$A \in \mathbb{C}^{n\times n}$

Theorem HYP.
Let A be such that "$\rho(A)$ is eigenvalue of A and there exists $k \geq 1$ such that A^k is non negative and irreducible" (HYP) (NOTE: *if A is non negative, then* HYP *on A is equivalent to* IRREDUCIBILITY *of A*). Then

(i) $\rho(A)$ is positive, is a simple eigenvalue of A and of A^T, and $\exists !$ $\mathbf{z}$, $\mathbf{w}$ both positive vectors such that $\|\mathbf{z}\|_1 = \|\mathbf{w}\|_1 = 1$ and $A\mathbf{z} = \rho(A)\mathbf{z}$, $A^T\mathbf{w} = \rho(A)\mathbf{w}$.
(ii) There exists a diagonal matrix D with positive diagonal entries such that DAD^{-1} is $\rho(A)$-stochastic by columns (or by rows). Note that $(DAD^{-1})_{ij} \neq 0$ iff $(A)_{ij} \neq 0$, and $(DAD^{-1})_{ij}$ and $(A)_{ij}$ have the same argument [A and DAD^{-1} have the same pattern].
(iii) If $[A^k]_{ii}$ is positive for some i, then the remaining $n-1$ eigenvalues of A have absolute value less than $\rho(A)$.
(iv) If A^k is positive, then the remaining $n-1$ eigenvalues of A have absolute value less than $\rho(A)$.
(v) If the remaining $n-1$ eigenvalues of A have absolute value less than $\rho(A)$, then $\frac{1}{\rho(A)^j}A^j \rightarrow \frac{\mathbf{z}\mathbf{w}^T}{\mathbf{w}^T\mathbf{z}}$, and therefore there exists s such that A^s is positive.
(vi) A is similar to a $\rho(A)$-stochastic by rows and by columns matrix.

PROOF. SEE Appendix.

NOTE. Let A be a non negative $n \times n$ matrix. Then A is primitive ($A \geq 0$, A irreducible, $\rho(A)$ dominates the remaining eigenvalues of A) iff there exists m such that A^m is positive.

EXERCISE (by Fra). Let A be irreducible, with hermitian pattern (in the sense that $a_{ij} \neq 0$ iff $a_{ji} \neq 0$, and, in such case, $a_{ij}a_{ji} \in \mathbb{R}^+$), and such that A^2 is non negative and irreducible. Prove that then there exists m such that A^m is positive.

Corollary HYP. Let A be a stochastic by columns $n \times n$ matrix, i.e. $\sum_i a_{ij} = 1$ $\forall\, j$ ($a_{ij} \in \mathbb{C}$). Assume that there exists $k \geq 1$ such that $A^k \geq 0$. Then 1 is eigenvalue of A and $1 = \rho(A)$ (SEE Appendix). If, moreover, A^k is irreducible, then all assertions (i)–(vi) hold with $\rho(A) = 1$ and $\mathbf{w}^T = \mathbf{e}^T = [1\ 1\ \cdots\ 1]$.

The result stated in the latter Corollary justifies the researches of Riccardo.

At the end of the Appendix, are reported some considerations on $n\times n$ stochastic by columns matrices A (with complex entries), from which one deduces:
- If $m_a^A(1) = m_g^A(1)$, then there exists $\mathbf{z}$ such that $A\mathbf{z} = \mathbf{z}$, $\mathbf{z}^T\mathbf{e} \neq 0$.
- If $m_a^A(1) > m_g^A(1)$ and $\exists\, k$ such that $A^k \geq 0$, then ($1 = \rho(A)$ is eigenvalue of A, see above, and) A must have an eigenvalue $\lambda \neq 1$ such that $|\lambda| = 1$.

2×2 THEOREM. Let A be a square $n \times n$ matrix that can be partitioned as follows

$$A = \begin{bmatrix} M & 0 \\ N & L \end{bmatrix}$$

where M (L) is square and the number of its columns (rows) is equal to the number of columns (rows) of N. M can have complex entries, $N \geq 0$, $L \geq 0$. Assume that M satisfies HYP (this implies $\rho(M)$ positive). Assume also that $\rho(L) < \rho(M)$ (this implies $(\rho(M)I - L)^{-1}N \geq 0$).

Then $\rho(A)$ $(= \rho(M))$ is positive, is a simple eigenvalue of A and of A^T, and exists a unique $\mathbf{z} \geq \mathbf{0}$ such that $\|\mathbf{z}\|_1 = 1$ and $A\mathbf{z} = \rho(A)\mathbf{z}$:

$$\mathbf{z} = \begin{bmatrix} \tilde{\mathbf{z}} \\ (\rho(M)I - L)^{-1}N\tilde{\mathbf{z}} \end{bmatrix}, \ \tilde{\mathbf{z}} > \mathbf{0}, \ M\tilde{\mathbf{z}} = \rho(M)\tilde{\mathbf{z}},$$

and a unique $\mathbf{w} \geq \mathbf{0}$ such that $\|\mathbf{w}\|_1 = 1$ and $A^T\mathbf{w} = \rho(A)\mathbf{w}$:

$$\mathbf{w} = \begin{bmatrix} \tilde{\mathbf{w}} \\ \mathbf{0} \end{bmatrix}, \ \tilde{\mathbf{w}} > \mathbf{0}, \ M^T\tilde{\mathbf{w}} = \rho(M)\tilde{\mathbf{w}}.$$

Moreover, there exists a diagonal matrix D with positive diagonal entries such that DMD^{-1} is $\rho(M)$-stochastic by columns. As a consequence, by the third Gershgorin theorem, if $[M^k]_{ii} > 0$ for some i, then the remaining $order(M) - 1$ eigenvalues of M (the remaining $n - 1$ eigenvalues of A) have absolute value smaller than $\rho(M)$ $(= \rho(A))$, and thus, if $j \to +\infty$,

$$\frac{1}{\rho(M)^j}A^j \ \to \ \frac{\mathbf{z}\mathbf{w}^T}{\mathbf{w}^T\mathbf{z}} \ \left(A^j = \begin{bmatrix} M^j & 0 \\ \sum_{i=0}^{j} L^i N M^{j-i} & L^j \end{bmatrix} \right).$$

Proof. See the Appendix.

3×3 THEOREM. Let A be a square $n \times n$ matrix that can be partitioned as follows

$$A = \begin{bmatrix} L_1 & 0 & 0 \\ N_1 & M & 0 \\ S & N_2 & L_2 \end{bmatrix}$$

with M, L_1, L_2 square. M and S can have complex entries, $L_1 \geq 0$, $N_1 \geq 0$, $L_2 \geq 0$, $N_2 \geq 0$.
Assume that M satisfies HYP (note that this implies $\rho(M)$ positive). Assume also that $\rho(L_1) < \rho(M)$, $\rho(L_2) < \rho(M)$ (note that this implies $(\rho(M)I - L_2)^{-1}N_2 \geq 0$, $(\rho(M)I - L_1^T)^{-1}N_1^T \geq 0$).

Then $\rho(A)$ $(= \rho(M))$ is positive, is a simple eigenvalue of A and of A^T, and exists a unique $\mathbf{z} \geq \mathbf{0}$ such that $\|\mathbf{z}\|_1 = 1$ and $A\mathbf{z} = \rho(A)\mathbf{z}$:

$$\mathbf{z} = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{z}} \\ (\rho(M)I - L_2)^{-1}N_2\tilde{\mathbf{z}} \end{bmatrix}, \quad \tilde{\mathbf{z}} > \mathbf{0}, \; M\tilde{\mathbf{z}} = \rho(M)\tilde{\mathbf{z}},$$

and a unique $\mathbf{w} \geq \mathbf{0}$ such that $\|\mathbf{w}\|_1 = 1$ and $A^T\mathbf{w} = \rho(A)\mathbf{w}$:

$$\mathbf{w} = \begin{bmatrix} (\rho(M)I - L_1^T)^{-1}N_1^T\tilde{\mathbf{w}} \\ \tilde{\mathbf{w}} \\ \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{w}} > \mathbf{0}, \; M^T\tilde{\mathbf{w}} = \rho(M)\tilde{\mathbf{w}}.$$

Moreover, there exists a diagonal matrix D with positive diagonal entries such that DMD^{-1} is $\rho(M)$-stochastic by columns. As a consequence, by the third Gershgorin theorem, if $[M^k]_{ii} > 0$ for some i, then the remaining $order(M) - 1$ eigenvalues of M (the remainining $n-1$ eigenvalues of A) have absolute value smaller than $\rho(M)$ $(= \rho(A))$, and thus, if $j \to +\infty$,

$$\frac{1}{\rho(M)^j}A^j \;\to\; \frac{\mathbf{z}\mathbf{w}^T}{\mathbf{w}^T\mathbf{z}} \quad \left(A^j = \begin{bmatrix} L_1^j & 0 & 0 \\ & M^j & 0 \\ & & L_2^j \end{bmatrix} \right).$$

Proof. Left to the reader.

Consider a $n \times n$ matrix A of the form

$$A = \begin{bmatrix} M & 0 \\ N & L \end{bmatrix} \qquad (**)$$

with M, L square, non negative, N non negative, M and N with the same number of columns, $M_{ii} = 0\ \forall\, i$, $\sum_i M_{ij} + \sum_k N_{kj} = 1\ \forall\, j$, and L with the structure

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ L_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ L_{r1} & \cdots & L_{r,r-1} & 0 \end{bmatrix}$$

where the diagonal zeros are null square matrices not necessarily of the same order, and $\sum_{s=t+1}^{r} \sum_i [L_{st}]_{ij} = 1\ \forall\, j\ \forall\, t = 1, \ldots, r-1$. Then

$$\rho(A) = \rho(M) = \begin{cases} 1 & N = 0\ (\Rightarrow\ M\ \textit{stochbycol}) \\ \leq 1 & N \neq 0,\ M \text{ reducible} \\ < 1 & N \neq 0,\ M \text{ irreducible} \end{cases}$$

Assume also that no column of M is null.

These assumptions are satisfied by QP^TQ^T where P is the transition matrix of the web and Q is the permutation putting together and down all null rows and null sub-rows of P, in the sense that

$$QPQ^T = \begin{bmatrix} M^T & N^T \\ 0 & L^T \end{bmatrix}$$

with M, L square, each row of M^T non null, and L with a strictly lower triangular block structure (see below for a precise definition of Q). By the 2×2 Theorem, if the square matrix M satisfies HYP (iff M irreducible ($M \geq 0!$)), since $\rho(L) = 0 < \rho(M) = \rho(P)$, then it is uniquely defined $\mathbf{z}$, $\mathbf{z} \geq \mathbf{0}$, $\|\mathbf{z}\|_1 = 1$, such that $QP^TQ^T\mathbf{z} = \rho(P)\mathbf{z}$, $P^T(Q^T\mathbf{z}) = \rho(P)(Q^T\mathbf{z})$ with $\rho(P) < 1$, unless $N = 0$ in which case $\rho(P) = 1$, i.e., if we set $\mathbf{p} = Q^T\mathbf{z}$, we have $p_j = \sum_{i:\, i \to j} \frac{(1/\rho(P))}{\deg(i)} p_i$, $\|\mathbf{p}\|_1 = 1$, $p_i \geq 0$.

Note that $p_i = (Q^T\mathbf{z})_i = z_{q_i}$ is null whenever $q_i = \mathit{order}(M) + s$ where the sth row of $(I - \frac{1}{\rho(M)}L)^{-1}N$ is null. If we want $p_i > 0\ \forall\, i$, then it is enough to perturb one zero entry of each null row of $(I - \frac{1}{\rho(M)}L)^{-1}N$. In order to do this, it is sufficient to perturb one zero entry of each null row of N

$\big[$ in fact,

$$(I - \frac{1}{\rho(M)}L)^{-1}N = N + \frac{1}{\rho(M)}LN + \ldots + \frac{1}{\rho(M)^{r-1}}L^{r-1}N$$

where $L^iN \geq 0$ $\big]$

(f.i. $0 \to \frac{1}{\deg(\cdot)+1}$), say the one in position r, and maintain non null but reduce (f.i. $\frac{1}{\deg(\cdot)} \to \frac{1}{\deg(\cdot)+1}$) the nonzero entries in the r column of N and of M so that the resulting M' and N' yet satisfy $\sum_i M'_{ij} + \sum_k N'_{kj} = 1\ \forall\, j$, and all other assumptions [$M' \geq 0$, $N' \geq 0$, M' satisfies HYP iff M' irreducible].

Observe also that $0 < \rho(P') = \rho(M') < \rho(P) = \rho(M) \leq 1$ ($= 1$ iff $N = 0$), where P' is defined by the following equality

$$Q(P')^T Q^T = \begin{bmatrix} M' & 0 \\ N' & L \end{bmatrix}$$

(since $M' \leq M$, $M' \neq M$, M is irreducible, we have $\rho(M') < \rho(M)$) and it is uniquely defined $\mathbf{z}'$, $\mathbf{z}' > \mathbf{0}$, $\|\mathbf{z}'\|_1 = 1$, such that $Q(P')^T Q^T \mathbf{z}' = \rho(P')\mathbf{z}'$, $(P')^T(Q^T\mathbf{z}') = \rho(P')(Q^T\mathbf{z}')$ with $\rho(P') < 1$, i.e., if we set $\mathbf{p}' = Q^T\mathbf{z}'$, we have $p'_j = \sum_{i:\, i \to j} \frac{(1/\rho(P'))}{\deg'(i)} p'_i$, $\|\mathbf{p}'\|_1 = 1$, $\mathbf{p}' > \mathbf{0}$.

Consider a $n \times n$ matrix A of the form

$$A = \begin{bmatrix} L_1 & 0 & 0 \\ N_1 & M & 0 \\ S & N_2 & L_2 \end{bmatrix} \qquad (***)$$

with M, L_1, L_2 square, non negative, N_1, N_2, S non negative, M and N_2 with the same number of columns, L_1, N_1, S with the same number of columns, $\sum_k [L_1]_{kj} + \sum_i [N_1]_{ij} + \sum_k [S]_{kj} = 1 \ \forall j$, $\sum_i M_{ij} + \sum_k [N_2]_{kj} = 1 \ \forall j$, $M_{ii} = 0$ $\forall i$, and L_2 with the structure

$$L_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ (L_2)_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (L_2)_{r1} & \cdots & (L_2)_{r,r-1} & 0 \end{bmatrix}$$

where the diagonal zeros are null square matrices not necessarily of the same order, and $\sum_{s=t+1}^{r} \sum_i ((L_2)_{st})_{ij} = 1 \ \forall j \ \forall t = 1, \ldots, r-1$, and L_1 with the structure

$$L_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ (L_1)_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (L_1)_{s1} & \cdots & (L_1)_{s,s-1} & 0 \end{bmatrix}$$

where the diagonal zeros are null square matrices not necessarily of the same order.

Then

$$\rho(A) = \rho(M) = \begin{cases} 1 & N_2 = 0 \ (\Rightarrow \ M \ \textit{stochbycol}) \\ \leq 1 & N_2 \neq 0, \ M \text{ reducible} \\ < 1 & N_2 \neq 0, \ M \text{ irreducible} \end{cases}$$

Assume M with no null row and no null column.

These assumptions are satisfied by QP^TQ^T where P is the transition matrix of the web and Q is the permutation putting together and down (together and on left) all null rows and null sub-rows (null columns and null sub-columns) of P, in the sense that

$$QPQ^T = \begin{bmatrix} L_1^T & N_1^T & S^T \\ 0 & M^T & N_2^T \\ 0 & 0 & L_2^T \end{bmatrix}$$

with M, L_1, L_2 square, each row and column of M^T non null, and L_1, L_2 with a strictly lower triangular block structure (see below for a precise definition of Q). By the 3×3 Theorem, if the square matrix M satisfies HYP (iff M irreducible ($M \geq 0!$)), since $\rho(L_2) = \rho(L_1) = 0 < \rho(M) = \rho(P)$, then it is uniquely defined $\mathbf{z}$, $\mathbf{z} \geq \mathbf{0}$, $\|\mathbf{z}\|_1 = 1$, such that $QP^TQ^T\mathbf{z} = \rho(P)\mathbf{z}$, $P^T(Q^T\mathbf{z}) = \rho(P)(Q^T\mathbf{z})$ with $\rho(P) < 1$, unless $N_2 = 0$ in which case $\rho(P) = 1$, i.e., if we set $\mathbf{p} = Q^T\mathbf{z}$, we have $p_j = \sum_{i:\, i \to j} \frac{(1/\rho(P))}{\deg(i)} p_i$, $\|\mathbf{p}\|_1 = 1$, $p_i \geq 0$.

Note that $p_i = (Q^T\mathbf{z})_i = z_{q_i}$ is null whenever $q_i \leq \mathit{order}(L_1)$ or $q_i = \mathit{order}(L_1) + \mathit{order}(M) + s$ where the sth row of $(I - \frac{1}{\rho(M)} L_2)^{-1} N_2$ is null. If we want $p_i > 0 \, \forall \, i : \, q_i > order(L_1)$, then it is enough to perturb one zero entry of each null row of $(I - \frac{1}{\rho(M)} L_2)^{-1} N_2$. In order to do this, it is sufficient to perturb one zero entry of each null row of N_2

$\big[$ in fact

$$(I - \frac{1}{\rho(M)} L_2)^{-1} N_2 = N_2 + \frac{1}{\rho(M)} L_2 N_2 + \ldots + \frac{1}{\rho(M)^{r-1}} L_2^{r-1} N_2$$

where $L_2^i N_2 \geq 0$ $\big]$

(f.i. $0 \to \frac{1}{\deg(\cdot)+1}$), say the one in position r, and maintain non null but reduce (f.i. $\frac{1}{\deg(\cdot)} \to \frac{1}{\deg(\cdot)+1}$) the nonzero entries in the r column of N_2 and of M so that the resulting M' and N_2' yet satisfy $\sum_i M'_{ij} + \sum_k (N_2)'_{kj} = 1 \, \forall \, j$, and all other assumptions [$M' \geq 0$, $N_2' \geq 0$, M' satisfies HYP iff M' irreducible].

Observe also that $0 < \rho(P') = \rho(M') < \rho(P) = \rho(M) \leq 1$ ($= 1$ iff $N_2 = 0$), where P' is defined by the following equality

$$Q(P')^T Q^T = \begin{bmatrix} L_1 & 0 & 0 \\ N_1 & M' & 0 \\ S & N' & L_2 \end{bmatrix}$$

(since $M' \leq M$, $M' \neq M$, M is irreducible, we have $\rho(M') < \rho(M)$) and it is uniquely defined $\mathbf{z}'$, $\mathbf{z}' \geq \mathbf{0}$ ($z'_i = 0$ iff $i \leq order(L_1)$), $\|\mathbf{z}'\|_1 = 1$, such that $Q(P')^T Q^T \mathbf{z}' = \rho(P')\mathbf{z}'$, $(P')^T (Q^T \mathbf{z}') = \rho(P')(Q^T \mathbf{z}')$ with $\rho(P') < 1$, i.e., if we set $\mathbf{p}' = Q^T \mathbf{z}'$, we have $p'_j = \sum_{i:\, i \to j} \frac{(1/\rho(P'))}{\deg'(i)} p'_i$, $\|\mathbf{p}'\|_1 = 1$, $\mathbf{p}' \geq \mathbf{0}$ ($p'_i = 0$ iff $q_i \leq order(L_1)$)].

Example: $A = QP^TQ^T$, P web transition matrix
Let P be the $n \times n$ transition matrix of an oriented graph with n vertices (f.i. the web graph), i.e. $P_{ij} = \frac{1}{\deg(i)}$ if there is a link from i to j ($\deg(i) = \#$ links from i), and $P_{ij} = 0$ otherwise. Note that P is a non negative matrix such that $\sum_j P_{ij} = 1$ if $\deg(i) > 0$, and $\sum_j P_{ij} = 0$ if $\deg(i) = 0$, i.e. P is a non negative quasi-stochastic by rows matrix.

Here below is the procedure generating QPQ^T

$$P = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Move the r_1 null rows of the upper-left $n \times n$ submatrix of P down:

$$R_1PR_1^T = \begin{bmatrix} & & & & & & \\ & & & & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}$$

Move the r_2 null rows of the upper-left $(n-r_1) \times (n-r_1)$ submatrix of $R_1PR_1^T$ down:

$$R_2R_1PR_1^TR_2^T = \begin{bmatrix} & & & & & & \\ & & & & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

Move the r_3 null rows of the upper-left $(n-r_1-r_2)\times(n-r_1-r_2)$ submatrix of $R_2R_1PR_1^TR_2^T$ down:

$$R_3R_2R_1PR_1^TR_2^TR_3^T = \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ 0 & \cdot & \cdot & \cdot & 0 & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \end{bmatrix}$$

In the upper-left $(n-r_1-r_2-r_3) \times (n-r_1-r_2-r_3)$ submatrix of $R_3R_2R_1PR_1^TR_2^TR_3^T$ there is no null row. Call it M^T and try to apply the 2×2 THEOREM.

Move the c_1 null columns of the upper-left $(n-r_1-r_2-r_3)\times(n-r_1-r_2-r_3)$ submatrix of $R_3R_2R_1PR_1^TR_2^TR_3^T$ on the left:

$$C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^T = \begin{bmatrix} 0 & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ 0 & & & & & & & \\ 0 & \cdot & \cdot & \cdot & 0 & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \end{bmatrix}$$

Move the c_2 null columns of the almost upper-left $(n-r_1-r_2-r_3-c_1)\times(n-r_1-r_2-r_3-c_1)$

submatrix of $C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^T$ on the left:

$$C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^TC_2^T = \begin{bmatrix} 0 & & & & & & \\ \cdot & 0 & & & & & \\ \cdot & \cdot & & & & & \\ 0 & 0 & & & & & \\ 0 & \cdot & \cdot & \cdot & 0 & & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

Move the c_3 null columns of the almost almost upper-left $(n - r_1 - r_2 - r_3 - c_1 - c_2) \times (n - r_1 - r_2 - r_3 - c_1 - c_2)$ submatrix of $C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^TC_2^T$ on the left:

$$C_3C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^TC_2^TC_3^T$$

$$= \begin{bmatrix} 0 & & & & & & \\ \cdot & 0 & & & & & \\ \cdot & \cdot & 0 & & & & \\ 0 & 0 & 0 & M^T & & & \\ 0 & \cdot & \cdot & \cdot & 0 & & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} = \begin{bmatrix} L_1^T & N_1^T & S^T \\ O & M^T & N_2^T \\ O & O & L_2^T \end{bmatrix}$$

In the almost almost almost upper-left $(n - r_1 - r_2 - r_3 - c_1 - c_2 - c_3) \times (n - r_1 - r_2 - r_3 - c_1 - c_2 - c_3)$ submatrix of $C_3C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^TC_2^TC_3^T$ there is no null column (besides no null row). Call it M^T and try to apply the 3×3 THEOREM.

APPENDIX

Perron-Frobenius theorem. Let M be a non negative ($M_{ij} \geq 0$), irreducible $n \times n$ matrix. Then $\rho(M)$ is positive, $\rho(M)$ is a simple eigenvalue of M, and there exists a unique positive vector $\mathbf{z}$ (z_i positive for all i) such that $\|\mathbf{z}\|_1 = 1$ and $M\mathbf{z} = \rho(M)\mathbf{z}$. If M is also stochastic by columns, then $1 = \rho(M)$.

PROOF of THEOREM HYP

(i) Since A^k is a non negative, irreducible $n \times n$ matrix, by the Perron-Frobenius theorem $\rho(A^k)$ is a positive simple eigenvalue of A^k (this implies that $\rho(A)$ is positive!) and there exists a unique positive vector $\mathbf{z}$ such that $\|\mathbf{z}\|_1 = \mathbf{e}^T\mathbf{z} = 1$, $A^k\mathbf{z} = \rho(A^k)\mathbf{z} = \rho(A)^k\mathbf{z}$. Let $\mathbf{y} \neq \mathbf{0}$ be an eigenvector of A corresponding to its eigenvalue $\rho(A)$, thus $A\mathbf{y} = \rho(A)\mathbf{y}$. Note that then $\mathbf{y}$ also satisfies the identities $A^j\mathbf{y} = \rho(A)^j\mathbf{y}$, $\forall\, j$, and in particular the identity $A^k\mathbf{y} = \rho(A)^k\mathbf{y}$. Since $m_g^{A^k}(\rho(A)^k) = 1$, this implies $\mathbf{y} = \alpha\mathbf{z}$, for some $\alpha \in \mathbb{C}$. So we have $A\mathbf{z} = \rho(A)\mathbf{z}$ and $m_a^A(\rho(A)) \geq m_g^A(\rho(A)) = 1$ (Stefano). Finally note that $m_a^A(\rho(A)) \leq m_a^{A^k}(\rho(A)^k) = 1$, thus $m_a^A(\rho(A)) = 1$. The assertion on $\mathbf{w}$ follows by observing that A satisfies HYP iff A^T satisfies HYP.

(ii) By (i) we know that there exists a unique positive vector $\mathbf{w}$ such that $\|\mathbf{w}\|_1 = 1$, $A^T\mathbf{w} = \rho(A)\mathbf{w}$. It follows that $\sum_i [A^T]_{ji} w_i = \rho(A) w_j$, and thus $\sum_i w_i [A]_{ij} w_j^{-1} = \rho(A)$ ($\forall\, j$). Now observe that the latter identity can be rewritten as follows $\sum_i [DAD^{-1}]_{ij} = \rho(A)$, $\forall\, j$, where $D = d(\mathbf{w})$ is a diagonal matrix with positive diagonal entries.

(iii) Let D be the matrix introduced in (ii). Then
$(A^k)_{ii}$ positive implies $(DA^kD^{-1})_{ii} = [(DAD^{-1})^k]_{ii}$ positive.
A^k non negative implies $DA^kD^{-1} = (DAD^{-1})^k$ non negative.
A^k irreducible implies $DA^kD^{-1} = (DAD^{-1})^k$ irreducible.
Note also that, since DAD^{-1} is $\rho(A)$-stochastic by columns, i.e. $(DAD^{-1})^T\mathbf{e} = \rho(A)\mathbf{e}$, we have that $(DAD^{-1})^k$ is $\rho(A)^k$-stochastic by columns, i.e. $((DAD^{-1})^k)^T\mathbf{e} = \rho(A)^k\mathbf{e}$.

Note that then all the Gershgorin circles $\mathcal{G}_j$ of $((DAD^{-1})^k)^T$ are in the set $\mathcal{B} = \{z \in \mathbb{C} : |z| \leq \rho(A)^k\}$ and their borders pass through the point $\rho(A)^k$. Moreover, the $\mathcal{G}_j$ coincide with $\mathcal{B}$ if $[(DAD^{-1})^k]_{jj} = 0$, otherwise they touch the circle $|z| = \rho(A)^k$ only in $\rho(A)^k$. So, we can apply the third Gershgorin theorem to the matrix $(DAD^{-1})^k$ and say that a complex number z, $|z| = \rho(A)^k$, not being inside any circle, can be an eigenvalue of $(DAD^{-1})^k$ only if $z = \rho(A)^k$, since $\rho(A)^k$ is the only point in $\cap_j \partial K_j$. This and the fact that $\rho(A)^k$ is a simple eigenvalue of $(DAD^{-1})^k$ imply that the remaining $n-1$ eigenvalues of $(DAD^{-1})^k$ must have absolute value smaller than $\rho(A)^k$, and thus, that exactly $n-1$ eigenvalues of DAD^{-1} must have absolute value smaller than $\rho(A)$.

(iv) It follows from (iii)

(v) Assume that A satisfies HYP. Let J be the Jordan form of A. Then there is a non singular matrix S such that

$$S^{-1}AS = J = \begin{bmatrix} \rho(A) & & \\ & \Big[|\lambda| = \rho(A),\ \lambda \neq \rho(A)\Big] & \\ & & \Big[|\lambda| < \rho(A)\Big] \end{bmatrix}.$$

Moreover, we can assume that the first column of S is exactly the vector $\mathbf{z}$ introduced in (i). Note that $\mathbf{e}_1^T S^{-1} A = \rho(A)\mathbf{e}_1^T S^{-1}$ and in the same time, of course, $\mathbf{w}^T A = \rho(A)\mathbf{w}^T$, where $\mathbf{w}$ is the other vector introduced in (i). Thus $\mathbf{e}_1^T S^{-1}$ must be equal to $\alpha \mathbf{w}^T$ for some $\alpha \in \mathbb{C}$ ($m_g^{A^T}(\rho(A)) = m_g^A(\rho(A)) = 1$). Then, since $\mathbf{e}_1^T S^{-1}\mathbf{z} = 1$, we must have $\alpha \mathbf{w}^T \mathbf{z} = 1$, that implies $\alpha = 1/\mathbf{w}^T\mathbf{z}$. In other words, if we assume that the first column of S is exactly the vector $\mathbf{z}$, then the first row of S^{-1} is exactly the vector $\frac{1}{\mathbf{w}^T\mathbf{z}}\mathbf{w}^T$.
Now consider a partition of S and S^{-1} according to the form of J:

$$S = \begin{bmatrix} \mathbf{z} & X & \tilde{X} \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} \frac{1}{\mathbf{w}^T\mathbf{z}}\mathbf{w}^T \\ Y \\ \tilde{Y} \end{bmatrix}$$

(note that X, $\tilde{X}$, Y, $\tilde{Y}$ must satisfy the identities $\mathbf{w}^T X = \mathbf{0}^T$, $\mathbf{w}^T \tilde{X} = \mathbf{0}^T$, $Y\mathbf{z} = \mathbf{0}$, $\tilde{Y}\mathbf{z} = \mathbf{0}$). Then

$$\frac{1}{\rho(A)^j} A^j = \frac{1}{\rho(A)^j} S J^j S^{-1}$$

$$= \begin{bmatrix} \mathbf{z} & X & \tilde{X} \end{bmatrix} \begin{bmatrix} 1 & & \\ & \frac{1}{\rho(A)^j}\Big[|\lambda| = \rho(A),\, \lambda \neq \rho(A)\Big]^j & \\ & & \frac{1}{\rho(A)^j}\Big[|\lambda| < \rho(A)\Big]^j \end{bmatrix} \begin{bmatrix} \frac{1}{\mathbf{w}^T\mathbf{z}}\mathbf{w}^T \\ Y \\ \tilde{Y} \end{bmatrix},$$

$$\frac{1}{\rho(A)^j} A^j = \frac{1}{\mathbf{w}^T\mathbf{z}}\mathbf{z}\mathbf{w}^T + X\frac{1}{\rho(A)^j}\Big[|\lambda| = \rho(A),\, \lambda \neq \rho(A)\Big]^j Y + \tilde{X}\frac{1}{\rho(A)^j}\Big[|\lambda| < \rho(A)\Big]^j \tilde{Y}.$$

If there is no eigenvalue λ of A such that $|\lambda| = \rho(A)$, $\lambda \neq \rho(A)$, then the last formula implies that, as $j \to +\infty$, the matrix $\frac{1}{\rho(A)^j}A^j$ tends to the rank one matrix $\frac{1}{\mathbf{w}^T\mathbf{z}}\mathbf{z}\mathbf{w}^T$, which is positive. Thus there must exists an s such that A^s is positive.

(vi) Let S be a non singular matrix. First notice that SAS^{-1} is $\rho(A)$-stochastic by columns and by rows iff $(SAS^{-1})^T\mathbf{e} = \rho(A)\mathbf{e}$, $(SAS^{-1})\mathbf{e} = \rho(A)\mathbf{e}$ iff $A^T(S^T\mathbf{e}) = \rho(A)(S^T\mathbf{e})$, $A(S^{-1}\mathbf{e}) = \rho(A)(S^{-1}\mathbf{e})$. Since A satisfies HYP, there exist positive vectors $\mathbf{z}$ and $\mathbf{w}'$ such that $A^T\mathbf{w}' = \rho(A)\mathbf{w}'$, $A\mathbf{z} = \rho(A)\mathbf{z}$, $\|\mathbf{z}\|_1 = 1$, $\sum_i w_i' z_i = n$. Now the problem is reduced to find S such that $S^T\mathbf{e} = \mathbf{w}'$ (1), $S\mathbf{z} = \mathbf{e}$ (2). The matrix $S = M + (\mathbf{e} - M\mathbf{z})\mathbf{e}^T$ satisfies (2) for all M, so it is enough to choose M such that (1) holds:

$$S^T\mathbf{e} = M^T\mathbf{e} + ((\mathbf{e} - M\mathbf{z})^T\mathbf{e})\mathbf{e} = \mathbf{w}'.$$

The latter equality is satisfied in particular by $M = d(\mathbf{w}')$, and such choice of M makes S non singular (check it!).

PROOF of COROLLARY HYP

Since A is stochastic by columns, we have $A^T\mathbf{e} = \mathbf{e}$, $\mathbf{e} = [1\ 1\ \cdots\ 1]^T$, so 1 is eigenvalue of A^T, and therefore of A (a matrix and its transpose have the same eigenvalues). If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k. Then

$$|\lambda^k| \leq \|A^k\|_1 = \max_j \sum_i |[A^k]_{ij}| = \max_j \sum_i [A^k]_{ij} = \max_j 1 = 1$$

(recall that μ eig of M implies $|\mu| \leq \|M\|_1$, and that A stochbycol implies A^j stochbycol for all j). Thus, $|\lambda|^k = |\lambda^k| \leq 1$, which implies $|\lambda| \leq 1$. So the absolute value of any eigenvalue of A is bounded by 1, and at least one of them (i.e. 1) has absolute value one.

PROOF of the 2×2 THEOREM

By the Perron-Frobenius theorem, $\rho(M^k)$ is a positive simple eigenvalue of M^k and there exists $\tilde{\mathbf{z}} > \mathbf{0}$ such that $M^k\tilde{\mathbf{z}} = \rho(M^k)\tilde{\mathbf{z}}$. $\rho(M)$ is positive since $\rho(M)^k = \rho(M^k) > 0$. Observe that $M\mathbf{y} = \rho(M)\mathbf{y}$, $\mathbf{y} \neq \mathbf{0}$, implies $M^k\mathbf{y} = \rho(M)^k\mathbf{y} = \rho(M^k)\mathbf{y}$, thus $\mathbf{y} = \alpha\tilde{\mathbf{z}}$, $M\tilde{\mathbf{z}} = \rho(M)\tilde{\mathbf{z}}$, and $m_g^M(\rho(M)) = 1$. Moreover, $m_a^M(\rho(M)) \leq m_a^{M^k}(\rho(M)^k) = m_a^{M^k}(\rho(M^k)) = 1$. So, $\rho(M)$ is positive, is a simple eigenvalue of M, and thus $\rho(A)$ $(= \rho(M))$ is positive, and is a simple eigenvalue of A. In fact,

$$\begin{bmatrix} M & 0 \\ N & L \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{z}} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} M\tilde{\mathbf{z}} \\ N\tilde{\mathbf{z}} + L\mathbf{x} \end{bmatrix} = \rho(M) \begin{bmatrix} \tilde{\mathbf{z}} \\ \mathbf{x} \end{bmatrix}$$

implies $\mathbf{x} = (\rho(M)I - L)^{-1}N\tilde{\mathbf{z}}$. Finally, of course, $\tilde{\mathbf{z}}$ can be chosen so that $\|\mathbf{z}\|_1 = 1$ where

$$\mathbf{z} = \begin{bmatrix} \tilde{\mathbf{z}} \\ (\rho(M)I - L)^{-1}N\tilde{\mathbf{z}} \end{bmatrix}.$$

Analogously, $\rho(M)$ is a simple eigenvalue of M^T and there exists $\tilde{\mathbf{w}} > \mathbf{0}$ such that $M^T\tilde{\mathbf{w}} = \rho(M)\tilde{\mathbf{w}}$, and thus $\rho(A)$ $(= \rho(M))$ is a simple eigenvalue of A^T. In fact,

$$\begin{bmatrix} M^T & N^T \\ 0 & L^T \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} M^T\tilde{\mathbf{w}} + N^T\mathbf{x} \\ L^T\mathbf{x} \end{bmatrix} = \begin{bmatrix} \rho(M)\tilde{\mathbf{w}} + N^T\mathbf{x} \\ L^T\mathbf{x} \end{bmatrix} = \rho(M) \begin{bmatrix} \tilde{\mathbf{w}} \\ \mathbf{x} \end{bmatrix}$$

implies $\mathbf{x} = \mathbf{0}$. Finally, of course, $\tilde{\mathbf{w}}$ can be chosen so that $\|\mathbf{w}\|_1 = 1$ where

$$\mathbf{w} = \begin{bmatrix} \tilde{\mathbf{w}} \\ \mathbf{0} \end{bmatrix}.$$

The proof of the remaining assertions is left to the reader (proceed as in the proof of (ii),(iii),(v) of Theorem HYP).

PROOF of the 3×3 THEOREM

Left to the reader.

In the following A is stochastic by columns and in $\mathbb{C}^{n\times n}$

Assume $m_a(1) = m_g(1) = q$ ($\Rightarrow q = m_a^{A^T}(1) = m_g^{A^T}(1)$). Let $\mathbf{z}_i \neq \mathbf{0}$ be linearly independent and such that $A\mathbf{z}_i = \mathbf{z}_i$, $i = 1, \ldots, q$, and consider the Jordan canonical form of A:

$$S = \begin{bmatrix} Z & X & \tilde{X} \end{bmatrix}, \; Z = \begin{bmatrix} \mathbf{z}_1 \cdots \mathbf{z}_q \end{bmatrix}, \; S^{-1} = \begin{bmatrix} E \\ Y \\ \tilde{Y} \end{bmatrix},$$

$$S^{-1}AS = \begin{bmatrix} I_q & & \\ & \begin{bmatrix} |\lambda| \geq 1, \, \lambda \neq 1 \end{bmatrix} & \\ & & \begin{bmatrix} |\lambda| < 1 \end{bmatrix} \end{bmatrix}.$$

Observe that the equalities $(\mathbf{e}_r^T S^{-1})A = (\mathbf{e}_r^T S^{-1})$, $r = 1, \ldots, q$, and $\mathbf{e}^T A = \mathbf{e}^T$ imply $\mathbf{e}^T = \sum_{i=1}^q \beta_i \mathbf{e}_i^T S^{-1}$. Moreover, E must be such that $I_q = EAZ = EZ$. Thus $\mathbf{e}^T Z = \sum_i \beta_i (\mathbf{e}_i^T S^{-1}) Z = \sum_i \beta_i \mathbf{e}_i^T$, and therefore $\beta_i = \mathbf{e}^T \mathbf{z}_i$. In other words, the following formula must hold:

$$\mathbf{e}^T = \sum_{i=1}^q (\mathbf{e}^T \mathbf{z}_i)(\mathbf{e}_i^T S^{-1}) \;\; [\text{if } q = 1 : \mathbf{e}^T = (\mathbf{e}^T \mathbf{z})(\mathbf{e}_1^T S^{-1})].$$

Note: the latter formula proves that at least one $\mathbf{e}^T \mathbf{z}_i$ must be nonzero [if $q = 1$: $\mathbf{e}^T \mathbf{z}$ must be nonzero] !
Then we have the following representation of A^r:

$$A^r = ZE + X \begin{bmatrix} |\lambda| \geq 1, \, \lambda \neq 1 \end{bmatrix}^r Y + \tilde{X} \begin{bmatrix} |\lambda| < 1 \end{bmatrix}^r \tilde{Y}$$

Such formula let us conclude that if the eigenvalues λ of A different from 1 are such that $|\lambda| < 1$, then $A^r \;\to\; ZE$, and, in particular, $A^r \mathbf{v}$ converges to a linear combination of the eigenvectors of 1 [if $q = 1$: $A^r \;\to\; \frac{\mathbf{z}\mathbf{e}^T}{\mathbf{e}^T \mathbf{z}}$ and $A^r \mathbf{v}$ converges to a multiple of $\mathbf{z}$; as a consequence, if $\mathbf{z} > \mathbf{0}$ (as in the example $A = \begin{bmatrix} -\frac{1}{2} & b \\ \frac{3}{2} & 1-b \end{bmatrix}$ $0 < b < \frac{1}{2}$) then $\exists\, r$ such that $A^r > 0$ ($A^2 > 0$)].

Example. For both the following matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & \frac{1}{2} \\ & & \frac{1}{2} \end{bmatrix}, \; A = \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix},$$

we have $m_a(1) = m_g(1) = 2$. For the first matrix 1 is dominant, whereas for the second one 1 is not dominant.

Assume now $m_a(1) = 2 > m_g(1) = 1$. In this case we shall do the following remark: if A^k is non negative for some k, then A must have an eigenvalue $\lambda \neq 1$ such that $|\lambda| = 1$. We conjecture that the latter remark holds in the more general case $m_a(1) > m_g(1)$ (it is not true if $m_a(1) = m_g(1)$, see the above Example).
Let $\mathbf{z}_1, \mathbf{z}_2 \neq \mathbf{0}$ and linearly independent be such that $A\mathbf{z}_1 = \mathbf{z}_1$, $A\mathbf{z}_2 = \mathbf{z}_1 + \mathbf{z}_2$. Consider the Jordan canonical form of A:

$$S = \begin{bmatrix} Z & X & \tilde{X} \end{bmatrix}, \; Z = \begin{bmatrix} \mathbf{z}_1 \; \mathbf{z}_2 \end{bmatrix}, \; S^{-1} = \begin{bmatrix} E \\ Y \\ \tilde{Y} \end{bmatrix},$$

$$S^{-1}AS = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & & \\ & \Big[|\lambda| \geq 1,\, \lambda \neq 1\Big] & \\ & & \Big[|\lambda| < 1\Big] \end{bmatrix}.$$

Then we have the following representation of A^r:

$$\begin{aligned} A^r &= Z \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} E + X \Big[|\lambda| \geq 1,\, \lambda \neq 1\Big]^r Y + \tilde{X} \Big[|\lambda| < 1\Big]^r \tilde{Y} \\ &= \mathbf{z}_1 \mathbf{e}_1^T S^{-1} + \frac{\mathbf{z}_2 \mathbf{e}^T}{\mathbf{e}^T \mathbf{z}_2} + r \frac{\mathbf{z}_1 \mathbf{e}^T}{\mathbf{e}^T \mathbf{z}_2} + X \Big[|\lambda| \geq 1,\, \lambda \neq 1\Big]^r Y + \tilde{X} \Big[|\lambda| < 1\Big]^r \tilde{Y} \end{aligned}$$

(prove it!).

Now let us prove the remark. So, assume $A^k \geq 0$ [$\Rightarrow 1 = \rho(A)$ is eigenvalue of A (by Corollary HYP)]. If all $\lambda \neq 1$ are such that $|\lambda| < 1$, then, chosen p such that $(\mathbf{z}_1)_p \neq 0$, we would have $|(A^{km})_{p,j}| \rightarrow +\infty$ as $m \rightarrow +\infty$, and this is not possible since A^{km} is non negative and stochastic by columns for all m.

Note: If $A \in \mathbb{C}^{n \times n}$ is stochastic by columns and S is the matrix transforming A in Jordan form, i.e.

$$S^{-1}AS = \begin{bmatrix} [1] & & \\ & [|\lambda| \geq 1,\, \lambda \neq 1] & \\ & & [|\lambda| = 1] \end{bmatrix},\ [1] = \begin{bmatrix} U_{q_1} & & \\ & \ddots & \\ & & U_{q_g} \end{bmatrix},\ U_s = \left.\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 1 \end{bmatrix}\right\} s,$$

then $\mathbf{e}^T(S\mathbf{e}_j) = 0$, if $j \neq q_1, q_1 + q_2, \ldots, q_1 + q_2 + \cdots + q_g$ (prove it!).