that the space

$$\begin{aligned} \mathcal{L} &= \{A \in \mathbb{C}^{n\times n} : a_{ij} = a_{si,sj}, \forall i, j, s \in \{1, \ldots, n\}\} \\ &= \{A \in \mathbb{C}^{n\times n} : a_{ij} = a_{1,i^{-1}j}, \forall i, j \in \{1, \ldots, n\}\} \end{aligned}$$

satisfies the assumptions (0). (Any space $\mathcal{L}$ of this type is usually called *group matrix algebra*; for example, circulants and the $\mathcal{L}$ in Exercise DG are group matrix algebras).

(3) Prove that the space $\mathcal{L}$ spanned by the matrices $J_1 = I$,

$$J_2 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & \mathbf{i} \\ & & -\mathbf{i} & \end{bmatrix}, \; J_3 = \begin{bmatrix} & & 1 & \\ & & & -\mathbf{i} \\ 1 & & & \\ & \mathbf{i} & & \end{bmatrix}, \; J_4 = \begin{bmatrix} & & & 1 \\ & & \mathbf{i} & \\ & -\mathbf{i} & & \\ 1 & & & \end{bmatrix},$$

satisfies the assumptions (0). Prove that $\mathcal{L}$ is not commutative.

EXERCISE Let $T$ be a symmetric Toeplitz matrix.

(1) Compute the first row of $C_T$ where $C$ is the space of circulant matrices

(2) Compute the first row of $\tau_T$ where $\tau$ is the space of tau matrices

(3) Compute the first row of $\mathcal{L}_T$ where $\mathcal{L}$ is the space in the point (3) of the previous exercise (so, $n = 4$)

(4) Compute the vector $(\mathbf{e}_1 + \mathbf{e}_6)^T \gamma_T$ where $\gamma$ is the space of all $6 \times 6$ gamma matrices (i.e. assume $n = 6$).

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Solving EXERCISES:

$C^S_{-1} + JC^{SK}_{-1}$ *is a commutative matrix algebra.* Assume $A_i \in C^S_{-1}$, $B_i \in C^{SK}_{-1}$. Note that $A_i$ and $B_i$ are also persymmetric. Then

$$\begin{aligned} (A_0 + JB_0)(A_1 + JB_1) &= A_0A_1 + A_0JB_1 + JB_0A_1 + JB_0JB_1 \\ &= A_0A_1 + JA_0B_1 + JB_0A_1 + B_0^TB_1. \end{aligned}$$

Since $A_0A_1$ is $(-1)$-circulant and $(A_0A_1)^T = A_1^TA_0^T = A_1A_0 = A_0A_1$, $A_0B_1$ is $(-1)$-circulant and $(A_0B_1)^T = B_1^TA_0^T = -B_1A_0 = -A_0B_1$ ($C_{-1}$ is closed under matrix multiplication and is commutative), we have $A_0A_1 \in C^S_{-1}$ and $A_0B_1 \in C^{SK}_{-1}$.

Since $B_0^TB_1$ is $(-1)$-circulant and $(B_0^TB_1)^T = B_1^TB_0 = -B_1B_0 = -B_1(-B_0^T) = B_1B_0^T = B_0^TB_1$, $B_0A_1$ is $(-1)$-circulant and $(B_0A_1)^T = A_1^TB_0^T = -A_1B_0 =$

$-B_0A_1$ ($C_{-1}$ is closed under transposition and matrix multiplication, and commutative), we have $B_0^TB_1 \in C_{-1}^S$ and $B_0A_1 \in C_{-1}^{SK}$.

*Proof of Theorem DD.*

$$\begin{aligned}
&\left[\sum H_{P_0}(\mathbf{x}_m)^T H_X(\mathbf{y}_m)\right]X - X\left[\sum H_{P_0}(\mathbf{x}_m)^T H_X(\mathbf{y}_m)\right] \\
&= \sum\left[H_{P_0}(\mathbf{x}_m)^T X - X H_{P_0}(\mathbf{x}_m)^T\right]H_X(\mathbf{y}_m) \\
&= b\sum\left[H_{P_0}()^T P_0 - P_0 H_{P_0}()^T\right]H_X(\mathbf{y}_m) \\
&= b\sum\Big[\begin{bmatrix} 0 & & \\ & (\mathbf{x}_m)_1 & \\ & & \\ 0 & (\mathbf{x}_m)_{n-1} & (\mathbf{x}_m)_1 \end{bmatrix} - \begin{bmatrix} (\mathbf{x}_m)_1 & & \\ & & \\ (\mathbf{x}_m)_{n-1} & (\mathbf{x}_m)_1 & \\ 0 & & 0 \end{bmatrix}\Big]H_X(\mathbf{y}_m) \\
&= b\sum\begin{bmatrix} -(\mathbf{x}_m)_1 & & \\ & & \\ -(\mathbf{x}_m)_{n-1} & & \\ 0 & (\mathbf{x}_m)_{n-1} & (\mathbf{x}_m)_1 \end{bmatrix} H_X(\mathbf{y}_m) \\
&= b\sum(-\mathbf{x}_m\mathbf{e}_1^T + \mathbf{e}_n\mathbf{x}_m^T J)H_X(\mathbf{y}_m) \\
&= b\sum(-\mathbf{x}_m\mathbf{y}_m^T + \mathbf{e}_n\mathbf{x}_m^T H_X(\mathbf{y}_m)^T J) = \dots
\end{aligned}$$

*Compute the first row of $C_T$.* Let $J_k \in \mathbb{C}^{n\times n}$ be the circulant matrices with first row $\mathbf{e}_k^T$, $k = 1, \dots, n$. Let us compute $(B^{-1}\mathbf{c})^T$ with respect to such basis (i.e. the first row of $C_A$) when $A$ is a symmetric Toeplitz matrix. So, we have

$$A = T = \begin{bmatrix} t_0 & t_1 & & t_{n-1} \\ t_1 & t_0 & & \\ & & & t_1 \\ t_{n-1} & & t_1 & t_0 \end{bmatrix},$$

$$\begin{aligned}
&(J_1, A)_F = nt_0, (J_2, A)_F = (n-1)t_1 + t_{n-1}, (J_3, A)_F = (n-2)t_2 + 2t_{n-2}, \\
&(J_4, A)_F = (n-3)t_3 + 3t_{n-3}, \dots (J_n, A)_F = t_{n-1} + (n-1)t_1,
\end{aligned}$$

and thus

$$(J_k, A)_F = (n-k+1)t_{k-1} + (k-1)t_{n-k+1},\ k = 1, \dots, n.$$

Since $B = nI$, we have $B^{-1} = \frac{1}{n}I$, and

$$[B^{-1}\mathbf{c}]_k = \frac{1}{n}((n-k+1)t_{k-1} + (k-1)t_{n-k+1}),\ k = 1, \dots, n.$$

*Compute the first row of $\tau_T$.* Let $J_k \in \mathbb{C}^{n\times n}$ be the $\tau$ matrices with first row $\mathbf{e}_k^T$, $k = 1, \dots, n$. Let us compute $(B^{-1}\mathbf{c})^T$ with respect to such basis (i.e. the first row of $\tau_A$) when $A$ is a symmetric Toeplitz matrix. So, we have

$$\begin{aligned}
&(J_1, A)_F = nt_0, (J_2, A)_F = 2(n-1)t_1, (J_3, A)_F = (n-2)t_0 + 2(n-2)t_2, \\
&(J_4, A)_F = 2(n-3)t_1 + 2(n-3)t_3, (J_5, A)_F = (n-4)t_0 + 2(n-4)t_2 + 2(n-4)t_4.
\end{aligned}$$

One can guess that

$$(J_k, A)_F = (n-k+1)[\delta_{k,o}t_0 + 2\sum_{j=1}^{[k/2]} t_{k-2j+1}],\ k = 1, \dots, n,$$

where $\delta_{k,o}=1$ if $k$ is odd and $\delta_{k,o}=0$ if $k$ is even. Since $B^{-1}=\frac{1}{2n+2}(3J_1-J_3)$ (prove such formula!), we have

$$\begin{array}{l}(B^{-1}\mathbf{c})_1=\frac{1}{2n+2}(3(J_1,A)_F-(J_3,A)_F),\ (B^{-1}\mathbf{c})_2=\frac{1}{2n+2}(2(J_2,A)_F-(J_4,A)_F),\\ (B^{-1}\mathbf{c})_k=\frac{1}{2n+2}(-(J_{k-2},A)_F+2(J_k,A)_F-(J_{k+2},A)_F),\\ (B^{-1}\mathbf{c})_{n-1}=\frac{1}{2n+2}(-(J_{n-3},A)_F+2(J_{n-1},A)_F),\ (B^{-1}\mathbf{c})_n=\frac{1}{2n+2}(-(J_{n-2},A)_F+3(J_n,A)_F).\end{array}$$

It is not difficult to conclude that

$$\begin{array}{l}(B^{-1}\mathbf{c})_1=\frac{1}{n+1}((n+1)t_0-(n-2)t_2),\\ (B^{-1}\mathbf{c})_k=\frac{1}{n+1}((n-k+3)t_{k-1}-(n-k-1)t_{k+1}),\ k=2,\dots,n-1,\\ (B^{-1}\mathbf{c})_n=\frac{1}{n+1}(3t_{n-1}).\end{array}$$

*Compute the sum of the first row and of the last row of* $\gamma_T$. We know that

$$\begin{array}{l}\gamma_T=\sum_{k=1}^{6}[B^{-1}\mathbf{c}]_kJ_k=Gd(G^TB^{-1}\mathbf{c})d(G^T(\mathbf{e}_1+\mathbf{e}_6))^{-1}G^{-1},\\ (\mathbf{e}_1+\mathbf{e}_6)^T\gamma_T=(B^{-1}\mathbf{c})^T,\ B_{ij}=(J_i,J_j)_F,\ c_i=(J_i,T)_F,\end{array}$$

where the $J_k$ are the matrices in $\gamma$ with the property $(\mathbf{e}_1+\mathbf{e}_6)^TJ_k=\mathbf{e}_k^T$, $k=1,\dots,6$ (they have been written explicitely above). So, we have to compute the vector $B^{-1}\mathbf{c}$. Recall that we expect (from theory) that $B$, $B^{-1}$ are matrices in $\gamma$:

$$B=3\begin{bmatrix}1&2&1&0&-1&-2\\2&1&2&1&0&-1\\1&2&1&2&1&0\\0&1&2&1&2&1\\-1&0&1&2&1&2\\-2&-1&0&1&2&1\end{bmatrix},\ B^{-1}=\frac{1}{12}\begin{bmatrix}-2&1&1&0&-1&-1\\1&-2&1&1&0&-1\\1&1&-2&1&1&0\\0&1&1&-2&1&1\\-1&0&1&1&-2&1\\-1&-1&0&1&1&-2\end{bmatrix},$$

$$\mathbf{c}=\begin{bmatrix}3t_0\\3t_0+5t_1-t_5\\3t_0+5t_1+4t_2-2t_4-t_5\\3t_0+5t_1+4t_2-2t_4-t_5\\3t_0+5t_1-t_5\\3t_0\end{bmatrix}.$$

Note that $B$, $B^{-1}$ are symmetric $(-1)$-circulant matrices, which is a stronger condition than the expected $B$, $B^{-1}\in\gamma$. Moreover, note that the vector $\mathbf{c}$ is centrosymmetric; thus, since $B^{-1}$ is centrosymmetric, i.e. $JB^{-1}=B^{-1}J$ (it is both symmetric and persymmetric!), the vector $B^{-1}\mathbf{c}$ is expected to be centrosymmetric. In fact, we have

$$\gamma_T(\mathbf{e}_1+\mathbf{e}_6)=B^{-1}\mathbf{c}=\frac{1}{12}\begin{bmatrix}-6t_0+5t_1+4t_2-2t_4-t_5\\8t_2-4t_4\\6t_0+5t_1-4t_2+2t_4-t_5\\6t_0+5t_1-4t_2+2t_4-t_5\\8t_2-4t_4\\-6t_0+5t_1+4t_2-2t_4-t_5\end{bmatrix}.$$

Note that a $\gamma$ matrix $A+JB$ with $(\mathbf{e}_1+\mathbf{e}_6)^T(A+JB)$ centrosymmetric must be in $C_{-1}^S$; more precisely, the following implication holds:

$$\begin{array}{l}(\mathbf{e}_1+\mathbf{e}_6)^T(A+JB)=[z_1\,z_2\,z_3\,z_3\,z_2\,z_1]\\ \Rightarrow\ a_2=z_3,\ a_1=z_2+z_3,\ a_0=z_1+z_2+z_3,\ b_i=0\ \forall i.\end{array}$$

In our case:

$$a_2 = \frac{1}{12}(6t_0+5t_1-4t_2+2t_4-t_5),\ a_1 = \frac{1}{12}(6t_0+5t_1+4t_2-2t_4-t_5),\ a_0 = \frac{1}{12}(10t_1+8t_2-4t_4-2t_5)$$

But $\gamma_T$ should coincide with the T.Chan $(C_1)_T$ ! ... is there something wrong ?

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*An alternative preconditioner for Toeplitz systems?*
Consider the $n-2 \times n-2$ $\tau$ matrix with first row $[a_1\ a_2\ \cdots\ a_{n-2}]$ and call it $\tau_{a_1 a_{n-2}}$. For example

$$n=5:\ \tau_{a_1a_3} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1+a_3 & a_2 \\ a_3 & a_2 & a_1 \end{bmatrix},\ n=6:\ \tau_{a_1a_4} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1+a_3 & a_2+a_4 & a_3 \\ a_3 & a_2+a_4 & a_1+a_3 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix}, \ldots.$$

Consider the $n \times n$ matrix

$$A_{a_1a_{n-2}} = \begin{bmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \tau_{a_1a_{n-2}} & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{bmatrix}.$$

For example,

$$n=5:\ A_{a_1a_3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_2 & a_1+a_3 & a_2 & 0 \\ 0 & a_3 & a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ n=6:\ A_{a_1a_4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & a_2 & a_1+a_3 & a_2+a_4 & a_3 & 0 \\ 0 & a_3 & a_2+a_4 & a_1+a_3 & a_2 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ldots.$$

We want to find $\tau_{A_{a_1a_{n-2}}}$, i.e. the $n \times n$ $\tau$ matrix defined by the following minimization property

$$\|\tau_{A_{a_1a_{n-2}}} - A_{a_1a_{n-2}}\|_F = \min\{\|X - A_{a_1a_{n-2}}\|_F :\ X \in \tau,\ X\ n \times n\}.$$

Let $J_k$ be the $n \times n$ $\tau$ matrices defined by the conditions $\mathbf{e}_1^T J_k = \mathbf{e}_k^T$, $k = 1, \ldots, n$, and set $\tau_{A_{a_1a_{n-2}}} = \tau(\mathbf{z}) := \sum_k z_k J_k$. We want to find $\mathbf{z}$. We know that if $B_{ij} = (J_i, J_j)_F$, $c_i = (J_i, A_{a_1a_{n-2}})_F$, then $\mathbf{z} = B^{-1}\mathbf{c}$.

For example for $n=5$:

$$B = \begin{bmatrix} 5 & 0 & 3 & 0 & 1 \\ 0 & 8 & 0 & 4 & 0 \\ 3 & 0 & 9 & 0 & 3 \\ 0 & 4 & 0 & 8 & 0 \\ 1 & 0 & 3 & 0 & 5 \end{bmatrix},\ c = \begin{bmatrix} 3a_1+a_3 \\ 4a_2 \\ 3a_1+3a_3 \\ 4a_2 \\ 3a_3+a_1 \end{bmatrix},\ B^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 3 \end{bmatrix}.$$

Then

$$\mathbf{z} = B^{-1}\mathbf{c} = \frac{1}{6}\begin{bmatrix} 3a_1 \\ 2a_2 \\ a_1+a_3 \\ 2a_2 \\ 3a_3 \end{bmatrix}$$

$n = 6$:

Since $B_{ij} = (J_i, J_j)$ and $B \in \tau$, one can easily obtain the first row of $B$ and write down $B$:

$$B = \begin{bmatrix} 6 & 0 & 4 & 0 & 2 & 0 \\ 0 & 10 & 0 & 6 & 0 & 2 \\ 4 & 0 & 12 & 0 & 6 & 0 \\ 0 & 6 & 0 & 12 & 0 & 4 \\ 2 & 0 & 6 & 0 & 10 & 0 \\ 0 & 2 & 0 & 4 & 0 & 6 \end{bmatrix}, \; c = \begin{bmatrix} 4a_2 + 2a_3 \\ 6a_2 + 2a_4 \\ 6a_3 + 4a_1 \\ 6a_2 + 4a_4 \\ 6a_3 + 2a_1 \\ 4a_4 + 2a_2 \end{bmatrix}, \; B^{-1} = \frac{1}{2 \cdot 6 + 2} \begin{bmatrix} 3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 3 \end{bmatrix}.$$

Then

$$\mathbf{z} = B^{-1}\mathbf{c} = \frac{1}{7} \begin{bmatrix} 4a_1 \\ 3a_2 \\ a_1 + 2a_3 \\ 2a_2 + a_4 \\ 3a_3 \\ 4a_4 \end{bmatrix}$$

*Exercise.* Prove that

$$B^{-1} = \frac{1}{2n+2}(3J_1 - J_3)$$

(first find $B$ and $\mathbf{w}$ such that $\mathbf{w}^T B = \mathbf{e}_1^T$, then $B^{-1} = \tau(\mathbf{w})$).

By observing the $\mathbf{z}$ obtained for $n = 5$, $n = 6$ (see above), and in the cases $n = 7$ and $n = 8$,

$$\begin{aligned} \mathbf{z}^T &= \tfrac{1}{8}[5a_1 \; 4a_2 \; a_1 + 3a_3 \; 2a_2 + 2a_4 \; 3a_3 + a_5 \; 4a_4 \; 5a_5], \\ \mathbf{z}^T &= \tfrac{1}{9}[6a_1 \; 5a_2 \; a_1 + 4a_3 \; 2a_2 + 3a_4 \; 3a_3 + 2a_5 \; 4a_4 + a_6 \; 5a_5 \; 6a_6], \end{aligned}$$

one can conjecture a formula for $\mathbf{z}$ in case $n$ is generic:
*Exercise.* Prove that $\tau_{A_{a_1 a_{n-2}}} = \tau(\mathbf{z})$ with

$$\begin{aligned} \mathbf{z}^T = \tfrac{1}{n+1}\Big[ & \; [0 \; 0 \; a_1 \; 2a_2 \; \cdots \; (n-4)a_{n-4} \; (n-3)a_{n-3} \; (n-2)a_{n-2}] \\ & + [(n-2)a_1 \; (n-3)a_2 \; (n-4)a_3 \; \cdots \; 2a_{n-3} \; a_{n-2} \; 0 \; 0] \Big]. \end{aligned} \qquad (**)$$

APPLICATION. Let $T = (t_{|i-j|})_{i,j=1}^n$ be a symmetric Toeplitz matrix. It is easy to realize that

$$T = \tau_{t_0 t_{n-1}} - A_{t_2 t_{n-1}}, \; A_{t_2 t_{n-1}} = \begin{bmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \tau_{t_2 t_{n-1}} & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{bmatrix}.$$

From the equality

$$T = \tau_{t_0 t_{n-1}} - \tau_{A_{t_2 t_{n-1}}} + \tau_{A_{t_2 t_{n-1}}} - A_{t_2 t_{n-1}}$$

follows the inequality:

$$\|T - \tau_T\|_F \le \|\tau_{A_{t_2 t_{n-1}}} - A_{t_2 t_{n-1}}\|_F = \|T - (\tau_{t_0 t_{n-1}} - \tau_{A_{t_2 t_{n-1}}})\|_F.$$

However, the $\tau$ matrix $\tau_{t_0 t_{n-1}} - \tau_{A_{t_2 t_{n-1}}}$ could be for some reasons (...) a preconditioner for $T$ better than $\tau_T$. Note that $\tau_{A_{t_2 t_{n-1}}} = \tau(\mathbf{z})$ where $\mathbf{z}^T$ is like in (**) but with $a_i$ replaced by $t_{i+1}$, $i = 1, \dots, n-2$.

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*Displacement decompositions.*
Let $\mathcal{L}$ be in $\mathbb{V}$, and $X$ a matrix in $\mathcal{L}$. Assume that $\mathcal{L}$ is commutative.

Denote by $\mathcal{L}(\mathbf{z})$ the matrix of $\mathcal{L}$ whose $\mathbf{v}$-row is $\mathbf{z}^T$, and assume that $\mathcal{L}(\mathbf{z})^T Q = \tilde{Q}\mathcal{L}(\mathbf{z})$, $\forall\, \mathbf{z} \in \mathbb{C}^n$. (For example, the latter condition is satisfied with $Q = \tilde{Q} = I$ if $\mathcal{L}$ is symmetric, or with $Q = \tilde{Q} = J$ if $\mathcal{L}$ is persymmetric).

Assume $AX - XA = \sum_m \mathbf{x}_m \mathbf{y}_m^T$. Then

$$\sum_m \mathbf{x}_m^T \mathcal{L}(\mathbf{y}_m)^T = \mathbf{0}^T$$

($\sum_m \mathbf{x}_m^T \mathcal{L}(\mathbf{y}_m)^T \mathbf{e}_k = \ldots = 0$), and thus

$$\begin{array}{rcl} AX - XA & = & \sum_m \mathbf{x}_m \mathbf{y}_m^T \\ & = & \sum_m \mathbf{x}_m \mathbf{v}^T \mathcal{L}(\mathbf{y}_m) \\ & = & \sum_m \mathbf{x}_m \mathbf{w}^T \mathcal{L}(\mathbf{w})^{-1} \mathcal{L}(\mathbf{y}_m) \\ & = & \sum_m \mathbf{x}_m \mathbf{w}^T \mathcal{L}(\mathbf{y}_m) \mathcal{L}(\mathbf{w})^{-1} \\ & = & \sum_m \mathbf{x}_m \mathbf{w}^T \mathcal{L}(\mathbf{y}_m) \mathcal{L}(\mathbf{w})^{-1} - \mathbf{b} \sum_m \mathbf{x}_m^T \mathcal{L}(\mathbf{y}_m)^T Q \mathcal{L}(\mathbf{w})^{-1} \\ & = & \sum_m \mathbf{x}_m \mathbf{w}^T \mathcal{L}(\mathbf{y}_m) \mathcal{L}(\mathbf{w})^{-1} - \mathbf{b} \sum_m \mathbf{x}_m^T \tilde{Q} \mathcal{L}(\mathbf{y}_m) \mathcal{L}(\mathbf{w})^{-1} \\ & = & \sum_m (\mathbf{x}_m \mathbf{w}^T - \mathbf{b} \mathbf{x}_m^T \tilde{Q}) \mathcal{L}(\mathbf{y}_m) \mathcal{L}(\mathbf{w})^{-1} \\ & = & \sum_m (\mathbf{x}_m \mathbf{w}^T - \mathbf{b} \mathbf{x}_m^T \tilde{Q}) \mathcal{L}(\mathcal{L}(\mathbf{w})^{T^{-1}} \mathbf{y}_m) \end{array}$$

($\mathbf{w}$ must be chosen such that $\mathcal{L}(\mathbf{w})$ is non singular; it can be simply chosen equal to $\mathbf{v}$ so that $\mathcal{L}(\mathbf{w}) = I$).

We want to find $Z_m$ ($Z_m = R_m + E_m$) such that

$$Z_m X - X Z_m = \mathbf{x}_m \mathbf{w}^T - \mathbf{b} \mathbf{x}_m^T \tilde{Q} \qquad (*)$$

so that $A - \sum_m Z_m \mathcal{L}(\mathcal{L}(\mathbf{w})^{T^{-1}} \mathbf{y}_m)$ must be a matrix commuting with $X$. ...

Note that the left hand side in (*) has zero trace. Thus, if $\mathcal{L}$ is symmetric, since $\tilde{Q} = I$, then $\mathbf{b}$ must be chosen equal to $\mathbf{w}$ (otherwise the right hand side may have nonzero trace). If $\mathcal{L}$ is persymmetric, since $\tilde{Q} = J$, then $\mathbf{b}$ must be chosen equal to $J\mathbf{w}$ (for the same reason).

*Symmetric case*:

$$Z_m X - X Z_m = \mathbf{x}_m \mathbf{w}^T - \mathbf{w} \mathbf{x}_m^T.$$

...

*Persymmetric case*:

$$Z_m X - X Z_m = \mathbf{x}_m \mathbf{w}^T - J \mathbf{w} \mathbf{x}_m^T J.$$

...

NOTE: Assume moreover $\mathcal{L}$ both symmetric and persymmetric. Then we have also

$$\begin{array}{rcl} JAJX - XJAJ & = & JAXJ - JXAJ = J(AX - XA)J \\ & = & \sum_m J\mathbf{x}_m\mathbf{y}_m^T J \\ & = & \sum_m J\mathbf{x}_m\mathbf{v}^T\mathcal{L}(\mathbf{y}_m)J \\ & = & \sum_m J\mathbf{x}_m\mathbf{w}^T\mathcal{L}(\mathbf{w})^{-1}\mathcal{L}(\mathbf{y}_m)J \\ & = & \sum_m J\mathbf{x}_m\mathbf{w}^T J\mathcal{L}(\mathbf{y}_m)\mathcal{L}(\mathbf{w})^{-1} \\ & = & \sum_m J\mathbf{x}_m\mathbf{w}^T J\mathcal{L}(\mathbf{y}_m)\mathcal{L}(\mathbf{w})^{-1} - \mathbf{c}\sum_m \mathbf{x}_m^T\mathcal{L}(\mathbf{y}_m)S\mathcal{L}(\mathbf{w})^{-1} \\ & = & \sum_m J\mathbf{x}_m\mathbf{w}^T J\mathcal{L}(\mathbf{y}_m)\mathcal{L}(\mathbf{w})^{-1} - \mathbf{c}\sum_m \mathbf{x}_m^T\tilde{S}\mathcal{L}(\mathbf{y}_m)\mathcal{L}(\mathbf{w})^{-1} \\ & = & \sum_m (J\mathbf{x}_m\mathbf{w}^T J - \mathbf{c}\mathbf{x}_m^T\tilde{S})\mathcal{L}(\mathbf{y}_m)\mathcal{L}(\mathbf{w})^{-1} \\ & = & \sum_m (J\mathbf{x}_m\mathbf{w}^T J - \mathbf{c}\mathbf{x}_m^T\tilde{S})\mathcal{L}(\mathcal{L}(\mathbf{w})^{-1}\mathbf{y}_m) \end{array}$$

By summing this result with the previous one, we obtain

$$(A+JAJ)X - X(A+JAJ) = \sum_m (\mathbf{x}_m\mathbf{w}^T + J\mathbf{x}_m\mathbf{w}^T J - \mathbf{b}\mathbf{x}_m^T\tilde{Q} - \mathbf{c}\mathbf{x}_m^T\tilde{S})\mathcal{L}(\mathcal{L}(\mathbf{w})^{-1}\mathbf{y}_m)$$

($\mathbf{w}$ must be chosen such that $\mathcal{L}(\mathbf{w})$ is non singular; it can be simply chosen equal to $\mathbf{v}$ so that $\mathcal{L}(\mathbf{w}) = I$).

We want to find $Z_m$ ($Z_m = R_m + E_m$) such that

$$Z_m X - X Z_m = \mathbf{x}_m\mathbf{w}^T + J\mathbf{x}_m\mathbf{w}^T J - \mathbf{b}\mathbf{x}_m^T\tilde{Q} - \mathbf{c}\mathbf{x}_m^T\tilde{S} \qquad (*')$$

(see page 209 of DiF,Zell, LAA 268 (1998) for a matrix $Z_m$ satisfying (*')) so that $A+JAJ-\sum_m Z_m\mathcal{L}(\mathcal{L}(\mathbf{w})^{-1}\mathbf{y}_m)$ must be a matrix commuting with $X$. ... Note that if $A$ is centrosymmetric, i.e. A=JAJ, (as in the case of $A$ =symmetric Toeplitz matrix), then one can conclude that $2A - \sum_m Z_m\mathcal{L}(\mathcal{L}(\mathbf{w})^{-1}\mathbf{y}_m)$ must be a matrix commuting with $X$, from which we have a representation for $A$ ...

Note that the left hand side in (*) has zero trace. Thus, if $S = \tilde{S} = I$ then $\mathbf{c} = \mathbf{w}$ and $Q = \tilde{Q}$ can be chosen either equal to $I$ (in such case $\mathbf{b} = \mathbf{w}$) or equal to $J$ (in such case $\mathbf{b} = J\mathbf{w}$). If $S = \tilde{S} = J$ then $\mathbf{c} = J\mathbf{w}$ and $Q = \tilde{Q}$ can be chosen either equal to $I$ (in such case $\mathbf{b} = \mathbf{w}$) or equal to $J$ (in such case $\mathbf{b} = J\mathbf{w}$). Compare with page 209 of DiF,Zell, LAA 268 (1998).

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*Displacement decompositions involving spaces in* $\mathbb{V}$

1) Look for $\tau = \tau_1(\mathbf{z})$ matrices of rank one.

2) Verify if there exist $\mathbf{v}$ such that $\tau_{\mathbf{v}}(A^T\mathbf{v}) = \tau_1(A^T\mathbf{v})\tau_1(\mathbf{v})^{-1} = \tau_1(\mathbf{z})$ with $\mathbf{z}$ as in 1), and such that $\tau_1(\mathbf{v})$ is non singular (or equivalently that $\tau$ is a in $\mathbb{V}$ for such $\mathbf{v}$).

3) Find a displacement decomposition of the type

$$A = \square + \tau_v(A^T\mathbf{v}),\ \mathbf{v}^T\square = \mathbf{0}^T$$

where $\mathbf{v}$ is as in 2) (so that $\tau_v(A^T\mathbf{v})$ has rank 1 !). Attempt:

$$\mathbf{v}^T\sum_{m=1}^{\alpha} Z_m\mathcal{L}_m = \sum_{m=1}^{\alpha}(\mathbf{v}^T Z_m)\mathcal{L}_m = \sum_{m=1}^{\alpha}(\mathbf{v}^T \begin{bmatrix} \frac{1}{v_1}[\cdots_{1m}] \\ \frac{1}{v_2}[\cdots_{2m}] \\ \\ \frac{1}{v_n}[\cdots_{nm}] \end{bmatrix})\mathcal{L}_m$$

$$[\cdots_{1m}] + [\cdots_{2m}] + \ldots + [\cdots_{nm}] = \mathbf{x}_m^T$$

*$\tau$ matrices of rank* 1

First observe that the entry $(1,1)$ of a non null $\tau$ matrix $A$ of rank 1 must be nonzero. In fact, if $A_{11}=0$, then also $A_{12}=0$ because $A_{12}\neq 0$ would imply $A_{21}\neq 0$ ($A$ is symmetric!) and so $A$ would have rank at least 2. Thus $A_{11}=0$ implies $A_{12}=0$ and, by symmetry, $A_{21}=0$. But then also $A_{13}$ must be zero because $A_{13}\neq 0$ would imply $A_{31}\neq 0$ and so $A$ would have rank at least 2. Going on one proves that the first row and column of $A$ must be null, so $A$ itself must be the null matrix.

A faster proof is the following: if $A\in\tau$, $A=\mathbf{u}\mathbf{v}^T$ and $A_{11}=0$, then $u_1v_1=0$, and thus either $u_1$ or $v_1$ must be zero; but $u_1=0$ ($v_1$=0) implies $\mathbf{e}_1^TA=\mathbf{0}^T$ ($A\mathbf{e}_1=\mathbf{0}$), that is, $A=0$ since $A\in\tau$.

$n$ odd:

$\tau$ matrices $A$ of rank one for $n=3$: since $A$ is in $\tau$, is symmetric, is persymmetric, and its second column is a multiple of its first column, it must be (unless a multiplier) of the type

$$A=\begin{bmatrix} 1 & \alpha & \\ \alpha & 1+\beta & \\ \beta & \alpha & \end{bmatrix},\ \alpha^2=1+\beta,\ \alpha\beta=\alpha.$$

We have two possible alternatives, $\alpha=0$, $\beta=-1$, and $\beta=1$, $\alpha=\pm\sqrt{2}$ ($\alpha^2-2=0$):

$$A=\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix},\ A=\begin{bmatrix} 1 & \pm\sqrt{2} & 1 \\ \pm\sqrt{2} & 2 & \pm\sqrt{2} \\ 1 & \pm\sqrt{2} & 1 \end{bmatrix}.$$

So, there are three $3\times 3$ rank one $\tau$ matrices. Note that they are orthogonal with respect the inner product $(\cdot,\cdot)_F$ or, equivalently, the three vectors that define them are orthogonal.

$\tau$ matrices $A$ of rank one for $n=5$: since $A$ is in $\tau$, is symmetric, is persymmetric, and its second column is a multiple of its first column, it must be (unless a multiplier) of the type

$$A=\begin{bmatrix} 1 & \alpha & & & \\ \alpha & 1+\beta & & & \\ \beta & \alpha+\gamma & & & \\ \gamma & \beta+\delta & & & \\ \delta & \gamma & & & \end{bmatrix},\ \alpha^2=1+\beta,\ \alpha\beta=\alpha+\gamma,\ \alpha\gamma=\beta+\delta,\ \alpha\delta=\gamma.$$

First observe that $\delta\neq 0$ since $\delta=0$ would imply $\gamma=\beta=0$, $\alpha=0$, $\beta=-1$. So, the conditions become

$$\delta\alpha^2=\delta+\beta\delta,\ \alpha\beta=\alpha+\alpha\delta,\ \alpha^2\delta=\beta+\delta,\ \alpha\delta=\gamma$$

The first and the third imply $\beta=\beta\delta$. We have two possible alternatives. For $\beta=0$ we have the equation $\alpha^2-1=0$ and thus $\alpha=\pm 1$, $\beta=0$, $\gamma=\mp 1$, $\delta=-1$. For $\beta\neq 0$ we have the equation $\alpha^3-3\alpha=0$ and thus either $\alpha=0$,

$\delta = 1$, $\beta = -1$, $\gamma = 0$ or $\alpha = \pm\sqrt{3}$, $\delta = 1$, $\beta = 2$, $\gamma = \pm\sqrt{3}$.

$$A = \begin{bmatrix} 1 & \pm 1 & 0 & & \\ \pm 1 & 1 & 0 & & \\ 0 & 0 & 0 & & \\ \mp 1 & -1 & 0 & & \\ -1 & \mp 1 & 0 & & \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & & \\ 0 & 0 & 0 & & \\ -1 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ 1 & 0 & -1 & & \end{bmatrix}, \begin{bmatrix} 1 & \pm\sqrt{3} & 2 & & \\ \pm\sqrt{3} & 3 & \pm 2\sqrt{3} & & \\ 2 & \pm 2\sqrt{3} & 4 & & \\ \pm\sqrt{3} & 3 & \pm 2\sqrt{3} & & \\ 1 & \pm\sqrt{3} & 2 & & \end{bmatrix}.$$

So, there are five $5 \times 5$ rank one $\tau$ matrices. Note that they are orthogonal with respect the inner product $(\cdot, \cdot)_F$ or, equivalently, the five vectors that define them are orthogonal.

$\tau$ matrices $A$ of rank one for $n = 7$: since $A$ is in $\tau$, is symmetric, is persymmetric, and its second column is a multiple of its first column, it must be (unless a multiplier) of the type

$$A = \begin{bmatrix} 1 & \alpha & & & & & \\ \alpha & 1+\beta & & & & & \\ \beta & \alpha+\gamma & & & & & \\ \gamma & \beta+\delta & & & & & \\ \delta & \gamma+\sigma & & & & & \\ \sigma & \delta+\rho & & & & & \\ \rho & \sigma & & & & & \end{bmatrix}, \ \alpha^2 = 1+\beta, \ \alpha\beta = \alpha+\gamma, \ \alpha\gamma = \beta+\delta, \ \alpha\delta = \gamma+\sigma, \ \alpha\sigma = \delta+\rho, \ \alpha\rho = \sigma.$$

First observe that $\rho \neq 0$ since $\rho = 0$ would imply $\sigma = \delta = \gamma = \beta = 0$, $\alpha = 0$, $\beta = -1$. So, the conditions become

$$\rho\alpha^2 = \rho + \beta\rho, \ \alpha\beta = \alpha + \gamma, \ \alpha\gamma = \beta + \delta, \ \alpha\delta = \gamma + \alpha\rho, \ \alpha^2\rho = \delta + \rho, \ \alpha\rho = \sigma$$

The second implies $\gamma = \alpha(\beta - 1)$. The first and the fifth imply $\delta = \beta\rho$. So that the third and the fourth become $\alpha\gamma = \beta(1 + \rho)$, $\alpha\rho(\beta - 1) = \gamma$. It follows that $\alpha\rho(\beta - 1) = \alpha(\beta - 1)$.

We have three possible alternatives. For $\alpha = 0$ we have $\alpha = 0$, $\beta = -1$, $\gamma = 0$, $\delta = 1$, $\sigma = 0$, $\rho = -1$. For $\beta = 1$ we have $\alpha = \pm\sqrt{2}$, $\beta = 1$, $\gamma = 0$, $\delta = -1$, $\sigma = \mp\sqrt{2}$, $\rho = -1$. For $\rho = 1$ we have $\alpha^2 = 2 \pm \sqrt{2}$, $\beta = 1 \pm \sqrt{2}$, $\gamma = \alpha(\pm\sqrt{2})$, $\delta = \beta$, $\sigma = \alpha$, $\rho = 1$.

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ -1 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ 1 & 0 & -1 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ -1 & 0 & 1 & 0 & & & \end{bmatrix}, \begin{bmatrix} 1 & \pm\sqrt{2} & 1 & 0 & & & \\ \pm\sqrt{2} & 2 & \pm\sqrt{2} & 0 & & & \\ 1 & \pm\sqrt{2} & 1 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ -1 & \mp\sqrt{2} & -1 & 0 & & & \\ \mp\sqrt{2} & -2 & \mp\sqrt{2} & 0 & & & \\ -1 & \mp\sqrt{2} & -1 & 0 & & & \end{bmatrix},$$

$$\begin{bmatrix} 1 & \alpha & 1 \pm \sqrt{2} & \alpha(\pm\sqrt{2}) & & & \\ \alpha & \alpha^2 & \alpha(1 \pm \sqrt{2}) & \alpha^2(\pm\sqrt{2}) & & & \\ 1 \pm \sqrt{2} & \alpha(1 \pm \sqrt{2}) & 3 \pm 2\sqrt{2} & \alpha(\pm\sqrt{2} + 2) & & & \\ \alpha(\pm\sqrt{2}) & \alpha^2(\pm\sqrt{2}) & \alpha(\pm\sqrt{2} + 2) & 2\alpha^2 & & & \\ 1 \pm \sqrt{2} & \alpha(1 \pm \sqrt{2}) & 3 \pm 2\sqrt{2} & \alpha(\pm\sqrt{2} + 2) & & & \\ \alpha & \alpha^2 & \alpha(1 \pm \sqrt{2}) & \alpha^2(\pm\sqrt{2}) & & & \\ 1 & \alpha & 1 \pm \sqrt{2} & \alpha(\pm\sqrt{2}) & & & \end{bmatrix}, \ \alpha^2 = 2 \pm \sqrt{2}.$$

So, there are seven $7 \times 7$ rank one $\tau$ matrices. Note that they are orthogonal with respect the inner product $(\cdot,\cdot)_F$ or, equivalently, the seven vectors that define them are orthogonal.

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Another important remark. If $A \in \tau$ is such that $\mathbf{e}_2^T A = \alpha \mathbf{e}_1^T A$, then $\forall i$ there exist $\xi_i$ such that $\mathbf{e}_i^T A = \xi_i \mathbf{e}_1^T A$. In other words, in order to make a $\tau$ matrix of rank one it is sufficient to impose that its second row (column) is a multiple of its first row (column).
proof: Let $\mathbf{z}^T$ be the first row of $A$. Then $A = \tau(\mathbf{z})$. The proof is by induction on $i$:

$$\begin{aligned}
\mathbf{e}_i^T A &= \mathbf{e}_i^T \tau(\mathbf{z}) = \mathbf{z}^T \tau(\mathbf{e}_i) = \mathbf{z}^T(\tau(\mathbf{e}_{i-1})\tau(\mathbf{e}_2) - \tau(\mathbf{e}_{i-2})) \\
&= \mathbf{e}_{i-1}^T \tau(\mathbf{z})\tau(\mathbf{e}_2) - \mathbf{e}_{i-2}^T \tau(\mathbf{z}) = \xi_{i-1}\mathbf{z}^T \tau(\mathbf{e}_2) - \xi_{i-2}\mathbf{z}^T \\
&= \xi_{i-1}\xi_2 \mathbf{z}^T - \xi_{i-2}\mathbf{z}^T = (\xi_{i-1}\xi_2 - \xi_{i-2})\mathbf{z}^T.
\end{aligned}$$

So, $\xi_1 = 1$, $\xi_2 = \alpha$, $\xi_i = \xi_{i-1}\xi_2 - \xi_{i-2}$, $i = 3, \ldots, n$.

$n$ even:

$n = 2$:

Assume that $A \in \mathbb{C}^{2\times 2}$ is in $\tau$. Then, since it is symmetric, is persymmetric, and its second column is a multiple of its first column, it must be (unless a multiplier) of the type

$$A = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}, \ \alpha^2 = 1.$$

$$(1): \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \alpha - 1 = 0$$

$$(-1): \qquad A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \alpha + 1 = 0$$

So, there are two $2 \times 2$ rank one $\tau$ matrices. Note that they are orthogonal with respect the inner product $(\cdot,\cdot)_F$ or, equivalently, the two vectors that define them are orthogonal.

$n = 4$:

Assume that $A \in \mathbb{C}^{4\times 4}$ is in $\tau$. Then, since it is symmetric, is persymmetric, and its second column is a multiple of its first column, it must be (unless a multiplier) of the type

$$A = \begin{bmatrix} 1 & \alpha & & \\ \alpha & 1+\beta & & \\ \beta & \alpha+\gamma & & \\ \gamma & \beta & & \end{bmatrix}, \ \alpha^2 = 1+\beta, \ \alpha\beta = \alpha+\gamma, \ \alpha\gamma = \beta.$$

Observe that $\gamma \neq 0$ since $\gamma = 0$ would imply $\alpha = \beta = 0$, $\beta = -1$. So, the conditions become $\gamma\alpha^2 = \gamma + \alpha\gamma^2$, $\alpha^2\gamma = \alpha + \gamma$, $\alpha\gamma = \beta$. They imply $\alpha\gamma^2 = \alpha$, and thus (since $\alpha \neq 0$) $\gamma^2 = 1$.

So, we have necessarily either $\gamma = 1$, $\beta = \alpha$ or $\gamma = -1$, $\beta = -\alpha$. Such cases will be referred respectively (1) and (−1):

$$(1): \qquad A = \begin{bmatrix} 1 & & & \\ \alpha & & & \\ \alpha & & & \\ 1 & & & \end{bmatrix},$$

$$\alpha(\alpha) = 1 + \alpha, \ \textit{or}$$

$$g_2^+(\alpha) = \alpha(\alpha - 1) - 1 = \alpha^2 - \alpha - 1 = 0, \ \alpha = \tfrac{1 \pm \sqrt{5}}{2}$$

($\alpha(\alpha)$ must be equal to $1+\alpha$ by the cross-sum condition applied for $(i, j) = (2, 1)$)

$$(-1): \qquad A = \begin{bmatrix} 1 & & & \\ \alpha & & & \\ -\alpha & & & \\ -1 & & & \end{bmatrix},$$

$$\alpha(\alpha) = 1 - \alpha \ \textit{or}$$

$$g_2^-(\alpha) = \alpha(\alpha + 1) - 1 = \alpha^2 + \alpha - 1 = 0, \ \alpha = \tfrac{-1 \pm \sqrt{5}}{2}$$

($\alpha(\alpha)$ must be equal to $1 + (-\alpha)$ by the cross-sum condition applied for $(i, j) = (2, 1)$).

Note that the zeros of $g_2^-$ are the opposite of the zeros of $g_2^+$.

So, there are four $4 \times 4$ rank one $\tau$ matrices. Note that they are orthogonal with respect the inner product $(\cdot, \cdot)_F$ or, equivalently, the four vectors that define them are orthogonal.

$n = 6$:

$$A = \begin{bmatrix} 1 & \alpha & & & & \\ \alpha & 1+\beta & & & & \\ \beta & \alpha+\gamma & & & & \\ \gamma & \beta+\delta & & & & \\ \delta & \gamma+\sigma & & & & \\ \sigma & \delta & & & & \end{bmatrix},$$

$$\alpha^2 = 1 + \beta, \ \alpha\beta = \alpha + \gamma, \ \alpha\gamma = \beta + \delta, \ \alpha\delta = \gamma + \sigma, \ \alpha\sigma = \delta.$$

The above conditions imply necessarily either $\sigma = 1$, $\delta = \alpha$, $\gamma = \beta$ or $\sigma = -1$, $\delta = -\alpha$, $\gamma = -\beta$. Such cases will be referred respectively (1) and (−1):

$$(1): \qquad A = \begin{bmatrix} 1 & & & & & \\ \alpha & & & & & \\ \alpha^2 - 1 & & & & & \\ \alpha^2 - 1 & & & & & \\ \alpha & & & & & \\ 1 & & & & & \end{bmatrix},$$

$$\alpha(\alpha^2 - 1) = \alpha + (\alpha^2 - 1) \ \textit{or}$$

$$g_3^+(\alpha) = \alpha(\alpha^2 - \alpha - 1) - (\alpha - 1) = \alpha^3 - \alpha^2 - 2\alpha + 1 = 0$$

($\alpha(\alpha^2 - 1)$ must be equal to $\alpha + (\alpha^2 - 1)$ by the cross-sum condition applied for

$(i,j) = (3,1))$

$$(-1): \qquad A = \begin{bmatrix} 1 & & & & & \\ \alpha & & & & & \\ \alpha^2-1 & & & & & \\ -(\alpha^2-1) & & & & & \\ -\alpha & & & & & \\ -1 & & & & & \end{bmatrix},$$

$$\alpha(\alpha^2-1) = \alpha - (\alpha^2-1) \ or$$

$$g_3^-(\alpha) = \alpha(\alpha^2+\alpha-1) - (\alpha+1) = \alpha^3+\alpha^2-2\alpha-1 = 0$$

($\alpha(\alpha^2-1)$ must be equal to $\alpha + (-(\alpha^2-1))$ by the cross-sum condition applied for $(i,j) = (3,1)$).

Note that the zeros of $g_3^-$ are the opposite of the zeros of $g_3^+$. Note, moreover, that the zeros of $g_3^+$ ($g_3^-$) are distinct.

So, there are six linearly independent $6\times 6$ rank one $\tau$ matrices. Computer says that if $\alpha_k^\pm$, $k = 1,2,3$, are the zeros of $g_3^\pm$, then

$$1 + \alpha_k^\pm \alpha_s^\pm + ((\alpha_k^\pm)^2 - 1)((\alpha_s^\pm)^2 - 1) = 0,$$

i.e. the vector $[1\ \alpha_k^\pm\ (\alpha_k^\pm)^2-1\ \pm((\alpha_k^\pm)^2-1)\ \pm\alpha_k^\pm\ \pm 1]^T$ is orthogonal to the vector $[1\ \alpha_s^\pm\ (\alpha_s^\pm)^2-1\ \pm((\alpha_s^\pm)^2-1)\ \pm\alpha_s^\pm\ \pm 1]^T$. It follows that such six $6\times 6$ rank one $\tau$ matrices are orthogonal with respect the inner product $(\cdot,\cdot)_F$ (because the six vectors that define them are orthogonal).

Question:

$$\left.\begin{array}{l} \alpha^3-\alpha^2-2\alpha+1=0 \\ \beta^3-\beta^2-2\beta+1=0 \end{array}\right\} \Rightarrow\ 1+\alpha\beta+(\alpha^2-1)(\beta^2-1) = 0$$

? Answer: since the roots are: 1.802, $-1.25$, 0.445 (Tommaso 18/11/2010), it seems yes

$n = 8$:

$$A = \begin{bmatrix} 1 & \alpha & & & & & & \\ \alpha & 1+\beta & & & & & & \\ \beta & \alpha+\gamma & & & & & & \\ \gamma & \beta+\delta & & & & & & \\ \delta & \gamma+\sigma & & & & & & \\ \sigma & \delta+\rho & & & & & & \\ \rho & \sigma+x & & & & & & \\ x & \rho & & & & & & \end{bmatrix},$$

$$\alpha^2 = 1+\beta,\ \alpha\beta = \alpha+\gamma,\ \alpha\gamma = \beta+\delta,\ \alpha\delta = \gamma+\sigma,\ \alpha\sigma = \delta+\rho,\ \alpha\rho = \sigma+x,\ \alpha x = \rho.$$

The above conditions imply necessarily either $x = 1$, $\rho = \alpha$, $\sigma = \beta$, $\delta = \gamma$ or $x = -1$, $\rho = -\alpha$, $\sigma = -\beta$, $\delta = -\gamma$. Such cases will be referred respectively (1) and $(-1)$:

$$(1): \qquad A = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2-1 \\ \alpha^3-2\alpha \\ \alpha^3-2\alpha \\ \alpha^2-1 \\ \alpha \\ 1 \end{bmatrix},$$

$$\alpha(\alpha^3-2\alpha) = (\alpha^2-1)+(\alpha^3-2\alpha) \textit{ or}$$
$$g_4^+(\alpha) = \alpha(\alpha^3-\alpha^2-2\alpha+1)-(\alpha^2-\alpha-1) = \alpha^4-\alpha^3-3\alpha^2+2\alpha+1$$
$$= (\alpha-1)(\alpha^3-3\alpha-1) = 0$$

($\alpha(\alpha^3-2\alpha)$ must be equal to $(\alpha^2-1)+(\alpha^3-2\alpha)$ by the cross-sum condition applied for $(i,j)=(4,1)$)

$$(-1): \qquad A = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2-1 \\ \alpha^3-2\alpha \\ -(\alpha^3-2\alpha) \\ -(\alpha^2-1) \\ -\alpha \\ -1 \end{bmatrix},$$

$$\alpha(\alpha^3-2\alpha) = (\alpha^2-1)-(\alpha^3-2\alpha) \textit{ or}$$
$$g_4^-(\alpha) = \alpha(\alpha^3+\alpha^2-2\alpha-1)-(\alpha^2+\alpha-1) = \alpha^4+\alpha^3-3\alpha^2-2\alpha+1$$
$$= (\alpha+1)(\alpha^3-3\alpha+1) = 0$$

($\alpha(\alpha^3-2\alpha)$ must be equal to $(\alpha^2-1)+(-(\alpha^3-2\alpha))$ by the cross-sum condition applied for $(i,j)=(4,1)$).

Note that the zeros of $g_4^-$ are the opposite of the zeros of $g_4^+$. ??? FROM HERE Note, moreover, that the zeros of $g_4^+$ ($g_4^-$) are distinct.

So, there are eight linearly independent $8\times 8$ rank one $\tau$ matrices. Computer says that if $\alpha_k^\pm$, $k=1,2,3,4$, are the zeros of $g_4^\pm$, then

$$1+\alpha_k^\pm\alpha_s^\pm+((\alpha_k^\pm)^2-1)((\alpha_s^\pm)^2-1)+((\alpha_k^\pm)^3-2\alpha_k^\pm)((\alpha_s^\pm)^3-2\alpha_s^\pm)=0,$$

i.e. the vector $[1\ \alpha_k^\pm\ (\alpha_k^\pm)^2-1\ (\alpha_k^\pm)^3-2\alpha_k^\pm\ \pm((\alpha_k^\pm)^3-2\alpha_k^\pm)\ \pm((\alpha_k^\pm)^2-1)\ \pm\alpha_k^\pm\ \pm 1]^T$ is orthogonal to the vector $[1\ \alpha_s^\pm\ (\alpha_s^\pm)^2-1\ (\alpha_s^\pm)^3-2\alpha_s^\pm\ \pm((\alpha_s^\pm)^3-2\alpha_s^\pm)\ \pm((\alpha_s^\pm)^2-1)\ \pm\alpha_s^\pm\ \pm 1]^T$. It follows that such six $8\times 8$ rank one $\tau$ matrices are orthogonal with respect the inner product $(\cdot,\cdot)_F$ (because the eight vectors that define them are orthogonal).TO HERE I HAVE TO CHECK ???

Set $p_0(\alpha)=1$, $p_1(\alpha)=\alpha$, $p_{i+1}(\alpha)=\alpha p_i(\alpha)-p_{i-1}(\alpha)$, $i=1,2,\ldots$ ($p_i(\alpha)$ is the characteristic polynomial of the $i\times i$ upper-left submatrix of $P_0+P_0^T$). We observe, in general, that for $n$ even generic we have the following $n$ rank one $\tau$ matrices:

$$(1): \ (\mathbf{u}_k^+)(\mathbf{u}_k^+)^T, \ \mathbf{u}_k^+ = \begin{bmatrix} \mathbf{x}_k^+ \\ J\mathbf{x}_k^+ \end{bmatrix}, \ \mathbf{x}_k^+ = \begin{bmatrix} p_0(\alpha_k^+) \\ p_1(\alpha_k^+) \\ \\ p_{\frac{n}{2}-1}(\alpha_k^+) \end{bmatrix}, \ k=1,\ldots,\tfrac{n}{2},$$

$\alpha_k^+$ zeri di $\alpha p_{\frac{n}{2}-1}(\alpha) = p_{\frac{n}{2}-1}(\alpha)+p_{\frac{n}{2}-2}(\alpha)$

If the polynomial

$$g^{+}_{\frac{n}{2}}(\alpha) = \alpha p_{\frac{n}{2}-1}(\alpha) - p_{\frac{n}{2}-1}(\alpha) - p_{\frac{n}{2}-2}(\alpha)$$

has distinct real zeros (I have to check if this is true), then the vectors $\mathbf{u}_k^+$ are linearly independent. (Such polynomial should coincide with the polynomial obtained in the particular cases (1) $n = 2, 4, 6, 8$).

$$(-1):\ (\mathbf{u}_k^-)(\mathbf{u}_k^-)^T,\ \mathbf{u}_k^- = \begin{bmatrix} \mathbf{x}_k^- \\ -J\mathbf{x}_k^- \end{bmatrix},\ \mathbf{x}_k^- = \begin{bmatrix} p_0(\alpha_k^-) \\ p_1(\alpha_k^-) \\ \\ p_{\frac{n}{2}-1}(\alpha_k^-) \end{bmatrix},\ k = 1, \dots, \tfrac{n}{2},$$

$\alpha_k^-$ zeri di $\alpha p_{\frac{n}{2}-1}(\alpha) = -p_{\frac{n}{2}-1}(\alpha) + p_{\frac{n}{2}-2}(\alpha)$

If the polynomial

$$g^{-}_{\frac{n}{2}}(\alpha) = \alpha p_{\frac{n}{2}-1}(\alpha) + p_{\frac{n}{2}-1}(\alpha) - p_{\frac{n}{2}-2}(\alpha)$$

has distinct real zeros (I have to check if this is true), then the vectors $\mathbf{u}_k^-$ are linearly independent. (Such polynomial should coincide with the polynomial obtained in the particular cases $(-1)$ $n = 2, 4, 6, 8$).

Note that the zeros of $g^+_{\frac{n}{2}}$ are the opposite of the zeros of $g^-_{\frac{n}{2}}$ (I have to check if this is true).

Note that $(\mathbf{u}_k^+)^T(\mathbf{u}_s^-) = 0\ \forall\, k, s$. So, if the "distinct condition" on the zeros is satisfied, then $\{(\mathbf{u}_k^+)(\mathbf{u}_k^+)^T, (\mathbf{u}_k^-)(\mathbf{u}_k^-)^T : k = 1, \dots, \frac{n}{2}\}$ is a set of $n$ linearly independent rank one $\tau$ matrices, and $\tau =$ Span $\{(\mathbf{u}_k^+)(\mathbf{u}_k^+)^T\}$ + Span $\{(\mathbf{u}_k^-)(\mathbf{u}_k^-)^T\}$ with Span $\{(\mathbf{u}_k^+)(\mathbf{u}_k^+)^T\}$ orthogonal to Span $\{(\mathbf{u}_k^-)(\mathbf{u}_k^-)^T\}$.

For $n = 4$ it also happens that $(\mathbf{u}_1^+)^T(\mathbf{u}_2^+) = 0 = (\mathbf{u}_1^-)^T(\mathbf{u}_2^-)$. So, for $n = 4$ we have 4 *orthogonal* rank one $\tau$ matrices.

Is for $n = 6$ yet true that $(\mathbf{u}_k^+)^T(\mathbf{u}_s^+) = 0 = (\mathbf{u}_k^-)^T(\mathbf{u}_s^-)$ $s \neq k$ $s, k = 1, \dots, 3$ ? In other words, is for $n = 6$ the basis $\{(\mathbf{u}_k^+)(\mathbf{u}_k^+)^T, (\mathbf{u}_k^-)(\mathbf{u}_k^-)^T : k = 1, 2, 3\}$ of $\tau$ an orthogonal basis ? SOLVED with Computer with "yes" (see above).

For $n = 8$?

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Call $\mathbf{u}_k$, $k = 1, \dots, n$, the above (orthogonal) vectors $\mathbf{u}_k^+$ and $\mathbf{u}_k^-$. Let $\mathbf{v}$ be such that $\tau_1(\mathbf{v}) = Sd(S^T\mathbf{v})d(S^T\mathbf{e}_1)^{-1}S^{-1}$ is invertible ($d(S^T\mathbf{v})$ non singular), so that the matrix $\tau_{\mathbf{v}}(\mathbf{z})$ is well defined. Note that $\mathbf{v}^T\mathbf{u}_k \neq 0\ \forall\, k$ ... (this fact is assured by the the assumption $d(S^T\mathbf{v})$ non singular, since, we shall see, the $\mathbf{u}_k$ are nothing else, unless a multiplier, the columns of $S$).

Since the $\mathbf{u}_k\mathbf{u}_k^T$ form an (orthogonal) basis for $\tau$, there exist $c_k$ such that $\tau_{\mathbf{v}}(\mathbf{z}) = \sum_{k=1}^n c_k\mathbf{u}_k\mathbf{u}_k^T$. We want to give a formula for such $c_k$.

$$\begin{aligned} Sd(S^T\mathbf{z})d(S^T\mathbf{v})^{-1}S^{-1} &= \textstyle\sum_{k=1}^n c_k Sd(S^T(\mathbf{u}_k\mathbf{u}_k^T)\mathbf{v})d(S^T\mathbf{v})^{-1}S^{-1} \\ &= \textstyle\sum_{k=1}^n c_k(\mathbf{u}_k^T\mathbf{v})Sd(S^T\mathbf{u}_k)d(S^T\mathbf{v})^{-1}S^{-1} \end{aligned}$$

if and only if

$$\mathbf{z} = \sum_{k=1}^n c_k(\mathbf{u}_k^T\mathbf{v})\mathbf{u}_k.$$

Note that the $\mathbf{u}_k^T\mathbf{v}$ must be all non zero; in fact, if one of them is zero, we would have that any vector $\mathbf{z}$ (with $n$ entries) can be written as a linear combination of only $n-1$ vectors. For the $c_k$ in the latter equality we can obtain an explicit formula; in fact, by the orthogonality of the $\mathbf{u}_k$,

$$\mathbf{u}_s^T\mathbf{z} = \sum_k c_k(\mathbf{u}_k^T\mathbf{v})\mathbf{u}_s^T\mathbf{u}_k = c_s(\mathbf{u}_s^T\mathbf{v})\mathbf{u}_s^T\mathbf{u}_s.$$

Thus

$$c_s = \frac{\mathbf{u}_s^T\mathbf{z}}{(\mathbf{u}_s^T\mathbf{v})(\mathbf{u}_s^T\mathbf{u}_s)}.$$

So

$$\tau_{\mathbf{v}}(\mathbf{z}) = \sum_{k=1}^{n} \frac{\mathbf{u}_k^T\mathbf{z}}{(\mathbf{u}_k^T\mathbf{v})(\mathbf{u}_k^T\mathbf{u}_k)}\mathbf{u}_k\mathbf{u}_k^T$$

Question: $\mathbf{u}_k^T\mathbf{u}_k = ?$
Question: is

$$\mathbf{v}^T = [1\ 1\ \cdots\ 1]\begin{bmatrix} \mathbf{u}_1 & & \mathbf{u}_n \end{bmatrix}^{-1} = [1\ 1\ \cdots\ 1]\begin{bmatrix} \frac{1}{\mathbf{u}_1^T\mathbf{u}_1}\mathbf{u}_1^T \\ \\ \frac{1}{\mathbf{u}_n^T\mathbf{u}_n}\mathbf{u}_n^T \end{bmatrix}$$

(which is such that $\mathbf{v}^T\mathbf{u}_k = 1\ \forall\, k$) such that $\tau_1(\mathbf{v})$ is invertible ? ... yes since in such case

$$[S^T\mathbf{v}]_i = [S^T[\frac{1}{\mathbf{u}_1^T\mathbf{u}_1}\mathbf{u}_1\ \cdots\ \frac{1}{\mathbf{u}_n^T\mathbf{u}_n}\mathbf{u}_n]\begin{bmatrix} 1 \\ \\ 1 \end{bmatrix}]_i \neq 0,\ \forall\, i$$

and the $\mathbf{u}_k$, we shall see, are nothing else, unless a multiplier, the columns of $S$.

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I think I have simply found again the sine transform $S$, that is the matrix $\sin(ij\pi/(n+1))$.

More precisely, the vectors $\mathbf{u}_k^+$ e $\mathbf{u}_k^-$ that define my $n$ rank one orthogonal $\tau$ matrices $((\mathbf{u}_k^+)(\mathbf{u}_k^+)^T$ e $(\mathbf{u}_k^-)(\mathbf{u}_k^-)^T$, $k = 1, ..., n/2)$ are nothing else, unless a multiplier, the columns of the sine matrix. I have observed this for small values of $n$.

Thus, it is obvious that the $(\mathbf{u}_k^+)(\mathbf{u}_k^+)^T$, $(\mathbf{u}_k^-)(\mathbf{u}_k^-)^T$, $k = 1, ..., n/2$, are a basis, orthogonal with respect to $(\cdot,\cdot)_F$, of $\tau$.
proof: if $A \in \tau$, then $A = SDS^T = \sum_i D_{ii}(column\ i\ of\ S)(column\ i\ of\ S)^T$.

Even the fact that such $n$ matrices $(\mathbf{u}_k^+)(\mathbf{u}_k^+)^T$, $(\mathbf{u}_k^-)(\mathbf{u}_k^-)^T$, $k = 1, ..., n/2$, are $\tau$ matrices, is not a novelty.
proof: since for any diagonal $D$ the matrix $SDS^T$ belongs $\tau$, it is sufficient to choose $D = \mathbf{e}_i\mathbf{e}_i^T$ in order to prove that the matrices $(column\ i\ of\ S)(column\ i\ of\ S)^T$ are in $\tau$.

Moreover, rank of $A \in \tau$ is 1 if and only if rank of $D$ in $A = SDS$ is 1 if and only if $D = \mathbf{e}_i\mathbf{e}_i^T$ unless a multiplier.

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Consider the unitary sine matrix $S$. Note that the columns $\mathbf{c}_j$ $(j = 1, \ldots, n)$ of the matrix

$$S\sqrt{(n+1)/2}\,\mathrm{diag}\,((1/\sin\frac{j\pi}{n+1}) : j = 1, \ldots, n)$$

are orthogonal and their first entries are $(\mathbf{c}_j)_1 = 1$, $(\mathbf{c}_j)_2 = 2\cos\frac{j\pi}{n+1}$. Note that the $(\mathbf{c}_j)_2$ coincide with the zeros of the two polynomials

$$\begin{aligned} g^+_{\frac{n}{2}}(\alpha) &= \alpha p_{\frac{n}{2}-1}(\alpha) - p_{\frac{n}{2}-1}(\alpha) - p_{\frac{n}{2}-2}(\alpha), \\ g^-_{\frac{n}{2}}(\alpha) &= \alpha p_{\frac{n}{2}-1}(\alpha) + p_{\frac{n}{2}-1}(\alpha) - p_{\frac{n}{2}-2}(\alpha). \end{aligned}$$

The $(\mathbf{c}_j)_2$ are also the eigenvalues of $P_0 + P_0^T$, i.e. the zeros of the polynomial $p_n(\alpha)$ defined by the sequence

$$p_0(\alpha) = 1,\ p_1(\alpha) = \alpha,\ p_{i+1}(\alpha) = \alpha p_i(\alpha) - p_{i-1}(\alpha),\ i = 1, \dots, n-1.$$

Moreover, they are twice the stationary points of the chebycev polynomial $T_{n+1}(\alpha)$. More precisely, if $T'_{n+1}(\alpha) = \sum_{s=0}^n a_s\alpha^s$, then the $(\mathbf{c}_j)_2$ are the zeros of the polynomial $q_n(\alpha) = \sum_{s=0}^n \frac{1}{2^s} a_s\alpha^s$. For example, since $T_0(\alpha) = 1$, $T_1(\alpha) = \alpha$, $T_2(\alpha) = 2\alpha T_1(\alpha) - T_0(\alpha) = 2\alpha^2 - 1$, $T_3(\alpha) = 2\alpha T_2(\alpha) - T_1(\alpha) = 4\alpha^3 - 3\alpha$, $T_4(\alpha) = 8\alpha^4 - 8\alpha^2 + 1$, $T_5(\alpha) = 16\alpha^5 - 20\alpha^3 + 5\alpha$, we have $T'_3(\alpha) = 12\alpha^2 - 3$, $T'_5(\alpha) = 5(16\alpha^4 - 12\alpha^2 + 1)$, so $q_2(\alpha) = 3\alpha^2 - 3 = 3(\alpha^2 - 1)$, $q_4(\alpha) = 5(\alpha^4 - 3\alpha^2 + 1)$.
Verify that $2\cos\frac{j\pi}{7}$, $j = 1, \dots, 6$, are the zeros of

$$\begin{aligned} &g_3^+(\alpha) = \alpha(\alpha^2 - \alpha - 1) - (\alpha - 1) = \alpha^3 - \alpha^2 - 2\alpha + 1 = 0 \ \textit{Tommaso} \\ &g_3^-(\alpha) = \alpha(\alpha^2 + \alpha - 1) - (\alpha + 1) = \alpha^3 + \alpha^2 - 2\alpha - 1 = 0 \end{aligned}$$

Verify that $2\cos\frac{j\pi}{9}$, $j = 1, \dots, 8$, are the zeros of

$$\begin{aligned} &g_4^+(\alpha) = \alpha(\alpha^3 - \alpha^2 - 2\alpha + 1) - (\alpha^2 - \alpha - 1) = \alpha^4 - \alpha^3 - 3\alpha^2 + 2\alpha + 1 = (\alpha - 1)(\alpha^3 - 3\alpha - 1) \\ &g_4^-(\alpha) = \alpha(\alpha^3 + \alpha^2 - 2\alpha - 1) - (\alpha^2 + \alpha - 1) = \alpha^4 + \alpha^3 - 3\alpha^2 - 2\alpha + 1 = (\alpha + 1)(\alpha^3 - 3\alpha + 1) \end{aligned}$$

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Set $p_{-1}(\alpha) = 0$, and

$$\begin{aligned} &p_0(\alpha) = 1,\ g_1^\pm(\alpha) = (\alpha \mp 1)p_0(\alpha) - p_{-1}(\alpha), \\ &p_1(\alpha) = \alpha,\ g_2^\pm(\alpha) = (\alpha \mp 1)p_1(\alpha) - p_0(\alpha), \\ &p_i(\alpha) = \alpha p_{i-1}(\alpha) - p_{i-2}(\alpha),\ g_{i+1}^\pm(\alpha) = (\alpha \mp 1)p_i(\alpha) - p_{i-1}(\alpha),\ i = 1, \dots \end{aligned}$$

Note that $p_i(\alpha)$ is the characteristic polynomial of the $i \times i$ upper-left submatrix of $P_0 + P_0^T$.

Introduce also the polynomials:

$$f_0^\pm(\alpha) = 1,\ f_1^\pm(\alpha) = \alpha \mp 1,\ f_i^\pm(\alpha) = \alpha f_{i-1}^\pm(\alpha) - f_{i-2}^\pm(\alpha),\ i = 2, \dots .$$

Note that $f_i^+(\alpha)$ ($f_i^-(\alpha)$) is the characteristic polynomial of the $i \times i$ upper-left submatrix of $P_0 + P_0^T + \mathbf{e}_1\mathbf{e}_1^T$ ($P_0 + P_0^T - \mathbf{e}_1\mathbf{e}_1^T$). Then $g_i^\pm = f_i^\pm$.
proof: for $i = 0$, $i = 1$ it is true:

$$\begin{aligned} g_1^\pm(\alpha) &= (\alpha \mp 1)1 - 0 = \alpha \mp 1 = f_1^\pm(\alpha), \\ g_2^\pm(\alpha) &= (\alpha \mp 1)\alpha - 1 = \alpha(\alpha \mp 1) - 1 = f_2^\pm(\alpha). \end{aligned}$$

Assume the thesis true, and let us show that $g_{i+1}^\pm = f_{i+1}^\pm$:

$$\begin{aligned} g_{i+1}^\pm(\alpha) &= (\alpha \mp 1)p_i(\alpha) - p_{i-1}(\alpha) = (\alpha \mp 1)(\alpha p_{i-1}(\alpha) - p_{i-2}(\alpha)) - p_{i-1}(\alpha) \\ &= \alpha(\alpha \mp 1)p_{i-1}(\alpha) - \alpha p_{i-2}(\alpha) \pm p_{i-2}(\alpha) - p_{i-1}(\alpha) \\ &= \alpha g_i^\pm(\alpha) \pm p_{i-2}(\alpha) - \alpha p_{i-2}(\alpha) + p_{i-3}(\alpha) \\ &= \alpha g_i^\pm(\alpha) - (p_{i-2}(\alpha)(\alpha \mp 1) - p_{i-3}(\alpha)) \\ &= \alpha g_i^\pm(\alpha) - g_{i-1}^\pm(\alpha) = \alpha f_i^\pm(\alpha) - f_{i-1}^\pm(\alpha) \\ &= f_{i+1}^\pm(\alpha) \end{aligned}$$

*For the characteristic polynomial* $p_{2i}$ *of the* $2i \times 2i$ *matrix* $P_0 + P_0^T$ *we have* $p_{2i}(\alpha) = f_i^-(\alpha) f_i^+(\alpha)$ (prove it!). So, the $2i$ zeros $2\cos\frac{j\pi}{2i+1}$, $j = 1, \dots, 2i$, of $p_{2i}$ are the zeros of $f_i^+$ and of $f_i^-$.

proof: the proof is by induction. The basis of the induction is true:

$$p_0(\alpha) = 1 = f_0^+(\alpha) f_0^-(\alpha),\ f_0^+(\alpha) = 1,\ f_0^-(\alpha) = 1,$$

$$p_2(\alpha) = \alpha^2 - 1 = f_1^+(\alpha) f_1^-(\alpha),\ f_1^+(\alpha) = \alpha - 1,\ f_1^-(\alpha) = \alpha + 1,$$

$$p_4(\alpha) = \alpha^4 - 3\alpha^2 + 1 = f_2^+(\alpha) f_2^-(\alpha),\ f_2^+(\alpha) = \alpha^2 - \alpha - 1,\ f_2^-(\alpha) = \alpha^2 + \alpha - 1.$$

Assume $p_{2j}(\alpha) = f_j^+(\alpha) f_j^-(\alpha)$, $j = 0, 1, \dots, i-1$. Then

$$p_{2i}(\alpha) = \alpha p_{2i-1}(\alpha) - p_{2i-2}(\alpha) = (\alpha^2 - 1) p_{2i-2}(\alpha) - \alpha p_{2i-3}(\alpha) = \ *$$

(use the identity: $p_{2i-1} = \alpha p_{2i-2} - p_{2i-3}$). Since the identities

$$p_{2i-4}(\alpha) = \alpha p_{2i-5}(\alpha) - p_{2i-6}(\alpha),\ \alpha p_{2i-3}(\alpha) = \alpha^2 p_{2i-4}(\alpha) - \alpha p_{2i-5}(\alpha)$$

imply $p_{2i-4}(\alpha) + \alpha p_{2i-3}(\alpha) = \alpha^2 p_{2i-4}(\alpha) - p_{2i-6}(\alpha)$, we have the equality $p_{2i-3}(\alpha) = \frac{1}{\alpha}((\alpha^2 - 1) p_{2i-4}(\alpha) - p_{2i-6}(\alpha))$. So, $*$ becomes:

$$* = (\alpha^2 - 1) p_{2i-2}(\alpha) - ((\alpha^2 - 1) p_{2i-4}(\alpha) - p_{2i-6}(\alpha)) = (\alpha^2 - 1)(p_{2i-2}(\alpha) - p_{2i-4}(\alpha)) + p_{2i-6}(\alpha).$$

Now, by the inductive hypothesis,

$$\begin{aligned}
p_{2i}(\alpha) &= (\alpha^2 - 1)(f_{i-1}^+(\alpha) f_{i-1}^-(\alpha) - f_{i-2}^+(\alpha) f_{i-2}^-(\alpha)) + f_{i-3}^+(\alpha) f_{i-3}^-(\alpha) \\
&= \alpha^2 f_{i-1}^+(\alpha) f_{i-1}^-(\alpha) + f_{i-2}^+(\alpha) f_{i-2}^-(\alpha) - \alpha^2 f_{i-2}^+(\alpha) f_{i-2}^-(\alpha) \\
&\quad - f_{i-1}^+(\alpha) f_{i-1}^-(\alpha) + f_{i-3}^+(\alpha) f_{i-3}^-(\alpha) \\
&= \alpha^2 f_{i-1}^+(\alpha) f_{i-1}^-(\alpha) + f_{i-2}^+(\alpha) f_{i-2}^-(\alpha) - \alpha^2 f_{i-2}^+(\alpha) f_{i-2}^-(\alpha) \\
&\quad - f_{i-1}^+(\alpha) f_{i-1}^-(\alpha) + (\alpha f_{i-2}^+(\alpha) - f_{i-1}^+(\alpha))(\alpha f_{i-2}^-(\alpha) - f_{i-1}^-(\alpha)) \\
&= \alpha^2 f_{i-1}^+(\alpha) f_{i-1}^-(\alpha) + f_{i-2}^+(\alpha) f_{i-2}^-(\alpha) - \alpha(f_{i-2}^+(\alpha) f_{i-1}^-(\alpha) + f_{i-1}^+(\alpha) f_{i-2}^-(\alpha)) \\
&= (\alpha f_{i-1}^+(\alpha) - f_{i-2}^+(\alpha))(\alpha f_{i-1}^-(\alpha) - f_{i-2}^-(\alpha)) \\
&= f_i^+(\alpha) f_i^-(\alpha)
\end{aligned}$$

*Exercise*. In $n$ is even, then the eigenvalues of the $n \times n$ matrix $P_0 + P_0^T$ are the eigenvalues of the following two $\frac{n}{2} \times \frac{n}{2}$ matrices:

$$\begin{bmatrix} 1 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & & 1 \\ & & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & & 1 \\ & & 1 & 0 \end{bmatrix}.$$

---

Let us come back to the odd case. Recall:

$n = 3$

$$A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{bmatrix}$$

Note on the second entries: 0 is eigenvalue of $[0]$; $\pm\sqrt{2}$ are eigenvalues of $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

$n = 5$

$$\begin{bmatrix} 1 \\ \pm 1 \\ 0 \\ \mp 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \pm\sqrt{3} \\ 2 \\ \pm\sqrt{3} \\ 1 \end{bmatrix}$$

Note on the second entries: $\pm 1$ are eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; 0 and $\pm\sqrt{3}$ are eigenvalues of $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$n = 7$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ \pm\sqrt{2} \\ 1 \\ 0 \\ -1 \\ \mp\sqrt{2} \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 - 1 \\ \alpha^3 - 2\alpha \\ \alpha^2 - 1 \\ \alpha \\ 1 \end{bmatrix}, \ \alpha = \pm\sqrt{2 - \sqrt{2}}, \ \alpha = \pm\sqrt{2 + \sqrt{2}}$$

Note on the second entries: 0 and $\pm\sqrt{2}$ are eigenvalues of $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; $\pm\sqrt{2 - \sqrt{2}}$, $\pm\sqrt{2 + \sqrt{2}}$ are eigenvalues of $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

For $n$ odd generic we have

$$A = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 - 1 \\ \\ p_{\frac{n-3}{2}}(\alpha) \\ p_{\frac{n-1}{2}}(\alpha) \\ \rho p_{\frac{n-3}{2}}(\alpha) \\ \\ \rho(\alpha^2 - 1) \\ \rho\alpha \\ \rho \end{bmatrix}$$

and necessarily $\rho = \pm 1$. If $\rho = 1$ we have the further condition $\alpha p_{\frac{n-1}{2}}(\alpha) = 2p_{\frac{n-3}{2}}(\alpha)$, or $0 = g_{\frac{n+1}{2}}(\alpha) := \alpha p_{\frac{n-1}{2}}(\alpha) - 2p_{\frac{n-3}{2}}(\alpha)$. If $\rho = -1$, we have the further conditions $p_{\frac{n-1}{2}}(\alpha) + p_{\frac{n-5}{2}}(\alpha) = \alpha p_{\frac{n-3}{2}}(\alpha)$, $p_{\frac{n-1}{2}}(\alpha) - p_{\frac{n-5}{2}}(\alpha) = -\alpha p_{\frac{n-3}{2}}(\alpha)$ which imply $p_{\frac{n-1}{2}}(\alpha) = 0$. Viceversa, such conditions ($0 = g_{\frac{n+1}{2}}(\alpha)$, in case $\rho = 1$, and $p_{\frac{n-1}{2}}(\alpha) = 0$, in case $\rho = -1$) imply that the second column of $A$ is $\alpha$ times the first one.

So, we have the following *Rank one $\tau$ matrices.*

$$(\mathbf{u}_k^+)(\mathbf{u}_k^+)^T,\ \mathbf{u}_k^+ = \begin{bmatrix} \mathbf{x}_k^+ \\ p_{\frac{n-1}{2}}(\alpha_k^+) \\ J\mathbf{x}_k^+ \end{bmatrix},\ \mathbf{x}_k^+ = \begin{bmatrix} 1 \\ \alpha_k^+ \\ (\alpha_k^+)^2 - 1 \\ \\ p_{\frac{n-3}{2}}(\alpha_k^+) \end{bmatrix},\ k = 1, \ldots, \tfrac{n+1}{2},$$

$\alpha_k^+$ roots of $g_{\frac{n+1}{2}}(\alpha) = \alpha p_{\frac{n-1}{2}}(\alpha) - 2p_{\frac{n-3}{2}}(\alpha)$.

Prove that $g_{\frac{n+1}{2}}(\alpha)$ is the characteristic polynomial of the $\frac{n+1}{2} \times \frac{n+1}{2}$ matrix

$$\begin{bmatrix} 0 & 2 & 0 & \\ 1 & 0 & 1 & \\ 0 & 1 & & \\ & & & \end{bmatrix}.$$

$$(\mathbf{u}_k^-)(\mathbf{u}_k^-)^T,\ \mathbf{u}_k^- = \begin{bmatrix} \mathbf{x}_k^- \\ 0 \\ -J\mathbf{x}_k^- \end{bmatrix},\ \mathbf{x}_k^- = \begin{bmatrix} 1 \\ \alpha_k^- \\ (\alpha_k^-)^2 - 1 \\ \\ p_{\frac{n-3}{2}}(\alpha_k^-) \end{bmatrix},\ k = 1, \ldots, \tfrac{n-1}{2},$$

$\alpha_k^-$ roots of $p_{\frac{n-1}{2}}(\alpha)$

*Exercise.* Prove that $n$ odd $\Rightarrow p_n(\alpha) = p_{\frac{n-1}{2}}(\alpha) g_{\frac{n+1}{2}}(\alpha)$.

Set $f_0(\alpha) = 1,\ f_1(\alpha) = \alpha,\ f_2(\alpha) = \alpha^2 - 2,\ f_{i+1}(\alpha) = \alpha f_i(\alpha) - f_{i-1}(\alpha)$, $i = 2, 3, \ldots$. Note that $f_i(\lambda)$ is the characteristic polynomial of the upper-left $i \times i$ submatrix of $P_0 + \mathbf{e}_1\mathbf{e}_2^T$.

Set $g_0(\alpha) = 1,\ g_{i+1}(\alpha) = \alpha p_i(\alpha) - 2p_{i-1}(\alpha),\ i = 0, 1, \ldots$. Then $g_i(\alpha) = f_i(\alpha),\ i = 0, 1, \ldots$. So the roots of $g_i(\alpha)$ are those of $f_i(\alpha)$: $\pm\sqrt{2}$ $(i = 2)$, $0, \pm\sqrt{3}$ $(i = 3)$, $\pm\sqrt{2 - \sqrt{2}}, \pm\sqrt{2 + \sqrt{2}}$ $(i = 4)$, $\ldots$ or, more in general, $2\cos\frac{j\pi}{2i}$, $j = 1, 3, \ldots, 2i - 1$.

proof: the proof is by induction. $g_1(\alpha) = \alpha p_0(\alpha) - 2p_{-1}(\alpha) = \alpha = f_1(\alpha)$, $g_2(\alpha) = \alpha p_1(\alpha) - 2p_0(\alpha) = \alpha^2 - 2 = f_2(\alpha)$,

$$\begin{aligned} g_{i+1}(\alpha) &= \alpha p_i(\alpha) - 2p_{i-1}(\alpha) \\ &= \alpha(\alpha p_{i-1}(\alpha) - p_{i-2}(\alpha)) - 2(\alpha p_{i-2}(\alpha) - p_{i-3}(\alpha)) \\ &= \alpha(\alpha p_{i-1}(\alpha) - 2p_{i-2}(\alpha)) - \alpha p_{i-2}(\alpha) + 2p_{i-3}(\alpha) \\ &= \alpha g_i(\alpha) - g_{i-1}(\alpha) \\ &= \alpha f_i(\alpha) - f_{i-1}(\alpha) \\ &= f_{i+1}(\alpha). \end{aligned}$$

$p_{2i+1}(\alpha) = p_i(\alpha) f_{i+1}(\alpha),\ i = 0, 1, \ldots$.
proof: the proof is by induction. The basis of the induction is true:

$$p_1(\alpha) = \alpha = p_0(\alpha) f_1(\alpha),\ p_3(\alpha) = \alpha^3 - 2\alpha = \alpha(\alpha^2 - 2) = p_1(\alpha) f_2(\alpha),$$
$$p_5(\alpha) = \alpha^5 - 4\alpha^3 + 3\alpha = (\alpha^2 - 1)(\alpha^3 - 3\alpha) = p_2(\alpha) f_3(\alpha).$$

Assume $p_{2j+1}(\alpha) = p_j(\alpha) f_{j+1}(\alpha),\ j = 0, 1, \ldots, i - 1$. Then

$$p_{2i+1} = \alpha p_{2i} - p_{2i-1} = (\alpha^2 - 1)p_{2i-1} - \alpha p_{2i-2} = (\alpha^2 - 1)(p_{2i-1} - p_{2i-3}) + p_{2i-5}$$

(note that $p_{2i+1} = \alpha p_{2i} - p_{2i-1}$ and $\alpha p_{2i+2} = \alpha^2 p_{2i+1} - \alpha p_{2i}$ imply $p_{2i+1} + \alpha p_{2i+2} = \alpha^2 p_{2i+1} - p_{2i-1}$, and thus $p_{2i+2} = \frac{1}{\alpha}((\alpha^2 - 1)p_{2i+1} - p_{2i})$). By the inductive assumption:

$$\begin{array}{rcl} p_{2i+1} & = & (\alpha^2 - 1)(p_{i-1}f_i - p_{i-2}f_{i-1}) + p_{i-3}f_{i-2} \\ & = & \alpha^2 p_{i-1}f_i + p_{i-2}f_{i-1} - \alpha^2 p_{i-2}f_{i-1} \\ & & -p_{i-1}f_i + p_{i-3}f_{i-2} \\ & = & \alpha^2 p_{i-1}f_i + p_{i-2}f_{i-1} - \alpha^2 p_{i-2}f_{i-1} \\ & & -p_{i-1}f_i + (\alpha p_{i-2} - p_{i-1}) \\ & & (\alpha f_{i-1} - f_i) \\ & = & \alpha^2 p_{i-1}f_i + p_{i-2}f_{i-1} \\ & & -\alpha p_{i-2}f_i - \alpha p_{i-1}f_{i-1} \\ & = & (\alpha p_{i-1} - p_{i-2})(\alpha f_i - f_{i-1}) \\ & = & p_i f_{i+1}. \end{array}$$

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First row of the two tridiagonal matrices whose eigenvalues, collected together, give the eigenvalues of the $n \times n$ matrix $P_0 + P_0^T$ (the $i$th row, $i \geq 2$, is like the $i$th row, $i \geq 2$, of $P_0 + P_0^T$):

$n$ even:
$n/2 \times n/2$ 1 1 (+ or 1 in the bottom),
$n/2 \times n/2$ $-1$ 1 ($-$ or $-1$ in the bottom)

$n$ odd:
$(n-1)/2 \times (n-1)/2$ 0 1 ($-$ or $-1$ in the bottom),
$(n+1)/2 \times (n+1)/2$ 0 2 (+ or 1 in the bottom)

If we assume $\alpha_k^+$ and $\alpha_k^-$ both ordered in decreasing order ($\ldots \leq \alpha_2^+ \leq \alpha_1^+$, i.e. $\alpha_k^+ = 2\cos\frac{(2k-1)\pi}{n+1}$, $\ldots \leq \alpha_2^- \leq \alpha_1^-$, i.e. $\alpha_k^- = 2\cos\frac{(2k)\pi}{n+1}$), then, taking the first, the $[\frac{n+3}{2}]$th, the second, the $[\frac{n+5}{2}]$th, ..., columns of the following matrix

$$[\mathbf{u}_1^+ \ \cdots \ \mathbf{u}_{[\frac{n+1}{2}]}^+ \ \mathbf{u}_1^- \ \cdots \ \mathbf{u}_{[\frac{n}{2}]}^-]$$

one obtains the sine matrix normalized so that the entries on its first row are all equal to 1.

proof:

$$\begin{array}{l} \sin\frac{2j\pi}{n+1} / \sin\frac{j\pi}{n+1} = 2\cos\frac{j\pi}{n+1} \\ \sin\frac{3j\pi}{n+1} / \sin\frac{j\pi}{n+1} = (2\cos\frac{j\pi}{n+1})^2 - 1 \\ \cdots \end{array}$$

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*Conclusion.* Assume $n$ odd. If $A \in \tau$ is of rank one then there are $\alpha$ and $x$ such that

$$A\mathbf{e}_1 = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2-1 \\ \\ p_{\frac{n-3}{2}}(\alpha) \\ p_{\frac{n-1}{2}}(\alpha) = xp_{\frac{n-1}{2}}(\alpha) \\ xp_{\frac{n-3}{2}}(\alpha) \\ \\ x(\alpha^2-1) \\ x\alpha \\ x \end{bmatrix}.$$

Moreover, the following two conditions must be satisfied

$$(x-1)p_{\frac{n-1}{2}}(\alpha) = 0,\ \alpha p_{\frac{n-1}{2}}(\alpha) = (x+1)p_{\frac{n-3}{2}}(\alpha).$$

Note that $x = 1$ and $p_{\frac{n-1}{2}}(\alpha) = 0$ cannot be simultaneously verified (otherwise we would have $p_{\frac{n-3}{2}}(\alpha) = 0$ and roots of $p_{\frac{n-1}{2}}$ are different from those of $p_{\frac{n-3}{2}}$!). So, either

$$x = 1,\ 0 = f_{\frac{n+1}{2}} := \alpha p_{\frac{n-1}{2}}(\alpha) - 2p_{\frac{n-3}{2}}(\alpha)$$

or

$$x = -1,\ p_{\frac{n-1}{2}}(\alpha) = 0.$$

Assume $n$ even. If $A \in \tau$ is of rank one then there are $\alpha$ and $x$ such that

$$A\mathbf{e}_1 = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2-1 \\ \\ p_{\frac{n}{2}-2}(\alpha) \\ p_{\frac{n}{2}-1}(\alpha) \\ xp_{\frac{n}{2}-1}(\alpha) \\ xp_{\frac{n}{2}-2}(\alpha) \\ \\ x(\alpha^2-1) \\ x\alpha \\ x \end{bmatrix}.$$

Moreover, the following two conditions must be satisfied

$$\alpha p_{\frac{n}{2}-1}(\alpha) = p_{\frac{n}{2}-2}(\alpha) + xp_{\frac{n}{2}-1}(\alpha),\ \alpha xp_{\frac{n}{2}-1}(\alpha) = p_{\frac{n}{2}-1}(\alpha) + xp_{\frac{n}{2}-2}(\alpha)$$

which become (since $x \neq 0$)

$$\alpha xp_{\frac{n}{2}-1}(\alpha) = xp_{\frac{n}{2}-2}(\alpha) + x^2 p_{\frac{n}{2}-1}(\alpha),\ \alpha xp_{\frac{n}{2}-1}(\alpha) = p_{\frac{n}{2}-1}(\alpha) + xp_{\frac{n}{2}-2}(\alpha)$$

and thus imply $(x^2-1)p_{\frac{n}{2}-1}(\alpha) = 0$; but $p_{\frac{n}{2}-1}(\alpha) = 0$ would imply $p_{\frac{n}{2}-2}(\alpha) = 0$ (not possible! see above), so either

$$x = 1,\ f^+_{\frac{n}{2}}(\alpha) := (\alpha-1)p_{\frac{n}{2}-1}(\alpha) - p_{\frac{n}{2}-2}(\alpha) = 0$$

or

$$x = -1,\ f^-_{\frac{n}{2}}(\alpha) := (\alpha+1)p_{\frac{n}{2}-1}(\alpha) - p_{\frac{n}{2}-2}(\alpha) = 0.$$