DEGENERATIONS OF SCROLLS TO UNIONS OF PLANES

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ABSTRACT. In this paper we study degenerations of scrolls to union of planes, a problem already considered by G. Zappa in [23] and [24]. We prove, using techniques different from the ones of Zappa, a degeneration result to union of planes with the mildest possible singularities, for linearly normal scrolls of genus g and of degree $d \geq 2g+4$ in $\mathbb{P}^{d-2g+1}$. We also study properties of components of the Hilbert scheme parametrizing scrolls. Finally we review Zappa's original approach.

Dedicated to Professor G. Zappa on his 90th birthday

1. Introduction

In this paper we deal with the problem, originally studied by Guido Zappa in [23, 24], concerning the embedded degenerations of two-dimensional scrolls, to union of planes with the simplest possible singularities.

In [2] and [3], we have studied the properties of the so-called *Zappatic surfaces*, i.e. reduced, connected, projective surfaces which are unions of smooth surfaces with global normal crossings except at singular points, which are locally analytically isomorphic to the vertex of a cone over a union of lines whose dual graph is either a chain of length n, or a fork with $n-1$ teeth, or a cycle of order n, and with maximal embedding dimension. These singular points are respectively called *(good) Zappatic singularities* of type R_n, S_n and E_n (cf. Definition 2.1 below). A Zappatic surface is said to be *planar* if it is embedded in a projective space and all its irreducible components are planes.

An interesting problem is to find degenerations of surfaces to Zappatic surfaces with Zappatic singularities as simple as possible. This problem has been partly considered in [3]; e.g. in Corollary 8.10, it has been shown that, if X is a Zappatic surface which is the flat limit of a smooth scroll of sectional genus $g \geq 2$, then the Zappatic singularities of X cannot be too simple, in particular X has to have some point of type R_i or S_i, with $i \geq 4$, or of type E_j, with $j \geq 6$.

The main results in [23] can be stated in the following way:

Theorem 1.1. *(cf. §12 in* [23]*) Let F be a scroll of sectional genus g, degree $d \geq 3g+2$, whose general hyperplane section is a general curve of genus g. Then F is birationally equivalent to a scroll in $\mathbb{P}^r$, for some $r \geq 3$, which degenerates to a planar Zappatic surface with only points of type R_3 and S_4 as Zappatic singularities.*

Zappa's arguments rely on a rather intricate analysis concerning degenerations of hyperplane sections of the scroll and, accordingly, of the branch curve of a general projection of the scroll to a plane.

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We have not been able to check all the details of this very clever argument. However, we have been able to prove a slightly more general result using some basic smoothing technique (cf. [6]).

Our main result is the following (cf. Proposition 3.8, Constructions 4.1, 4.2, Remarks 4.20, 5.6 and Theorems 4.6, 5.4 later on):

Theorem 1.2. *Let $g \geq 0$ and either $d \geq 2$, if $g = 0$, or $d \geq 5$, if $g = 1$, or $d \geq 2g + 4$, if $g \geq 2$. Then there exists a unique irreducible component $\mathcal{H}_{d,g}$ of the Hilbert scheme of scrolls of degree d and sectional genus g in $\mathbb{P}^{d-2g+1}$, such that the general point of $\mathcal{H}_{d,g}$ represents a smooth scroll S which is linearly normal and moreover with $H^1(S, \mathcal{O}_S(1)) = 0$.*

Furthermore,

(i) *$\mathcal{H}_{d,g}$ is generically reduced and $dim(\mathcal{H}_{d,g}) = (d - 2g + 2)^2 + 7(g - 1)$,*
(ii) *$\mathcal{H}_{d,g}$ contains the Hilbert point of a planar Zappatic surface having only $d - 2g + 2$ points of type R_3 and $2g - 2$ points of type S_4 as Zappatic singularities,*
(iii) *$\mathcal{H}_{d,g}$ dominates the moduli space $\mathcal{M}_g$ of smooth curves of genus g.*

We also construct examples of scrolls S with same numerical invariants, which are not linearly normal in $\mathbb{P}^{d-2g+1}$, as well as examples of components of the Hilbert scheme of scrolls with same invariants, different from $\mathcal{H}_{d,g}$ and with general moduli (cf. Examples 5.11 and 5.12).

We shortly describe the contents of the paper. In § 2 we recall standard definitions and properties of Zappatic surfaces. In § 3 we focus on some degenerations of products of curves to planar Zappatic surfaces and we prove some results which go back to [24]. In particular, we consider Zappatic degerations of rational and elliptic normal scrolls and of abelian surfaces.

In § 4 we prove the greatest part of Theorem 1.2. First, we construct, with an inductive argument, planar Zappatic surfaces which have the same numerical invariants of scrolls of degree d and genus g in $\mathbb{P}^{d-2g+1}$ and having only $d - 2g + 2$ points of type R_3 and $2g - 2$ points of type S_4 as Zappatic singularities. Then we prove that these Zappatic surfaces can be smoothed to smooth scrolls which fill up the component $\mathcal{H}_{d,g}$ and we compute the cohomology of the hyperplane bundle and of the normal bundle. These computations imply that $\mathcal{H}_{d,g}$ is generically smooth, of the right dimension and its general point represents a linearly normal scroll.

Section 5 is devoted to study some properties of components of the Hilbert scheme of scrolls. In particular, we show that the component $\mathcal{H}_{d,g}$ is the unique component of the Hilbert scheme of scrolls of degree d and sectional genus g whose general point $[S]$ is linearly normal in $\mathbb{P}^{d-2g+1}$ and moreover with $H^1(S, \mathcal{O}_S(1)) = 0$. Furthermore, we give the examples mentioned above (cf. Examples 5.11 and 5.12).

In the last section, § 6, we briefly explain Zappa's original approach in [23]. Moreover, we make some comments and give some improvements on some interesting results from [23] concerning extendability of plane curves to scrolls which are not cones.

2. Notation and preliminaries

In this paper we deal with projective varieties defined over the complex field $\mathbb{C}$.

Let us recall the notions of Zappatic singularities, Zappatic surfaces and their dual graphs. We refer the reader for more details to our previous papers [2] and [3]. One word of warning: what we call *good Zappatic singularities* there, here we simply call *Zappatic singularities*, because no other type of Zappatic singularity will be considered in this paper.

Definition 2.1. Let us denote by R_n [resp. S_n, E_n] a graph which is a chain [resp. a fork, a cycle] with n vertices, $n \geq 3$, cf. Figure 1. Let C_{R_n} [resp. C_{S_n}, C_{E_n}] be a connected, projectively normal curve of degree n in $\mathbb{P}^n$ [resp. in $\mathbb{P}^n$, in $\mathbb{P}^{n-1}$], which is a *stick curve*, i.e. a reduced, union of lines with only double points, whose dual graph is R_n [resp. S_n, E_n].

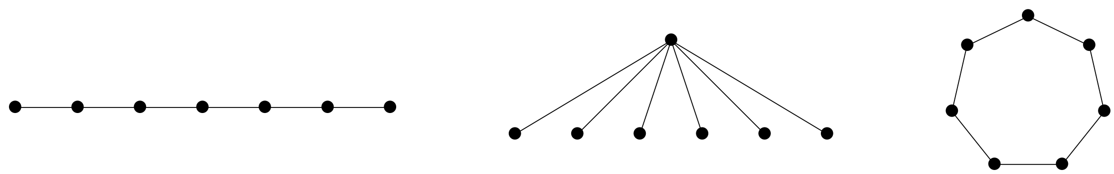

FIGURE 1. A chain R_n, a fork S_n with $n-1$ teeth, a cycle E_n.

We say that a point x of a projective surface X is a *point of type* R_n [resp. S_n, E_n] if (X, x) is locally analytically isomorphic to a pair (Y, y) where Y is the cone over a curve C_{R_n} [resp. C_{S_n}, C_{E_n}], $n \geq 3$, and y is the vertex of the cone (cf. Figure 2). We say that R_n-, S_n-, E_n-points are *Zappatic singularities*.

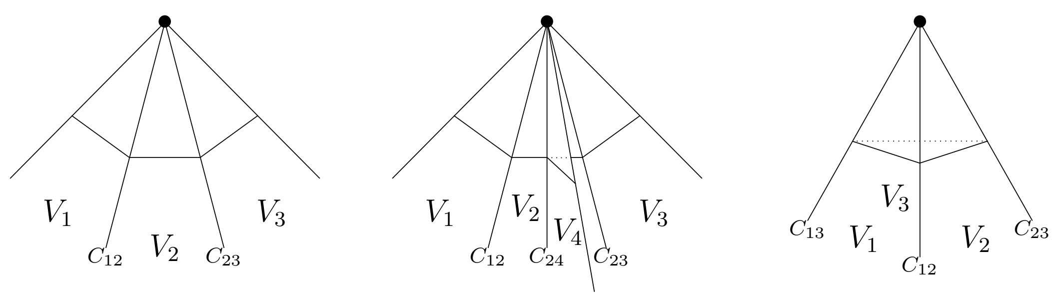

FIGURE 2. Examples: a R_3-point, a S_4-point and an E_3-point.

In this paper we will deal mainly with points of type R_3 and S_4. We will use the following:

Notation 2.2. If x is a point of type R_3 [of type S_4, resp.] of a projective surface X, we say that the component V_2 of X as in picture on the left [in the middle, resp.] in Figure 2 is the *central* component of X passing through x.

Definition 2.3. A projective surface $X = \bigcup_{i=1}^{v} V_i$ is called a *Zappatic surface* if X is connected, reduced, all its irreducible components $V_1, \ldots, V_v$ are smooth and:

- the singularities in codimension one of X are at most double curves which are smooth and irreducible along which two surfaces meet transversally;
- the further singularities of X are Zappatic singularities.

We set $C_{ij} = V_i \cap V_j$ if V_i and V_j meet along a curve, we set $C_{ij} = \emptyset$ otherwise. We set $C_i = V_i \cap \overline{X - V_i} = \bigcup_{j=1}^{v} C_{ij}$. We denote by $C = \mathrm{Sing}(X)$ the singular locus of X, i.e. the curve $C = \bigcup_{1 \leq i < j \leq v} C_{ij}$.

We denote by f_n [resp. r_n, s_n] the number of point of type E_n [resp. R_n, S_n] of X.

Remark 2.4. A Zappatic surface X is Cohen-Macaulay. Moreover it has global normal crossings except at the R_n- and S_n-points, for $n \geq 3$, and at the E_m-points, for $m \geq 4$.

We associate to a Zappatic surface X a *dual graph* G_X as follows.

Definition 2.5. Let $X = \bigcup_{i=1}^{v} V_i$ be a Zappatic surface. The dual graph G_X of X is given by:

- a vertex v_i for each irreducible component V_i of X;
- an edge l_{ij}, joining the vertices v_i and v_j, for each irreducible component of the curve $C_{ij} = V_i \cap V_j$;
- a *n-face* F_p for each point p of X of type E_n for some $n \geq 3$: the n edges bounding the face F_p are the n irreducible components of the double curve C of X concurring at p;
- an *open n-face* for each point p of X of type R_n for some $n \geq 3$; it is bounded by $n-1$ edges, corresponding to the $n-1$ irreducible components of the double curve of X concurring at p, and by a *dashed* edge, which we add in order to join the two extremal vertices;
- a *n-angle* for each p of X of type S_n, spanned by the $n-1$ edges that are the $n-1$ irreducible components of the double curves of X concurring at p.

By abusing notation, we will denote by G_X also the CW-complex associated to the dual graph G_X of X, formed by vertices, edges and n-faces.

Remark 2.6 (cf. [2])**.** When we deal with the dual graph of a *planar* Zappatic surface $X = \bigcup_{i=1}^{v} V_i$, we will not indicate open 3-faces with a dashed edge. Indeed, the graph itself shows where open 3-faces are located.

Some invariants of a Zappatic surface X have been computed in [2] and in [4], namely the Euler-Poincaré characteristic $\chi(\mathcal{O}_X)$, the *ω-genus* $p_\omega(X) = h^0(X, \omega_X)$, where ω_X is the dualizing sheaf of X, and, when X is embedded in a projective space $\mathbb{P}^r$, the *sectional genus* $g(X)$, i.e. the arithmetic genus of a general hyperplane section of X. In particular, for a planar Zappatic surface (for the general case, see [2, 4]) one has:

Proposition 2.7. *Let $X = \bigcup_{i=1}^{v} V_i$ be a planar Zappatic surface of degree v in $\mathbb{P}^r$ and denote by e the degree of $C = \mathrm{Sing}(X)$, i.e. the number of double lines of X. Then:*

$$g(X) = e - v + 1, \tag{2.8}$$

$$p_\omega(X) = h^0(X, \omega_X) = h_2(G_X, \mathbb{C}), \tag{2.9}$$

$$\chi(\mathcal{O}_X) = \chi(G_X) = v - e + \sum_{i \geq 3} f_i. \tag{2.10}$$

In this paper, a Zappatic surface will always be considered as the central fibre of an embedded degeneration, in the following sense.

Definition 2.11. Let Δ be the spectrum of a DVR (or equivalently the complex unit disk). A *degeneration* of surfaces parametrized by Δ is a proper and flat morphism $\pi : \mathcal{X} \to \Delta$ such that each fibre $\mathcal{X}_t = \pi^{-1}(t)$, $t \neq 0$ (where 0 is the closed point of Δ), is a smooth, irreducible, projective surface. A degeneration $\pi : \mathcal{X} \to \Delta$ is said to be *embedded* in $\mathbb{P}^r$ if $\mathcal{X} \subseteq \Delta \times \mathbb{P}^r$ and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} & \subseteq & \Delta \times \mathbb{P}^r \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_1} & \\ \Delta & & \end{array}$$

The invariants of the Zappatic surface $X = \mathcal{X}_0$, which is the central fibre of an embedded degeneration $\mathcal{X} \to \Delta$, determine the invariants of the general fibre $\mathcal{X}_t$, $t \neq 0$, as we proved in [2, 3, 4]. Again, we recall these results only for planar Zappatic surfaces and we refer to our previous papers for the general case.

Theorem 2.12. *Let $\mathcal{X} \to \Delta$ be an embedded degeneration in $\mathbb{P}^r$ such that the central fibre $X = \mathcal{X}_0$ is a planar Zappatic surface. Then, for any $0 \neq t \in \Delta$:*

$$g(\mathcal{X}_t) = g(X), \qquad p_g(\mathcal{X}_t) = p_\omega(X), \qquad \chi(\mathcal{O}_{\mathcal{X}_t}) = \chi(\mathcal{O}_X). \tag{2.13}$$

Moreover the self-intersection $K^2_{\mathcal{X}_t}$ of a canonical divisor of $\mathcal{X}_t$ is:

$$K^2_{\mathcal{X}_t} = 9v - 10e + \sum_{n\geq 3} 2nf_n + r_3 + k, \tag{2.14}$$

where k depends on the presence of points of type R_m and S_m, $m \geq 4$:

$$\sum_{m\geq 4}(m-2)(r_m + s_m) \leq k \leq \sum_{m\geq 4}(2m-5)r_m + \binom{m-1}{2} s_m.$$

Finally, let us recall the construction of rational normal scrolls.

Definition 2.15. Fix two positive integers a, b and set $r = a+b+1$. In $\mathbb{P}^r$ choose two disjoint linear spaces $\mathbb{P}^a$ and $\mathbb{P}^b$. Let C_a [resp. C_b] be a smooth, rational normal curve of degree a in $\mathbb{P}^a$ [resp. of degree b in $\mathbb{P}^b$] and fix an isomorphism $\phi : C_a \to C_b$. Then, the union in $\mathbb{P}^r$ of all the lines $\overline{p, \phi(p)}$, $p \in C_a$, is a smooth, rational, projectively normal surface which is called *scroll of type* (a, b) and it is denoted by $S_{a,b}$. Such a scroll is said to be *balanced* if either $b = a$ or $b = a + 1$.

Another way to define a scroll is as the embedding of a Hirzebruch surface $\mathbb{F}_n$, $n \geq 0$, which is the minimal ruled surface over $\mathbb{P}^1$ with a section of self-intersection $(-n)$. Setting F the ruling of $\mathbb{F}_n$ and C a section such that $C^2 = n$, the linear system $|C + aF|$ embeds $\mathbb{F}_n$ in $\mathbb{P}^{n+2a+1}$ as a scroll of type $(a, a+n)$, cf. e.g. [14]. In particular a balanced scroll in $\mathbb{P}^r$, $r \geq 3$, is the embedding either of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or of $\mathbb{F}_1$ depending on whether r is odd or even.

In the next section we will see, in particular, degenerations of rational scrolls to a planar Zappatic surface. In the subsequent section we will deal with scrolls of higher genus.

3. Degenerations of product of curves and of rational scrolls

Zappa suggested in [24] an interesting method for degenerating products of curves, which also gives a degeneration of rational and elliptic scrolls to planar Zappatic surfaces with only R_3-points.

Example 3.1 (Zappa)**.** Let $C \subset \mathbb{P}^{n-1}$ and $C' \subset \mathbb{P}^{m-1}$ be smooth curves. If C and C' may degenerate to stick curves, then the smooth surface

$$S = C \times C' \subset \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \subset \mathbb{P}^{nm-1},$$

embedded via the Segre map, degenerates to a Zappatic surface Y in $\mathbb{P}^{nm-1}$ whose irreducible components are quadrics and whose double curves are lines.

If it is possible to further, independently, degenerate each quadric of Y to the union of two planes, then one gets a degeneration of $S = C \times C'$ to a planar Zappatic surface. This certainly happens if each quadric of Y meets the other quadrics of Y along a union of at most four lines, at most two from each ruling (see Figure 3).

Therefore $S = C \times C'$ can degenerate to a planar Zappatic surface if C and C' are either rational or elliptic normal curves, since they degenerate to stick curves C_{R_n} and C_{E_n}, respectively. We will now describe these degenerations.

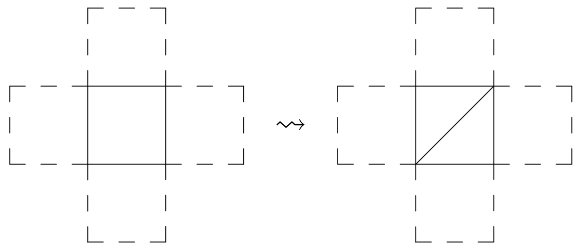

FIGURE 3. A quadric degenerating to the union of two planes

Example 3.2 (Rational scrolls). Let C be a smooth, rational normal curve of degree n in $\mathbb{P}^n$. Since C degenerates to a union of n lines whose dual graph is a chain, the smooth rational normal scroll $S = C \times \mathbb{P}^1 \subset \mathbb{P}^{2n+1}$ degenerates to a Zappatic surface $Y = \bigcup_{i=1}^{n} Y_i$ such that each Y_i is a quadric, Y has no Zappatic singularity and its dual graph G_Y is a chain of length n, see Figure 4.

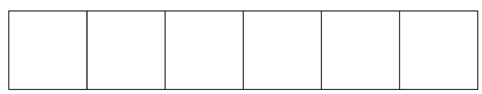

FIGURE 4. Chain of n quadrics as in Example 3.2

Each quadric Y_i meets $\overline{Y \setminus Y_i}$ either along a line or along two distinct lines of the same ruling. Thus, as we noted before, the quadric Y_i degenerates, in the $\mathbb{P}^3$ spanned by Y_i, to the union of two planes meeting along a line l_i, leaving the other line(s) fixed. Therefore, in $\mathbb{P}^{2n+1}$, the scroll S degenerates also to a planar Zappatic surface X of degree $2n$. The line l_i can be chosen generally enough so that X has $2n-2$ points of type R_3 as Zappatic singularities, for each i, i.e. its dual graph G_X is a chain of length $2n$, see Figure 5 (cf. Remark 2.6).

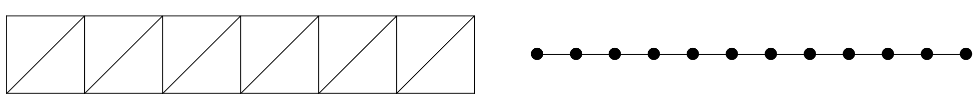

FIGURE 5. Planar Zappatic surface of degree $2n$ with a chain as dual graph

Example 3.3 (Elliptic scrolls). Let C be a smooth, elliptic normal curve of degree n in $\mathbb{P}^{n-1}$. Since C degenerates to a union of n lines whose dual graph is a cycle, the smooth elliptic normal scroll $S = C \times \mathbb{P}^1 \subset \mathbb{P}^{2n-1}$ degenerates to a Zappatic surface $Y = \bigcup_{i=1}^{n} Y_i$, such that each Y_i is quadric, Y has no Zappatic singularity and its dual graph G_Y is a cycle of length n, see the picture on the left in Figure 6.

Each quadric Y_i meets $\overline{Y \setminus Y_i}$ along two distinct lines r_i, r_i' of the same ruling. Hence, in the $\mathbb{P}^3$ spanned by Y_i, the quadric Y_i degenerates to the union of two planes meeting along a line l_i, leaving r_i, r_i' fixed. Choosing again a general l_i for each i, it follows that in $\mathbb{P}^{2n-1}$ the scroll S degenerates to a planar Zappatic surface X of degree $2n$ with $2n$ points of type R_3 as Zappatic singularities and its dual graph G_X is a cycle of length $2n$, see Figure 6.

Example 3.4 (Abelian surfaces). Let $C \subset \mathbb{P}^{n-1}$ and $C' \subset \mathbb{P}^{m-1}$ be smooth, elliptic normal curves of degree respectively n and m. Then C and C' degenerate to the stick curves C_{E_n} and

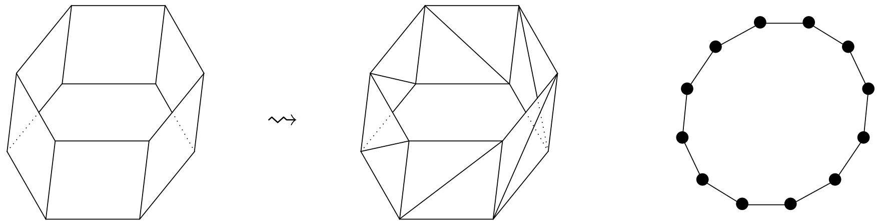

FIGURE 6. Cycle of n quadrics and of $2n$ planes as in Example 3.3

C_{E_m} respectively, hence the abelian surface $S = C \times C' \subset \mathbb{P}^{nm-1}$ degenerates to a Zappatic surface which is a union of mn quadrics with only E_4-points as Zappatic singularities, cf. e.g. the picture on the left in Figure 7, where the top edges have to be identified with the bottom ones, similarly the left edges have to be identified with the right ones. Thus the top quadrics meet the bottom quadrics and the quadrics on the left meet the quadrics on the right.

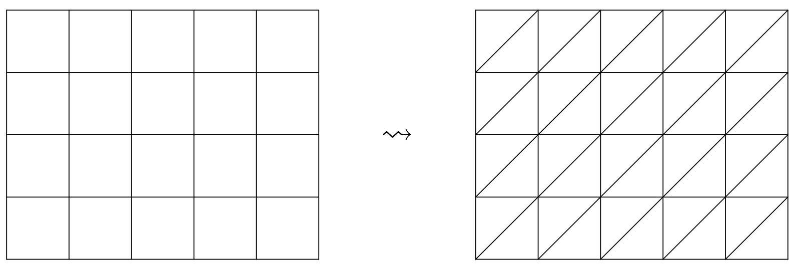

FIGURE 7. nm quadrics with E_4-points and $2nm$ planes with E_6-points

Again each quadric degenerates to the union of two planes. By doing this as depicted in Figure 7, one gets a degeneration of a general abelian surface with a polarization of type (n, m) to a planar Zappatic surface of degree $2nm$ with only E_6-points as Zappatic singularities.

Other examples of degenerations, similar to the one considered above, for $K3$ surfaces (the so called *pillow degenerations*) are considered in e.g. [8].

Remark 3.5. Going back to the general case, if either C or C' has genus greater than 1 and if they degenerate to stick curves, then the surface $S = C \times C'$ degenerates to a union of quadrics, as we said. Unfortunately it is not clear if it is possible to further independently degenerate each quadric to two planes.

From now on, until the end of this section, we deal with degenerations of rational normal scrolls only. Namely we will show that a general rational normal scroll degenerates to a planar Zappatic surface with Zappatic singularities of type R_3 only and we will see how "general" the scroll has to be in order to admit such degenerations (e.g., in Example 3.2, the scrolls are actually forced to have even degree).

There are several ways to construct these degenerations. We will start from the trivial family and then we will perform two basic operations: (1) blowing-ups and blowing-downs in the central fibre, (2) twisting the hyperplane bundle by a component of the central fibre.

Construction 3.6. *Let $S = S_{a,b}$ be a smooth, rational, normal scroll of type (a, b) in $\mathbb{P}^r$, where $r = a + b + 1 \geq 3$ and we assume that $b \geq a$. Then S degenerates to the union of a plane and a smooth, rational normal scroll $S_{a,b-1}$ meeting the plane along a ruling.*

Indeed, S is the embedding of the Hirzebruch surface $\mathbb{F}_n$, $n = b - a \geq 0$, via the linear system $|C + aF|$, where F is the ruling and C is a section of self-intersection n (clearly, if $n = 0$, we may choose F to be either one of the two rulings and C to be the other ruling). Set $H = C + aF$. Consider the trivial family $\mathcal{S} = \mathbb{F}_n \times \Delta \xrightarrow{\sigma} \Delta$. On $\mathcal{S}$ we have the hyperplane bundle $\mathcal{H}$ which coincides with H on each fibre of σ.

Now blow up $\mathcal{S}$ at a general point of the central fibre $\mathcal{S}_0$. Let V be the exceptional divisor and S' be the proper transform of $\mathcal{S}_0$. Then, $\mathcal{H} \otimes \mathcal{O}(-V)$ embeds V as a plane and maps S' to a scroll of type $(a, b-1)$, which meet each other along a ruling of S'. We explain these operations in Figure 8, where the dotted lines represent the hyperplane bundle. The last arrow is the so-called *type I transformation* on the vertical (-1)-curve (cf. [11]), which consists in blowing up the (-1)-curve and then blowing down the exceptional divisor, which is a $\mathbb{F}_0$, along the other ruling. The total effect on $\mathcal{S}_0$ is to perform an elementary transformation.

When $r = 3$ this process gives the degeneration of a smooth quadric to two planes meeting along a line.

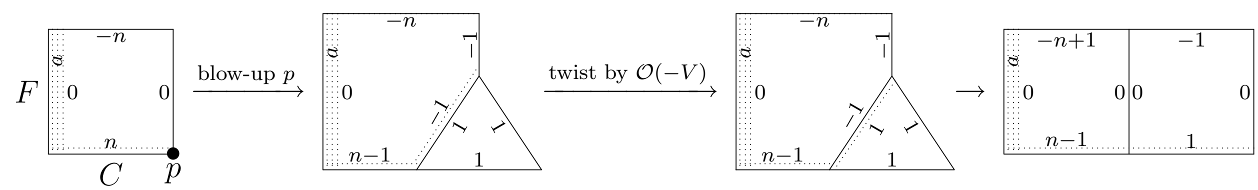

FIGURE 8. Degeneration of a scroll $S_{a,b}$ to the union of a plane and a scroll $S_{a,b-1}$

Construction 3.7. *Let $S = S_{a,b}$ be a smooth, rational, normal scroll of type (a, b) in $\mathbb{P}^r$, where $r = a+b+1$ and assume that $b \geq a > 1$. Then S degenerates to the union of a quadric and a smooth, rational normal scroll $S_{a-1,b-1}$ meeting the quadric along a ruling.*

Indeed, consider the Hirzebruch surface $\mathbb{F}_n$, $n = b - a \geq 0$, and the trivial family $\mathcal{S} = \mathbb{F}_n \times \Delta \xrightarrow{\sigma} \Delta$, with the hyperplane bundle $\mathcal{H}$, as in Construction 3.6.

Now blow up a ruling F_0 in the central fibre $\mathcal{S}_0$. Let W be the exceptional divisor and S' be the proper transform of $\mathcal{S}_0$. Then $\mathcal{H} \otimes \mathcal{O}(-W)$ embeds W as a quadric and S' as a scroll of type $(a-1, b-1)$, which meet along a ruling of S', cf. Figure 9.

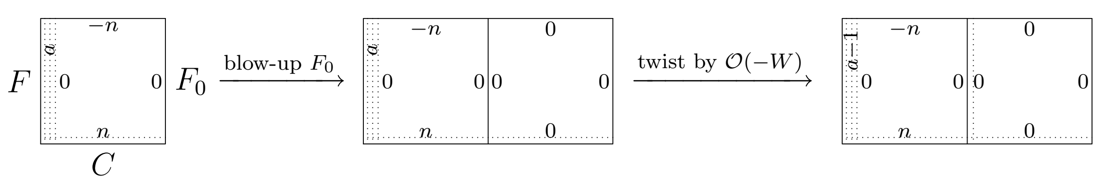

FIGURE 9. Degeneration of a scroll $S_{a,b}$ to the union of a quadric and a scroll $S_{a-1,b-1}$

By induction on the degree of the scroll and by using Constructions 3.6 and 3.7 for the inductive steps, we now show the following:

Proposition 3.8. *Let $d \geq 2$ and set $r = d + 1 \geq 3$. Let $X := X_{d,0}$ be a planar Zappatic surface of degree d in $\mathbb{P}^r$, whose dual graph is a chain, i.e. X has $d - 2$ points of type R_3 as Zappatic singularities. Then, the Hilbert point of X belongs to the irreducible component $\mathcal{H}_{d,0}$ of the Hilbert scheme parametrizing rational normal scrolls of degree d.*

Remark 3.9. It is well-known (cf. e.g. Lemma 3 in [6]) that $\mathcal{H}_{d,0}$ is generically reduced and of dimension $d^2 + 4d - 3$.

Proof of Proposition 3.8. We will directly show that a smooth, balanced scroll S degenerates to X.

Suppose first that r is even. Let $S = S(a, a+1)$ be a balanced scroll of degree d in $\mathbb{P}^r$, i.e. $a = (d-1)/2 = r/2 - 1$. Consider the trivial family $\mathbb{F}_1 \times \Delta$, where $\mathbb{F}_1$ is embedded in $\mathbb{P}^r$ by the linear system $|C + aF|$, such as in Constructions 3.6 and 3.7, cf. the picture on the left in Figure 10.

Now blow up a ruling in the central fibre, call $W \cong \mathbb{F}_0$ the exceptional divisor and twist the hyperplane bundle by $\mathcal{O}(-aW)$. In this way, one gets a degeneration of S to the union of a scroll of type (a, a) in $\mathbb{P}^{r-1}$ and a plane, meeting along a ruling, cf. Construction 3.7 and the picture in the middle of Figure 10.

Then blow up a general point (the bottom left corner in Figure 10) of the scroll, twist again by the opposite of the new surface and perform a type I transformation, as we did in Construction 3.6. By twisting again by the opposite of the new surface, counted with multiplicity $a-1$, one gets the configuration depicted on the right in Figure 10, namely the first two components are two planes, whereas the new component is a scroll of type $(a-1, a)$.

Going on by induction on a, by following the same process, one gets a chain of planes which is a planar Zappatic surface with only R_3-points, as wanted.

If r is odd, one starts from a $\mathbb{F}_0$ as in the central picture of Figure 10 and one may perform exactly the same operations in order to get a similar degeneration. ☐

Remark 3.10. In practice, Proposition 3.8 follows by Contructions 3.6 and 3.7 with a suitable induction. The explicit argument we made in the proof shows that there exists a flat degeneration of smooth, rational scrolls to X whose total space is singular only at the R_3-points of X. For another approach, the reader is also referred to [18].

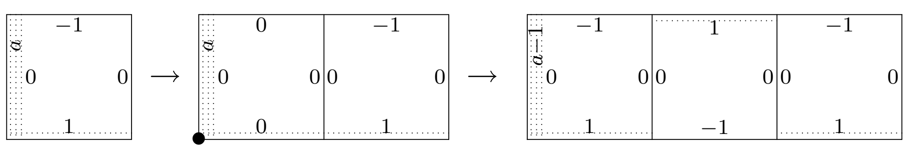

FIGURE 10. Degeneration of $S_{a,a+1}$ to a planar Zappatic surface with only R_3-points

Remark 3.11. Suppose to have a smooth scroll S which is the general fibre of an embedded degeneration in $\mathbb{P}^r$ to a Zappatic planar surface X. The ruling of S, considered as a curve Γ in the Grasmannian $\mathbb{G}(1, r)$, accordingly degenerates to a stick-curve Γ_0. This means that the ruling degenerates to a union of pencils of lines, one in each plane of X. Since Γ_0 is connected, each double line of X belongs to the pencil in either one of the two planes containing it. Hence, the centers of the pencils also belong to the double lines of X. Therefore, on each plane which contains more than one double line of X, all the double lines pass through the same Zappatic singularity which is the center of the pencil. However, the location of the centers of the pencils on the planes containing only one double line of X is not predictable.

We conclude this section by proving the following:

Proposition 3.12. *Let $S = S_{a,b}$ be a smooth, rational normal scroll in $\mathbb{P}^{a+b+1}$, with $b-a \geq 4$. Assume that S is the general fibre of a degeneration whose central fibre is a planar Zappatic surface X. Then X has worse singularities than R_3-points.*

Proof. By construction of the scroll S (cf. Definition 2.15), the minimum degree of a section of S is a and let C_a be the section of degree a. Suppose by contradiction that S is the general fibre of an embedded degeneration of surfaces whose central fibre is a planar Zappatic surface $X = \bigcup_{i=1}^{a+b} V_i$ in $\mathbb{P}^{a+b+1}$, with only R_3-points as Zappatic singularities. Then the dual graph G_X is a chain and we may and will assume that two planes V_i and V_j meet along a line if and only if $j = i \pm 1$.

While S degenerates to X, the ruling of S degenerates to a pencil of lines Λ_i on each plane V_i, $i = 1, \ldots, a+b$ (cf. Remark 3.11) and the section C_a degenerates to a chain of lines $l_1, \ldots, l_a$, with $l_i \subset V_{j_i}$, $i = 1, \ldots, a$, and we may and will assume that $j_1 < j_2 < \cdots < j_a$.

The pencil Λ_1 has to meet $\bigcup_{i=1}^{a} l_i$, hence V_1 has to have non-empty intersection with V_{j_1}, therefore the assumption that X has at most R_3-points implies that $j_1 \leq 3$. For each $k = 2, \ldots, a$, the lines l_k and l_{k-1} meet at a point, so the same argument implies that $j_k \leq j_{k-1} + 2$ (cf. Figure 11). It follows that $j_a \leq j_1 + 2(a-1) \leq 2a + 1$.

On the other hand, the pencil Λ_{a+b} has to meet $\bigcup_{i=1}^{a} l_i$, hence $j_a \geq a + b - 2$. In conclusion, one has that:

$$a + b - 2 \leq j_a \leq 2a + 1,$$

which contradicts the assumption that $b \geq a + 4$. □

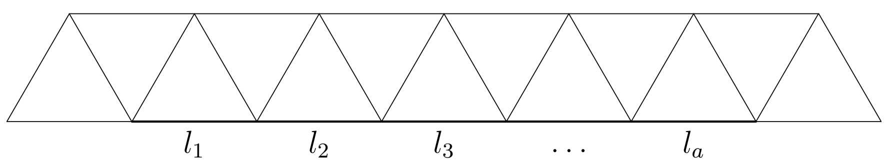

FIGURE 11. Degeneration of $S_{a,b}$, $b = a + 3$, to X with only R_3-points

For another approach to degenerations of rational scrolls to unions of planes, the reader is referred to [18].

Remark 3.13. By following the lines of the proof of Proposition 3.8 it is possible to prove that, given a, b positive integers such that $0 \leq b - a \leq 3$, there exist degenerations whose general fibre is a scroll of type $S(a, b)$ and whose central fibre is a planar Zappatic surface with only R_3-points as Zappatic singularities (cf. Figure 11). We will not dwell on this here.

4. Degenerations of scrolls: inductive constructions

In this section we produce families of smooth scrolls of any genus $g \geq 0$ which degenerate to planar Zappatic surfaces with Zappatic singularities of types R_3 and S_4 only.

We start by describing the planar Zappatic surfaces which will be the limits of our scrolls. We will construct these Zappatic surfaces by induction on g. From now on in this section, we will denote by $X_{d,g}$ a planar Zappatic surface consisting of d planes and whose sectional genus is g.

We start with the case $g = 1$.

Construction 4.1. *For any $d \geq 5$, there exists a planar Zappatic surface $X_{d,1} = \bigcup_{i=1}^{d} V_i$ in $\mathbb{P}^r$, with $r = d - 1$, whose dual graph is a cycle.*

Indeed, if $p_1, \ldots, p_d$ are the coordinate points of $\mathbb{P}^r$, we may let V_i, $i = 2, \ldots, d-1$, be the plane spanned by p_{i-1}, p_i, p_{i+1} and let $V_1 = \langle p_d, p_1, p_2 \rangle$, $V_d = \langle p_{d-1}, p_d, p_1 \rangle$. Then $X_{d,1} = \bigcup_{i=1}^{d} V_i$ is a planar Zappatic surface with dual graph a cycle and whose Zappatic singularities are points of type R_3 at $p_1, \dots, p_d$, cf. Figure 12, where one identifies the line $\langle p_d, p_1 \rangle$ on the left with the same line on the right.

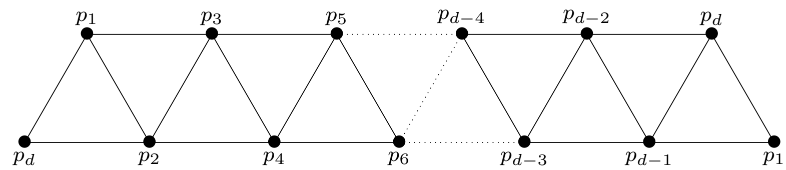

FIGURE 12. Planar Zappatic surface $X_{d,1}$ with dual graph a cycle

We will show in Theorem 4.6 that $X_{d,1}$ is the flat limit of a smooth scroll of genus 1 in $\mathbb{P}^r$. In order to do that, now we describe another way to construct $X_{d,1}$, which will also help to understand the next inductive steps.

Let $X_{d-2,0} = \bigcup_{i=1}^{d-2} V_i$ be a planar Zappatic surface of degree $d-2$ in $\mathbb{P}^r$, whose dual graph is a chain. We may and will assume that the planes V_i and V_j meet along a line if and only if $j = i \pm 1$.

Now choose a general line l_1 in V_1 and a general line l_2 in V_{d-2}, thus l_1 [resp. l_2] does not pass through the R_3-point $V_1 \cap V_2 \cap V_3$ [resp. $V_{d-4} \cap V_{d-3} \cap V_{d-2}$]. Clearly the lines l_1 and l_2 are skew and span a $\mathbb{P}^3$, call it Π. By a computation in coordinates one proves that, if $d \geq 6$, then $\Pi \cap X_0 = l_1 \cup l_2$. Therefore there exists a smooth quadric Q' in Π such that l_1, l_2 are lines of the same ruling on Q' and Q' meets X_0 transversally along $Q' \cap X_0 = l_1 \cup l_2$. On the other hand, if $d = 5$, then $\Pi \cap X_0 = l_1 \cup l_2 \cup l$, where l is a line in the central plane. Nonetheless it is still true that there exists a smooth quadric Q' which contains l_1 and l_2 and meets X_0 transversally.

Finally, in Π, the quadric Q' degenerates to two planes V_{d-1} and V_d, such that $l_i \subset V_{d-i+1}$, $i = 1, 2$. By construction, the planar Zappatic surface $X_{d,1} = X_{d-2,0} \cup V_{d-1} \cup V_d = \bigcup_{i=1}^{d} V_i$ has dual graph which is a cycle, hence it has only R_3-points as Zappatic singularities (cf. Example 3.3 and Figure 13). Note that, if $d \geq 6$, then there are pairs of disjoint planes in the cycle.

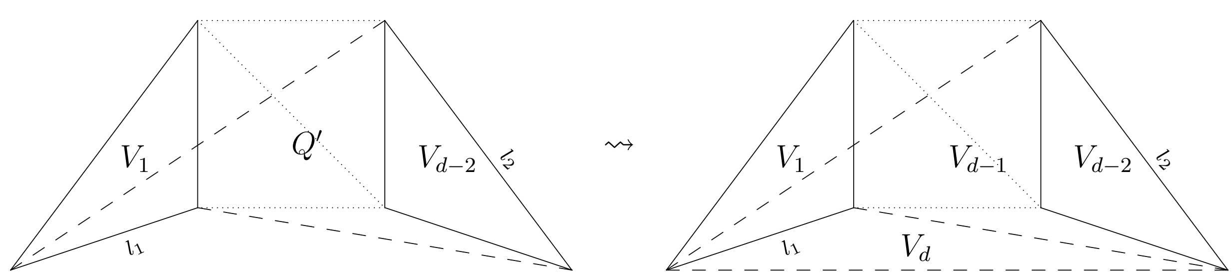

FIGURE 13. Construction of $X_{d,1}$ from $X_{d-2,0}$

Next, we complete the construction proceeding inductively.

Construction 4.2. *Fix integers g, d such that $g \geq 2$ and $d \geq 2g + 4$. Set $c = d - 2g - 4 \geq 0$ and $r = 5 + c = d - 2g + 1$. There is a planar Zappatic surface $X_{d,g} = \bigcup_{i=1}^{d} V_i$ in $\mathbb{P}^r$ such that:*

- *$X_{d,g}$ has $3g + 6 + c$ double lines, i.e. its dual graph $G_{X_{d,g}}$ has $3g + 6 + c$ edges;*

- *$X_{d,g}$ has $r+1$ points of type R_3 and $2(g-1)$ points of type S_4;*
- *for each i, V_i is the central plane through a point p of type either R_3 or S_4, i.e. V_i is the central component of $X_{d,g}$ passing through p as defined in Notation 2.2;*
- *there exist two R_3-points of $X_{d,g}$ whose central planes do not meet;*
- $\chi(\mathcal{O}_{X_{d,g}}) = 1-g$, $p_\omega(X_{d,g}) = 0$, $q(X_{d,g}) = g(X_{d,g}) = g$.

Taking into account Construction 4.1, which covers $g=1$ and $d \geq 5$, we can proceed by induction and assume that we have the surface $X_{d-2,g-1}$. Let V_1 and V_2 be disjoint planes in $X_{d-2,g-1}$ such that each one of them is the central plane for a R_3-point, say p_1 and p_2 respectively.

Now choose a line l_1 in V_1 [resp. l_2 in V_2] which is general among those passing through p_1 [resp. through p_2]. Then l_1 and l_2 are skew and span a $\mathbb{P}^3$, say Π, therefore there exists a smooth quadric Q' in Π containing l_1 and l_2 as lines of the same ruling, cf. Figure 14.

Now we prove the following:

Claim 4.3. *For general choices, Q' and $X_{d-2,g-1}$ meet transversally along $X_{d-2,g-1} \cap Q' = l_1 \cup l_2$.*

Proof. In order to prove the claim, it suffices to show that Π does not meet the remaining components of X_{g-1} along a curve, i.e. that Π does not meet V_i, $i \neq 1,2$, along a line. Before proving the claim, we make a remark. Suppose that there are two further planes, say V_3 and V_4, in $X_{d-2,g-1}$ contained in $\langle V_1, V_2 \rangle = \Sigma \cong \mathbb{P}^5$. Suppose also that the dual graph of the planar Zappatic surface $V_1 \cup V_3 \cup V_4 \cup V_2$ is a chain of length 4. Then the points $V_1 \cap V_3 \cap V_4$ and $V_3 \cap V_4 \cap V_2$ are of type R_3. Note that this certainly happens if $c=0$ and $g=2$ because in that case the dual graph of $X_{d-2,g-1}$ is a cycle of length six.

In this situation, a computation in coordinates in Σ shows that for a general choice of l_1 and l_2, $\Pi = \langle l_1, l_2 \rangle$ does not intersect either V_3 or V_4 along a line.

Now we prove the claim arguing by contradiction. Fix the line l_2 in V_2 and consider $\langle l_2, V_1 \rangle = \Omega \cong \mathbb{P}^4$. By moving l_1 in the pencil of lines of V_1 through p_1, one gets a pencil Φ of $\mathbb{P}^3$'s inside Ω and each of these $\mathbb{P}^3$'s meets a plane, say V_3, along a line. There are two possibilities: either $V_3 \subset \Omega$, or $V_3 \not\subseteq \Omega$.

In the former case, V_3 intersects V_1 at a point q. Let l_2 move in the pencil of lines of V_2 through p_2: one gets a pencil of $\mathbb{P}^4$'s in $\Sigma = \langle V_1, V_2 \rangle$, whose base-locus is $\langle p_1, V_2 \rangle \cong \mathbb{P}^3$ in which V_3 is contained. This implies that $q = p_1$, moreover V_3 intersects V_2 along a line which necessarily contains p_2. In conclusion, V_3 contains the line passing through p_1 and p_2. This yields the existence of a plane V_4 which forms, together with V_1, V_2 and V_3, a configuration in Σ of four planes as the one discussed above. This is a contradiction.

Suppose now that $V_3 \not\subseteq \Omega$. Then V_3 meets along a line the base locus of the pencil Φ, which is the plane $\langle p_1, l_2 \rangle$. By moving l_2, we see that V_3 has to contain the line through p_1 and p_2 and we get a contradiction as before. □

In Π, the smooth quadric Q' degenerates to the union of two planes, say $V_{d-1} \cup V_d$, where $l_i \subset V_{d-i+1}$, $i = 1,2$. Consider the planar Zappatic surface $X_{d,g} = X_{d-2,g-1} \cup V_{d-1} \cup V_d$ of degree d in $\mathbb{P}^r$. Thus, we added to $X_{d-2,g-1}$ two planes and three double lines $V_1 \cap V_d$, $V_d \cap V_{d-1}$ and $V_{d-1} \cap V_2$. Moreover, the points p_1 and p_2 become points of type S_4 for X_g and we added two further points of type R_3 at $V_1 \cap V_d \cap V_{d-1}$ and $V_d \cap V_{d-1} \cap V_2$, cf. Figure 14. Finally, one checks that each one of the planes V_{d-1} and V_d is disjoint from some other plane in the configuration. This ends the construction.

Next, we will prove that the Zappatic surfaces $X_{d,g}$ we constructed are limits of smooth scrolls of genus g. First we make a remark.

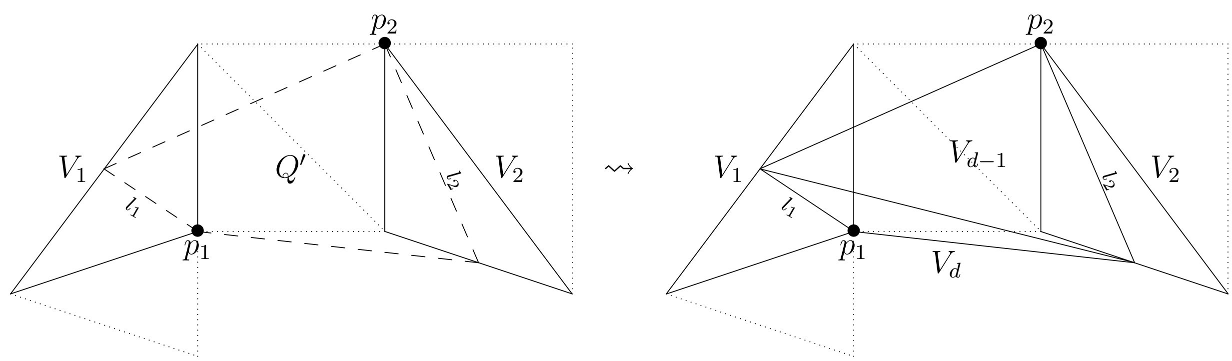

FIGURE 14. Construction of $X_{d,g}$ from $X_{d-2,g-1}$

Remark 4.4. If $X_{d,g}$ is the flat limit of a family of smooth surfaces Y, then Theorem 2.12 implies that:

$$g(Y) = g, \qquad p_g(Y) = 0, \qquad \chi(\mathcal{O}_Y) = 1 - g, \qquad 8(1-g) \le K_Y^2 \le 6(1-g). \tag{4.5}$$

Theorem 4.6. *Let $g \ge 0$ and $d \ge 2g+4$ be integers. Let $r = d - 2g + 1$. The Hilbert point corresponding to the planar Zappatic surface $X_{d,g}$ belongs to an irreducible component $\mathcal{H}_{d,g}$ of the Hilbert scheme of scrolls of degree d and genus g in $\mathbb{P}^r$, such that:*

(i) *the general point of $\mathcal{H}_{d,g}$ represents a smooth, linearly normal scroll $Y \subset \mathbb{P}^r$;*
(ii) *$\mathcal{H}_{d,g}$ is generically reduced, $dim(\mathcal{H}_{d,g}) = h^0(Y, \mathcal{N}_{Y/\mathbb{P}^r}) = (r+1)^2 + 7(g-1)$, and moreover $h^1(Y, \mathcal{N}_{Y/\mathbb{P}^r}) = h^2(Y, \mathcal{N}_{Y/\mathbb{P}^r}) = 0$.*

Proof of Theorem 4.6: beginning. We prove Theorem 4.6 by induction on g. The case $g = 0$ has been treated in Proposition 3.8. By induction on g, we may assume that $X_{d-2,g-1}$ is the flat limit of a smooth scroll S of degree $d-2$ and genus $g-1$ in $\mathbb{P}^r$, which is represented by a smooth point of a component $\mathcal{H}_{d-2,g-1}$ of the Hilbert scheme of dimension $(r+1)^2 + 7(g-2)$.

We can now choose l_1 and l_2 as in Constructions 4.1 and 4.2 so that they are limits of rulings F_1 and F_2, respectively, on S (cf. Remark 3.11).

Let Q be a smooth quadric containing F_1 and F_2, whose limit is Q'. By the properties of $X_{d-2,g-1}$ and of Q' (see Claim 4.3), it follows that S and Q meet transversally along $S \cap Q = F_1 \cup F_2$.

The inductive step is a consequence of the following lemma. □

Lemma 4.7. *In the above setting, consider the union*

$$R := S \cup Q.$$

Let $\mathcal{N}_R$ and $\mathcal{T}_R$ be the normal and the tangent sheaf of R in $\mathbb{P}^r$, respectively; then, one has:

$$H^1(\mathcal{N}_R) = H^2(\mathcal{N}_R) = 0, \tag{4.8}$$

$$h^0(\mathcal{N}_R) = (r+1)^2 + 7(g-1) = d^2 - 4dg + 4d + 4g^2 - g - 3. \tag{4.9}$$

Furthermore the natural map $H^0(\mathcal{N}_R) \to H^0(T^1)$, induced by the exact sequence

$$0 \to \mathcal{T}_R \to \mathcal{T}_{\mathbb{P}^r}|_R \xrightarrow{\tau} \mathcal{N}_R \to T^1 := Coker(\tau) \to 0, \tag{4.10}$$

is surjective.

Proof. We will compute the cohomology of $\mathcal{N}_R$, by using a similar technique as in section 2.2 of [6] (see Lemma 3 therein).

Let $\Gamma := S \cap Q = F_1 \cup F_2$ be the double curve of R. Since R has global normal crossings, the sheaf T^1 in (4.10) is locally free, of rank 1 on the singular locus Γ of R and, by [10], it is

$$T^1 \cong \mathcal{N}_{\Gamma/S} \otimes \mathcal{N}_{\Gamma/Q}.$$

Since Γ is the union of two lines of the same ruling on both Q and S, it follows that

$$T^1 \cong \mathcal{O}_\Gamma. \tag{4.11}$$

Let us consider the inclusions $\iota_S : \mathcal{N}_S \to \mathcal{N}_R|_S$ and $\iota_Q : \mathcal{N}_Q \to \mathcal{N}_R|_Q$. Lemma 2 in [6] shows that $T^1 \cong \mathrm{coker}(\iota_S)$ and $T^1 \cong \mathrm{coker}(\iota_Q)$. For readers' convenience, we recall here the proof. By a local computation, one sees that the cokernel K of ι_S is locally free of rank 1 on Γ. In the diagram

$$\begin{array}{ccccccc} \mathcal{T}_{\mathbb{P}^r}|_R & \longrightarrow & \mathcal{N}_R & \longrightarrow & T^1 & \longrightarrow & 0 \\ \downarrow & & & \searrow & & & \\ & & & & \mathcal{N}_R|_S & & \\ & & & \nearrow^{\iota_S} & & \searrow & \\ \mathcal{T}_{\mathbb{P}^r}|_S & \longrightarrow & \mathcal{N}_S & & & & 0 \end{array} \tag{4.12}$$

the horizontal and diagonal rows are exact, hence the commutativity of the pentagon shows that T^1 surjects onto K. Since both are locally free sheaves of rank 1, one concludes that $T^1 \cong K$. The same argument works for Q.

Hence the following sequences are exact:

$$0 \to \mathcal{N}_S \to \mathcal{N}_R|_S \to T^1 \to 0, \tag{4.13}$$

$$0 \to \mathcal{N}_Q(-\Gamma) \to \mathcal{N}_R|_Q(-\Gamma) \to T^1(-\Gamma) \to 0. \tag{4.14}$$

Moreover, one has the exact sequence

$$0 \to \mathcal{N}_R|_Q \otimes \mathcal{O}_R(-\Gamma) \to \mathcal{N}_R \to \mathcal{N}_R|_S \to 0, \tag{4.15}$$

so that, in order to prove (4.8), it suffices to show that

$$H^i(\mathcal{N}_R|_S) = 0, \qquad \text{for } 1 \leq i \leq 2, \tag{4.16}$$

$$H^i(\mathcal{N}_R|_Q \otimes \mathcal{O}_R(-\Gamma)) = 0, \qquad \text{for } 1 \leq i \leq 2. \tag{4.17}$$

By induction on g, one knows that $H^i(\mathcal{N}_S) = 0$, $i = 1, 2$. By (4.11), one has that $H^i(T^1) = H^i(\mathcal{O}_\Gamma) = 0$, $i = 1, 2$, because Γ is the union of two distinct lines. Hence the sequence (4.13) implies (4.16).

Note that $H^i(T^1(-\Gamma)) = H^i(\mathcal{O}_\Gamma(-\Gamma)) = 0$, $i = 1, 2$. Taking into account the exact sequence (4.14), the proof of (4.17) is concluded if one shows that

$$H^i(\mathcal{N}_Q(-\Gamma)) = 0, \qquad \text{for } 1 \leq i \leq 2. \tag{4.18}$$

Since Q lies in a $\mathbb{P}^3$, one has that

$$\mathcal{N}_Q \cong \mathcal{O}_Q(2) \oplus \mathcal{O}_Q(1)^{\oplus(r-3)}.$$

Recall that F_1 and F_2 are lines of the same ruling, so $F_1 \sim F_2$ and $\mathcal{O}_Q(-\Gamma) \cong \mathcal{O}_Q(-2F_1)$. Let G be the other ruling of Q and H be the general hyperplane section of Q, hence $H \sim G + F_1$ and one has that:

$$\mathcal{N}_Q \otimes \mathcal{O}_Q(-\Gamma) \cong \mathcal{O}_Q(2G) \oplus \mathcal{O}_Q(G - F_1)^{\oplus(r-3)} \tag{4.19}$$

and one sees that $h^i(\mathcal{O}_Q(2G)) = h^i(\mathcal{O}_Q(G - F)) = 0$, for $i = 1,\ 2$, which proves (4.18). The proof of (4.8) is thus concluded.

We now prove formula (4.9). By (4.8), one has that $h^0(\mathcal{N}_R) = \chi(\mathcal{N}_R)$, which one computes by using (4.13), (4.14) and (4.15):

$$\chi(\mathcal{N}_R) = \chi(\mathcal{N}_R|_S) + \chi(\mathcal{N}_R|_Q \otimes \mathcal{O}_Q(-\Gamma)) = \chi(\mathcal{N}_S) + \chi(T^1) + \chi(\mathcal{N}_Q(-\Gamma)) + \chi(T^1(-\Gamma)).$$

By (4.11), one has that $\chi(T^1) = \chi(T^1(-\Gamma)) = 2$. By (4.19), one has $\chi(\mathcal{N}_Q(-\Gamma)) = 3$. Finally, by induction

$$\chi(\mathcal{N}_S) = (r+1)^2 + 7(g-2),$$

which concludes the proof of (4.9).

It remains to show that the map $H^0(\mathcal{N}_R) \to H^0(T^1)$ is surjective. Since $H^1(\mathcal{N}_S) = 0$, the map $H^0(\mathcal{N}_R|_S) \to H^0(T^1)$ is surjective by (4.13). Finally (4.17) implies that $H^0(\mathcal{N}_R)$ surjects onto $H^0(\mathcal{N}_R|_S)$, which concludes the proof of the lemma. □

We are finally ready for the

Proof of Theorem 4.6: conclusion. By Lemma 4.7, one has that $H^1(\mathcal{N}_R) = 0$, which means that R corresponds to a smooth point $[R]$ of the Hilbert scheme of surfaces with degree d and sectional genus g in $\mathbb{P}^{d-2g+1}$. Therefore, $[R]$ belongs to a single reduced component $\mathcal{H}_{d,g}$ of the Hilbert scheme of dimension $h^0(\mathcal{N}_R)$. The last assertion of Lemma 4.7 implies that a general tangent vector to $\mathcal{H}_{d,g}$ at the point $[R]$ represents a first-order embedded deformation of R which smooths the double curve Γ. Therefore, the general point in $\mathcal{H}_{d,g}$ represents a smooth, irreducible surface Y. Thus Y degenerates to R and also to the planar Zappatic surface $X_{d,g}$ (cf. Proposition 3.8 and Constructions 4.1, 4.2).

Classical adjunction theory (cf. e.g. [15] and § 7 in [9]) implies that Y is a scroll: otherwise, if H is the hyperplane section of Y, one has $K_Y + H$ nef and therefore $0 < d \leq 4(g-1) + K_Y^2$ contradicting $K_Y^2 \leq 6(1-g)$ in (4.5).

Finally, the assertion about linear normality is trivial for $g = 0$ and is clear by induction and construction, for $g > 0$. □

Remark 4.20. By using the same first part of the proof of Theorem 4.6, one can observe that Construction 4.2 can be carried on also when $d = 2g + 3$.

Indeed, in this case, $X_{d,g}$ is a union of planes lying in $\mathbb{P}^4$ which is not a Zappatic surface if $g \geq 2$, since there are singular points where only two planes of the configuration meet, which are not Zappatic singularities. The only difference in the construction is that, since there are no pairs of disjoint planes, we have to choose l_1 and l_2 on two planes V_1 and V_2 which meet at a point but not along a line. Moreover the proof of the existence of the quadric meeting transversally the union of planes along $l_1 \cup l_2$ is a bit more involved.

Nonetheless, as in the proof of Theorem 4.6, one can show that $X_{d,g}$ is a flat limit of a family of linearly normal scrolls in $\mathbb{P}^4$ for any genus $g \geq 0$ and degree $d = 2g + 3$. These scrolls are smooth only if $g = 0, 1$, whereas they have isolated double points if $g \geq 2$.

We finish this section by mentioning two more examples of configurations of planes forming a planar Zappatic surface, with only points of type R_3 and S_4, which are degenerations of smooth scrolls. The advantage of this construction is that they are slightly simpler than Construction 4.2. The disadvantage is that they work only for larger values of the degree.

Example 4.21. *Fix arbitrary integers g, d such that $g \geq 2$ and $d > 4g$. Set $r = d - 2g + 1$. Let $X_{d-2g,0} = \bigcup_{i=1}^{d-2g} V_i$ be a planar Zappatic surface in $\mathbb{P}^r$ whose dual graph is a chain. One can attach $2g$ planes to $X_{d-2g,0}$ in order to get a planar Zappatic surface $Y_{d,g}$ of degree d and sectional genus g in $\mathbb{P}^r$ with $d - 2g + 2$ points of type R_3 and $2g - 2$ points of type S_4.*

Indeed, we may assume that V_i meets V_j along a line if and only if $j = i \pm 1$. Denote by $p_2, \dots, p_{d-2g-1}$ the points of type R_3 of $X_{d-2g,0}$, where $p_i = V_{i-1} \cap V_i \cap V_{i+1}$, $i = 2, \dots, d-2g-1$.

Choose a general line $l_{1,1}$ in V_1 [resp. $l_{1,2}$ in V_{d-2g}], i.e. a line not passing through p_2 [resp. p_{d-2g-1}]. For $i = 2, \dots, g$, choose a line $l_{i,1}$ in V_i [resp. a line $l_{i,2}$ in $V_{d-2g+1-i}$], which is general among those lines passing through p_i [resp. through $p_{d-2g+1-i}$].

The generality assumption implies that all the lines $l_{i,1}, l_{i,2}$, $1 \le i \le g$, are pairwise skew. For every $i = 1, \dots, g$, there is a smooth quadric surface Q'_i which contains $l_{i,1}$ and $l_{i,2}$, in the $\mathbb{P}^3$ spanned by them. In this $\mathbb{P}^3$ the quadric Q'_i degenerates to two distinct planes, say $V_{i,1}$ and $V_{i,2}$, leaving $l_{i,1}$ and $l_{i,2}$ fixed: the plane $V_{i,1}$ contains $l_{i,1}$ whereas $V_{i,2}$ contains $l_{i,2}$. Then $Y = Y_{d,g} := X_{d-2g,0} \cup \bigcup_{i=1}^{g} (V_{i,1} \cup V_{i,2})$ is a planar Zappatic surface in $\mathbb{P}^r$. Note that we added to the points $p_2, \dots, p_{d-2g-1}$ new Zappatic singularities at the points:

(i) $q_{i,j}$, with $1 \le i \le g$, $1 \le j \le 2$, where $q_{i,1} = V_i \cap V_{i,1} \cap V_{i,2}$ and $q_{i,2} = V_{i,1} \cap V_{i,2} \cap V_{d-2g+1-i}$,
(ii) $p_1 = V_1 \cap V_2 \cap V_{1,1}$ and $p_{d-2g} = V_{d-2g} \cap V_{d-2g-1} \cap V_{1,2}$

Then Y is a planar Zappatic surface with the following properties:

- the dual graph G_Y has d vertices and $d + g - 1$ edges;
- Y has $2g - 2$ points of type S_4, namely $p_2, \dots, p_g, p_{d-3g+1}, \dots, p_{d-2g-1}$;
- Y has $d - 2g + 2$ points of type R_3, namely $q_{i,j}$, $1 \le i \le g$, $1 \le j \le 2$, p_1, p_{d-2g} and $p_{g+1}, \dots, p_{d-3g}$.
- $\chi(\mathcal{O}_X) = 1 - g$, $p_\omega(X) = 0$, $q(X) = g(X) = g$,

(cf. Figure 15).

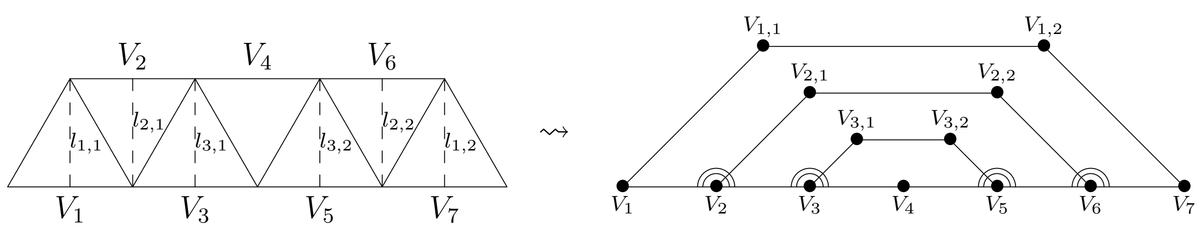

FIGURE 15. Construction of $Y_{d,g}$ from $X_{d-2g,0}$ for $g = 3$ and $d = 4g + 1 = 13$

Recall that $X_{d-2g,0}$ is the flat limit of a smooth, rational normal scroll S of degree $d - 2g$ in $\mathbb{P}^{d-2g+1}$. If $F_{i,j}$, $1 \le i \le g$, $1 \le j \le 2$, is the ruling of S whose limit is $l_{i,j}$ and Q_i a smooth quadric containing $F_{i,1}$, $F_{i,2}$, whose limit is Q'_i, then one can show, by using similar techniques as in the proof of Theorem 4.6, that the union of the rational normal scroll S and the g quadrics Q_i is a flat limit of a family of smooth, linearly normal scrolls of degree d and genus g in $\mathbb{P}^{d-2g+1}$, which is contained in a the same component $\mathcal{H}_{d,g}$ of Theorem 4.6 (cf. Theorem 5.4 and Remark 5.5 below).

With a slight modification of the previous construction, one can cover also the case $d = 4g$. We do not dwell on this here.

Example 4.22. *Fix integers g, d such that $g \ge 1$ and $d \ge 3g + 2$. By induction on g, we will construct a planar Zappatic surface $Z_{d,g} = \bigcup_{i=1}^{d} V_i$ in $\mathbb{P}^{d-2g+1}$ such that:*

- *$Z_{d,g}$ has $d - 2g + 1$ double lines, i.e. $G_{Z_{d,g}}$ has $d - 2g + 1$ edges;*
- *$Z_{d,g}$ has $d - 2g + 2$ points of type R_3 and $2g - 2$ points of type S_4;*
- *for each i, V_i is the central plane through a point p of type either R_3 or S_4;*
- *there exist two R_3-points of $Z_{d,g}$ whose central planes do not meet, unless $g = 1$ and $d = 5$;*

- $\chi(\mathcal{O}_{Z_{d,g}}) = 1 - g$, $p_\omega(Z_{d,g}) = 0$, $q(Z_{d,g}) = g(Z_{d,g}) = g$.

The base of the induction is the case $g = 1$. In this case, $Z_{d,1}$ is the surface $X_{d,1}$ considered in Construction 4.1. Now we assume $g > 1$ and we describe the inductive step.

Consider the surface $Z_{d-3,g-1}$, which sits in $\mathbb{P}^{d-2g}$, which we suppose to be embedded as a hyperplane in $\mathbb{P}^{d-2g+1}$.

If $g = 2$ and $d = 8$, choose two distinct planes V_1 and V_2 of $Z_{5,1} = X_{5,1}$, which do not meet along a line. Otherwise, choose two distinct planes V_1 and V_2 of $Z_{d-3,g-1}$ which are central for two R_3-points, say p_1 and p_2, and which span a $\mathbb{P}^5$.

Choose a line l_1 in V_1 [resp. l_2 in V_2] which is general among those lines passing through p_1 [resp. through p_2]. Consider a general $\mathbb{P}^4$ in $\mathbb{P}^{d-2g+1}$ containing l_1 and l_2.

One can show that, in this $\mathbb{P}^4$, there is a smooth, rational normal cubic scroll R' which contains l_1 and l_2 and such that R' meets transversally $Z_{d-3,g-1}$ along $R' \cap Z_{d-3,g-1} = l_1 \cup l_2$.

In this $\mathbb{P}^4$, the cubic scroll R' degenerates to a planar Zappatic surface $X_{3,0}$, consisting of three planes, say V_{d-2}, V_{d-1} and V_d, such that $l_1 \subset V_d$ and $l_2 \subset V_{d-2}$.

We define $Z_{d,g} = Z_{d-3,g-1} \cup X_{3,0}$. We added three planes and four double lines; the points p_1 and p_2 becomes of type S_4 for $Z_{d,g}$ and we added three points of type R_3 at $V_1 \cap V_{d-1} \cap V_d$, at $V_2 \cap V_{d-2} \cap V_{d-1}$ and at $V_{d-2} \cap V_{d-1} \cap V_d$. It is clear the existence of two R_3-points whose central planes do not meet.

Arguing by induction, one may assume that $Z_{d-3,g-1}$ is the flat limit of a smooth, linearly normal scroll S of degree $d-3$ and genus $g-1$ in $\mathbb{P}^{d-2g}$. If F_i, $i = 1, 2$, is the ruling of S whose limit is l_i and R is a smooth, cubic scroll containing F_1, F_2 as ruling and whose limit is R', one can show, by using the same proof of Theorem 4.6, that the union $S \cup R$ is the flat limit of a family of smooth, linearly normal scrolls of degree d and genus g in $\mathbb{P}^{d-2g+1}$, which is contained in the same component $\mathcal{H}_{d,g}$ of Theorem 4.6 (cf. Theorem 5.4 and Remark 5.5).

5. Hilbert schemes of scrolls

In this section we prove that $\mathcal{H}_{d,g}$, as determined in Theorem 4.6, is the unique irreducible component of the Hilbert scheme of scrolls of degree d and genus g in $\mathbb{P}^{d-2g+1}$ whose general point parametrizes a smooth, linearly normal scroll (cf. Theorem 5.4). This component $\mathcal{H}_{d,g}$ dominates $\mathcal{M}_g$ (cf. Remark 5.6).

This, together with Construction 4.2 and Theorem 4.6, proves Theorem 1.2 in the introduction.

On the other hand, we will also construct families of scrolls Y of degree d and genus g in $\mathbb{P}^r$, with $r > d - 2g + 1$, with $h^1(Y, \mathcal{O}_Y(1)) \neq 0$ (cf. Example 5.11). We will also show that projections of such scrolls may fill up components of the Hilbert scheme, different from $\mathcal{H}_{d,g}$, which may even dominate $\mathcal{M}_g$ (cf. Example 5.12).

Let C be a smooth curve of genus g and let $F \xrightarrow{\rho} C$ be a *geometrically ruled surface* on C, i.e. $F = \mathbb{P}(\mathcal{F})$, for some rank-two vector bundle $\mathcal{F}$ on C. Furthermore, we assume that $\mathcal{F}$ is very ample, i.e. F is embedded in $\mathbb{P}^r$, for some $r \geq 3$, via the $\mathcal{O}_F(1)$ bundle as a scroll of degree $d = deg(\mathcal{F})$. From now on, H will denote the hyperplane section of F. A general hyperplane section H is isomorphic to C, so that we will set L_F the line bundle on $C \cong H$ which is the restriction of the hyperplane bundle. We will denote by R a general ruling of F, and more precisely by R_x the ruling mapping to the point x in C.

Let $Y := C \times \mathbb{P}^1$. If L is a line bundle on C, we will set

$$\tilde{L} := \pi_1^*(L) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1)), \tag{5.1}$$

where π_i denotes the projection on the i^{th}-factor, $1 \leq i \leq 2$.

Proposition 5.2. *Let C be a smooth curve of genus $g \geq 0$ and let $F := \mathbb{P}(\mathcal{F})$ be a geometrically ruled surface on C. Assume that $deg(\mathcal{F}) = d$.*

Then there is a birational map

$$\varphi : Y \dashrightarrow F$$

which is the composition of d elementary transformations at distinct points of a set $\Gamma := \{y_1, \ldots, y_d\} \subset Y$ lying on d distinct rulings of Y. Moreover,

(i) $\varphi^*(\mathcal{O}_F(H)) = \tilde{L_F}$;

(ii) $\varphi^*(|\mathcal{O}_F(H)|) = |\tilde{L_F} \otimes \mathcal{I}_{\Gamma/Y}|$.

Proof. The argument is similar to the one in [12], Prop. 6.2, and in [16]. Indeed, let Π be a general linear subspace of codimension two in $\mathbb{P}^r$ which is the base locus of a pencil $\mathcal{P} \cong \mathbb{P}^1$ of hyperplanes. By abusing notation, we will denote by $\mathcal{P}$ the corresponding pencil of hyperplane sections of F. More specifically, we will denote by H_t the hyperplane section corresponding to the point $t \in \mathbb{P}^1$. Then we denote by $Z := \{z_1, \ldots, z_d\} = F \cap \mathcal{P}$; note that Z is formed by distinct points on distinct rulings.

The map $\varphi : Y \dashrightarrow F$ is defined by sending the general point $(x, t) \in Y$ to the point $R_x \cap H_t \in F$. One verifies that φ is birational and that the indeterminacy locus on F is Z. In order to describe the map φ on Y, note that each point z_i maps to a point $x_i \in C$ and determines a unique value $t_i \in \mathbb{P}^1$ such that H_{t_i} contains the ruling R_{x_i}, $1 \leq i \leq d$. The indeterminacy locus of φ on Y is $\Gamma := \{y_1, \ldots, y_d\}$, where $y_i = (x_i, t_i)$, $1 \leq i \leq d$.

As shown in [12], φ is the composition of the elementary transformations based at the points of Γ. The rest of the assertion immediately follows. □

Let $\Gamma = \{y_1, \ldots, y_d\} \subset Y$ be a subset formed by d distinct points. We consider the line bundle on C

$$L_\Gamma := \mathcal{O}_C(x_1 + \ldots + x_d), \tag{5.3}$$

where $\pi_1(y_i) = x_i$, $1 \leq i \leq d$.

Theorem 5.4. *Let $g \geq 0$ and $d > 2g + 3$ be integers. Then there exists a unique irreducible component $\mathcal{H}_{d,g}$ of the Hilbert scheme, parametrizing scrolls of degree d and genus g in $\mathbb{P}^{d-2g+1}$, whose general point represents a smooth scroll $F \subset \mathbb{P}^{d-2g+1}$ which is linearly normal and moreover with $h^1(F, \mathcal{O}_F(1)) = 0$.*

Proof. Let $U \subset Hilb^d(Y)$ be the open subset formed by all $\Gamma = \{y_1, \ldots, y_d\} \subset Y$ containing d points lying on d distinct fibres and imposing d independent conditions on $|\tilde{L_\Gamma}|$, which means

$$dim(|\tilde{L_\Gamma} \otimes \mathcal{I}_{\Gamma/Y}|) = dim(|\tilde{L_\Gamma}|) - d.$$

Note that, by the Kunneth formula, $h^0(\tilde{L_\Gamma}) = 2h^0(L_\Gamma) = 2(d - g + 1)$. Thus, $dim(|\tilde{L_\Gamma} \otimes \mathcal{I}_{\Gamma/Y}|) = d - 2g + 1$. The linear system $|\tilde{L_\Gamma}|$ determines a rational map

$$\varphi : Y \dashrightarrow \mathbb{P}^{d-2g+1}.$$

By Proposition 5.2, every smooth scroll F of degree d and genus g in $\mathbb{P}^{d-2g+1}$ is the image of such a map. Therefore, for general Γ in U, the map φ is birational onto its image F, which is a smooth scroll of degree d and genus g whose Hilbert point $[F]$ belongs to a unique well-determined component $\mathcal{H}_{d,g}$ of the Hilbert scheme.

Note that by (ii) of Proposition 5.2, $h^1(\mathcal{O}_F(1)) = h^1(\tilde{L_\Gamma} \otimes \mathcal{I}_{\Gamma/Y}) = 0$; therefore, by the Riemann-Roch Theorem, $h^0(\mathcal{O}_F(1)) = d - 2g + 2$. □

Remark 5.5. Observe that the irreducible component $\mathcal{H}_{d,g}$ determined in Theorem 4.6 coincides with the one determined in Theorem 5.4. The case $d = 2g + 3$ can also be covered with similar arguments. In that case, we have surfaces in $\mathbb{P}^4$ which are no longer smooth, but they have $2g(g - 1)$ double points as dictated by the *double point formula*. Nonetheless, the statement of Theorem 5.4 still holds by substituting F with its normalization.

Remark 5.6. The dimension count for $\mathcal{H}_{d,g}$ which has been done in Thereom 4.6 also stems from the proof of Theorem 5.4, which provides a parametric representation of $\mathcal{H}_{d,g}$. Indeed, the number of parameters on which the general point of $\mathcal{H}_{d,g}$ depends, is given by the following count:

- $3g - 3$ parameters for the class of the curve C in $\mathcal{M}_g$, plus
- $2d$ parameters for the general point in U, plus
- $(r + 1)^2 - 1$ parameters for projective transformations in $\mathbb{P}^r$, where $r = d - 2g + 1$, minus
- $2(r - 1) = 2d - 4g$ parameters for the choice of a codimension-two subspace Π in $\mathbb{P}^r$, minus
- 3 parameters for projective isomorphisms of the pencil of hyperplanes through Π with $\mathbb{P}^1$.

This computation shows that $\mathcal{H}_{d,g}$ has *general moduli*, in the sense that the base of the general scroll $[F] \in \mathcal{H}_{d,g}$ is a general point of $\mathcal{M}_g$.

Observe that this can also be viewed as a consequence of Theorem 4.6 and more specifically of the fact that $h^1(\mathcal{O}_F(1)) = 0$ for $[F]$ a general point of the generically smooth component $\mathcal{H}_{d,g}$.

Indeed, if $F \subset \mathbb{P}^r$, $r = d - 2g + 1$, is a smooth scroll, from the Euler sequence restricted to F,

$$0 \to \mathcal{O}_F \to H^0(\mathcal{O}_F(1))^\vee \otimes \mathcal{O}_F(1) \to \mathcal{T}_{\mathbb{P}^r|F} \to 0,$$

we get that $h^1(\mathcal{T}_{\mathbb{P}^r|F}) = 0$. Therefore, from the normal sequence of F in $\mathbb{P}^r$

$$0 \to \mathcal{T}_F \to \mathcal{T}_{\mathbb{P}^r|F} \to \mathcal{N}_{F/\mathbb{P}^r} \to 0,$$

we get the surjection

$$H^0(\mathcal{N}_{F/\mathbb{P}^r}) \twoheadrightarrow H^1(\mathcal{T}_F).$$

Since F is a $\mathbb{P}^1$-bundle over C, from the differential of the map $F \xrightarrow{\rho} C$, we get a surjection

$$H^1(\mathcal{T}_F) \twoheadrightarrow H^1(\mathcal{T}_C),$$

hence

$$H^0(\mathcal{N}_{F/\mathbb{P}^r}) \twoheadrightarrow H^1(\mathcal{T}_C).$$

which shows that $\mathcal{H}_{d,g}$ dominates $\mathcal{M}_g$.

Next, we consider the problem of the existence of components of the Hilbert schemes of scrolls of degree d and genus g in $\mathbb{P}^r$, with $r > d - 2g + 1$. First, it is easy to determine an upper-bound for r. This subject has been deeply studied by C. Segre (cf. [20] and [12]). For the following lemma, compare [20], § 14.

Lemma 5.7. *Let $g \geq 1$ be an integer. Let C be a smooth curve of genus g and let $F = \mathbb{P}(\mathcal{F})$ be a ruled surface on C and $d = deg(\mathcal{F}) \geq 2g + 1$. Assume that there exists a smooth curve in $|\mathcal{O}_F(1)|$. Then,*

$$h^0(\mathcal{O}_F(1)) \leq d - g + 2.$$

The equality holds if and only if $\mathcal{F} = \mathcal{O}_C \oplus L$, in which case $\mathcal{O}_F(1)$ maps F to a cone over a projectively normal curve of degree d and genus g in $\mathbb{P}^{d-g}$.

Proof. The bound on $h^0(\mathcal{O}_F(1))$ follows by the Riemann-Roch Theorem on C. If the equality holds, then C is linearly normally embedded as a curve of degree d and genus g in $\mathbb{P}^{d-g}$. It is well-known that this curve is projectively normal (cf. [5], [17] and [19]). Therefore F is mapped to a surface X which is projectively normal, since its general hyperplane section is (cf. [13], Theorem 4.27).

On the other hand, X is a scroll of positive genus. Therefore X cannot be smooth, and it has some isolated singularities. This forces X to be a cone (cf. Claim 4.4 in [7]). Hence, the assertion follows. □

Remark 5.8. Let C be a smooth curve of genus g and let $F = \mathbb{P}(\mathcal{F})$ be a ruled surface on C and $d = deg(\mathcal{F}) \geq 2g+1$. Then

$$d - 2g + 2 \leq h^0(\mathcal{O}_F(1)) \leq d - g + 2, \tag{5.9}$$

where the lower bound is immediately implied by the Riemann-Roch Theorem whereas the upper bound is given by the previous lemma. Equivalently,

$$0 \leq h^1(\mathcal{O}_F(1)) \leq g, \tag{5.10}$$

where the upper-bound is realized by the cones and the lower-bound by the general scrolls in the component $\mathcal{H}_{d,g}$ considered above.

Any intermediate value i of $h^1(\mathcal{O}_F(1))$, $1 \leq i \leq g$, can be actually realized. An easy construction is via decomposable bundles as the following example shows.

Example 5.11. Let $g \geq 3$ and let $d \geq 4g-1$ be integers. Let i be any integer between 1 and g. Let C be a smooth, projective curve of genus g with a line bundle L such that $|L|$ is base-point-free and $h^1(L) = i$. Let D be a general divisor of degree $d - deg(L)$. Notice that, since $deg(L) \leq 2g-2$ and $d \geq 4g-1$, then $deg(D) \geq 2g+1$ and the linear series $|D|$ is very ample.

Consider $\mathcal{F} = L \oplus \mathcal{O}_C(D)$. If $F = \mathbb{P}(\mathcal{F})$ then $\mathcal{O}_F(1)$ is base-point-free and $h^1(\mathcal{O}_F(1)) = i$.

For large values of i, $\mathcal{O}_F(1)$ is never very ample. For instance, for $i = g-1$, C is forced to be hyperelliptic and $L = g^1_2$. Thus, the image of F via $|\mathcal{O}_F(1)|$ has a double line.

Similarly, if $i = g-2$, either C is hyperelliptic and $L = 2g^1_2$, or C is trigonal and $L = g^1_3$ or $g = 3$ and $L = \omega_C$. In the former case, the image of F has a double conic; in the second case, the image of S has a triple line. Only in the third case, the image of C via $|L|$ is smooth.

The analysis is subtle and we do not dwell here on this.

Now we consider the question of whether there are other components, different from $\mathcal{H}_{d,g}$, of the Hilbert scheme of surfaces in $\mathbb{P}^{d-2g+1}$ whose general point corresponds to a smooth scroll of degree d and genus g. The answer to this question is affirmative; in fact one can construct such components even with general moduli. In the next example, we show one possible construction of a component with general moduli. The reader may easily generate other similar constructions.

Example 5.12. Let C be a curve with general moduli of genus $g = 4l + \epsilon$, where $0 \leq \epsilon \leq 3$. Let L be a very-ample, special line bundle of degree $m := 3 + g - l$ with $h^0(L) = 4$. Note that such a L varies in a family of dimension $\rho := \rho(g, 3, m) = \epsilon$.

Let d be an integer with either $d \geq 2g+10$, if $\epsilon = 0, 1$, or $d \geq 2g+11$, if $\epsilon = 2, 3$. Set $r = d - 2g + 1$.

Let N be a general line bundle on C of degree $d - m$. Note that $d - m > g + 7 + l$. Hence N is very ample (cf. e.g. [1]) and $h^0(N) = d - m - g + 1$.

Set $\mathcal{F} = L \oplus N$ and $X = \mathbb{P}(\mathcal{F})$. Then $R + 1 := h^0(\mathcal{O}_X(1)) = h^0(L) + h^0(M) = r + 1 + l$.

Since $\mathcal{O}_X(1)$ is very ample, X is linearly normal embedded in $\mathbb{P}^R$ as a smooth scroll of degree d and genus g, which can be generically projected to $\mathbb{P}^r$ to a smooth scroll X' with the same degree and genus, which belongs a certain component $\mathcal{H}$ of the Hilbert scheme. As in the proof of Theorem 4.6, the general member of $\mathcal{H}$ is a scroll of the same degree and genus.

The dimension of $\mathcal{H}$ can be easily bounded from below by the sum of the following quantities:

- $3g-3$, which are the parameters on which C depends,
- g, which are the parameters on which N depends,
- ϵ, which are the parameters on which L depends,
- $(r+1)l = dim(\mathbb{G}(r,R))$, which are the parameters for the projections,
- $(r+1)^2-1 = dim(PGL(r+1,\mathcal{C}))$.

The hypothesis on d implies that $dim(\mathcal{H}) \geq dim(\mathcal{H}_{d,g})$, which shows that $\mathcal{H}$ is different from $\mathcal{H}_{d,g}$.

Remark 5.13. The question of understanding how many components of the Hilbert scheme of scrolls there are, and the corresponding image to the moduli space of curves of genus g, is an intriguing one. The previous example suggests that the a complete answer could be rather complicated. It also leaves open the question whether $\mathcal{H}_{d,g}$ is the only component with general moduli for $2g+4 \leq d \leq 2g+10$.

6. Comments on Zappa's original approach

In [23], Zappa stated a result about embedded degenerations of scrolls of sectional genus $g \geq 2$ to unions of planes. His result, in our terminology, reads as Theorem 1.1 in the introduction.

Zappa's arguments rely on a rather intricate analysis of algebro-geometric and topological type of degenerations of hyperplane sections of the scroll and, accordingly, of the branch curve of a general projection of the scroll to a plane.

We have not been able to check all the details of this very clever argument. This is one of the reason why we preferred to solve the problem in a different way, which is the one we exposed in the previous sections. Our approach has the advantage of proving a result in the style of Zappa, but with better hypotheses about the degree of the scrolls.

However, the idea which Zappa exploits, of degenerating the branch curve of a general projection to a plane, is a classical one which goes back to Enriques, Chisini, etc, and certainly deserves attention. We hope to come back to these ideas in the future.

In reading Zappa's paper [23], our attention has been attracted also by another ingredient he uses which looks interesting on its own. It gives extendability conditions for a curve on a scroll which is not a cone. We finish this paper by briefly reporting on this. At the the end of the section we briefly summarize Zappa' s argument for the degenerations of the scroll.

Let $F \subset \mathbb{P}^3$ be a scroll, which is not a cone over a plane curve. We do not assume F to be smooth. Equivalently, we can look at F as a curve $\mathcal{C}$ in the Grassmannian $\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$, which is isomorphic to the Klein hyperquadric in $\mathbb{P}^5$ via the Plücker embedding.

Let Π be a general plane and let $\Gamma := F \cap \Pi$. Consider

$$\nu : C \to \Gamma$$

the normalization map. Then, there is a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\Phi} & \mathcal{C} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5 \\ & {\scriptstyle\nu}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \Gamma \subset \Pi & \end{array}$$

where Φ maps a general point $x \in C$ to the unique line of F passing through $\nu(x)$, and π maps each point $l \in \mathcal{C}$, corresponding to a ruling L of F, to the point $L \cap \Gamma$.

Zappa proves the following nice lemma:

Lemma 6.1. *(cf. §1 in* [23]*) In the above setting:*

$$\nu^*(\mathcal{O}_\Gamma(1)) \cong \Phi^*(\mathcal{O}_{\mathcal{C}}(1)).$$

More specifically, π is the projection of $\mathcal{C}$ from the plane $\Pi^* \subset \mathbb{G}(1,3)$, filled up by all lines of Π.

Proof. The assertion follows from the fact that, if r is a line in Π, then $\pi^*(r)$ is the section of the tangent hyperplane to $\mathbb{G}(1,3)$ at the point of Π^* corresponding to r. Such a hyperplane contains Π^*, and conversely any hyperplane containing Π^* is of this type. □

Zappa notes that an interesting converse of the previous lemma holds.

Proposition 6.2. *(cf. § 2 in* [23]*) An irreducible plane curve Γ is a section of a scroll $F \subset \mathbb{P}^3$ of degree d if and only if Γ is the projection of a curve $\mathcal{C}$ of degree d, lying on a smooth quadric $Q \subset \mathbb{P}^5$, and the center of the projection is a plane contained in Q.*

Proof. One implication is Lemma 6.1. Let us prove the other implication.

Suppose that Γ is the projection of $\mathcal{C} \subset Q \subset \mathbb{P}^5$ from a plane $\bar{\Pi} \subset Q$. Since all smooth quadrics in $\mathbb{P}^5$ are projectively equivalent, we may assume that Q is the Klein hyperquadric. The assertion follows by reversing the argument of the proof of Lemma 6.1. □

Proposition 6.2 can be extended in the following way. Let Γ be a plane curve of degree d and geometric genus g, such that $d \geq g+6$. Set $i = h^1(C, \nu^*(\Gamma(1)))$. Then, one has the birational morphism

$$C \xrightarrow{|\nu^*(\mathcal{O}_\Gamma(1))|} \bar{\mathcal{C}} \subset \mathbb{P}^r, \tag{6.3}$$

where $r = d - g + i > 5$ and the following linear projection:

$$\bar{\mathcal{C}} \xrightarrow{\bar{\pi}} \Gamma \subset \mathbb{P}^2.$$

Proposition 6.4. *(cf. § 3 in* [23]*) In the above setting, Γ is a plane section of a scroll F in $\mathbb{P}^3$, which is not a cone, if and only if $\bar{\mathcal{C}}$ lies on a quadric of rank 6 in $\mathbb{P}^r$ which contains the center of the projection $\bar{\pi}$.*

Proof. This is an immediate consequence of Proposition 6.2 and can be left to the reader. □

Zappa uses Proposition 6.4 to prove that any plane curve of degree $d >> g$ is the plane section of a scroll F which is not a cone. The next proposition is essentially Zappa's result in § 7 of [23], with an improvement on the bound on d: Zappa's bound is $d \geq 3g+2$.

Lemma 6.5. *Let $g \geq 0$ and let $d \geq 2g+2$ be integers. Let $\bar{\mathcal{C}}$ be an irreducible, smooth curve of degree d and genus g in $\mathbb{P}^r$, $r = d - g$. Then there exists a quadric of $\mathbb{P}^r$, of rank at most 6, which contains $\bar{\mathcal{C}}$ and a general $\mathbb{P}^{r-3}$.*

Proof. Note that a quadric Q of $\mathbb{P}^r$ contains a $\mathbb{P}^{r-3}$ if and only if Q has rank at most 6.

Consider the short exact sequence

$$0 \to \mathcal{I}_{\bar{C}/\mathbb{P}^r}(2) \to \mathcal{O}_{\mathbb{P}^r}(2) \to \mathcal{O}_{\bar{\mathcal{C}}}(2) \to 0.$$

Since $d \geq 2g+2$, one has $h^0(\mathcal{O}_{\bar{\mathcal{C}}}(2)) = 2d - g + 1$ and $\bar{\mathcal{C}}$ is projectively normal (cf. [5], [17], [19]). Thus

$$h^0(\mathcal{I}_{\bar{C}/\mathbb{P}^r}(2)) = \binom{r+2}{2} - (2d - g + 1). \tag{6.6}$$

Let Σ be a general $\mathbb{P}^{r-3}$ in $\mathbb{P}^r$. Then, from (6.6), one has

$$h^0(\mathcal{I}_{\bar{\mathcal{C}}\cup\Sigma/\mathbb{P}^r}(2)) \geq \binom{r+2}{2} - \binom{r-1}{2} - (2d-g+1) = d - 2g - 1 > 0.$$

□

We need the following lemma:

Lemma 6.7. *Let $\bar{\mathcal{C}} \subset \mathbb{P}^r$ be as in Proposition 6.5 and assume that, if $g = 0$, $d \geq 3$. Let Σ be a $\mathbb{P}^{r-3}$. The general quadric in the linear system $|\mathcal{I}_{\bar{\mathcal{C}}\cup\Sigma/\mathbb{P}^r}(2)|$ has rank $k > 3$.*

Proof. Suppose by contradiction that all quadrics containing $\bar{\mathcal{C}}$ and Σ have rank 3. Let us define

$$R_3(\bar{\mathcal{C}}) := \{Q \in \mathbb{P}(H^0(\mathcal{I}_{\bar{C}/\mathbb{P}^r}(2))) \mid \mathrm{rank}(Q) \leq 3\}.$$

By an easy count of parameters our assumption implies that:

$$\dim R_3(\bar{\mathcal{C}}) \geq 3d - 4g - 7.$$

Next, we will show that this inequality is not possible.

In order to do that, we apply results from [22]. Zamora proves in [22], cf. Lemma 1.2, that there is a one-to-one correspondence between quadrics $Q \in R_3(\bar{\mathcal{C}})$ and pairs (g^1_a, g^1_b) of linear series on $\bar{\mathcal{C}}$, with $a \leq b$, such that:

(i) $a + b = \deg \bar{\mathcal{C}} = d$,
(ii) $|g^1_a + g^1_b| = |\mathcal{O}_{\bar{\mathcal{C}}}(1)|$,
(iii) $g^1_a + B_b = g^1_b + B_a$, where B_a (B_b, resp.) is the base locus of the g^1_a (g^1_b, respectively).

Let Q be the general member of an irreducible component W of maximal dimension of $R_3(\bar{\mathcal{C}})$ and let (g^1_a, g^1_b) be the corresponding pair of linear series on $\bar{\mathcal{C}}$.

Zamora's result implies that there is a base-point-free linear series g^1_h on $\bar{\mathcal{C}}$ such that

$$g^1_a = g^1_h + B_a,\ g^1_b = g^1_h + B_b,$$

so that

$$|\mathcal{O}_{\bar{\mathcal{C}}}(1)| = |2g^1_h + B_a + B_b|.$$

Note that, once the divisor $B_a + B_b$ has been fixed, the line bundle L corresponding to g^1_h belongs to a zero-dimensional set in $Pic^h(\bar{\mathcal{C}})$. Set $\delta = deg(B_a + B_b)$, so that $d = 2h + \delta$.

Suppose now that L is non-special. Then,

$$3d - 4g - 7 \leq dim(W) \leq \delta + 2(h - g - 1) = d - 2g - 2,$$

which gives a contradiction.

Now assume that L is special, so that $|L| = g^r_h$, with $2r \leq h$. In this case

$$3d - 4g - 7 \leq dim(W) \leq \delta + 2(r-1) \leq \delta + h - 2,$$

which leads to a contradiction. □

As a consequence of the previous lemma, we have:

Theorem 6.8. *Let Γ be an irreducible, plane curve of degree d and geometric genus $g \geq 0$. If $d \geq \max\{g+5, 2g+2\}$, then Γ is a plane section of a scroll in $\mathbb{P}^3$, which is not a cone.*

Proof. Let $\bar{\mathcal{C}} \subset \mathbb{P}^r$ be the curve corresponding to Γ in $\mathbb{P}^2$. Then Γ is the projection of $\bar{\mathcal{C}}$ from $\Sigma = \mathbb{P}^{r-3}$ disjoint from $\bar{\mathcal{C}}$. By Lemma 6.5, there is a quadric Q containing $\bar{\mathcal{C}} \cup \Sigma$. If $rank(Q) := k$ is 6, we finished by Proposition 6.4. By Lemma 6.7, we know that $k \geq 4$.

If $k = 5$, then the vertex V of Q is a $\mathbb{P}^{r-5}$. By projecting from V, Q maps to a smooth quadric Q' in $\mathbb{P}^4$ containing $\mathcal{C}'$, the projection of $\bar{\mathcal{C}}$, and Σ', the projection of Σ; the line Σ' is skew with respect to $\mathcal{C}'$. Of course Γ is the projection of $\mathcal{C}'$ from Σ'. Let us embedd $\mathbb{P}^4$ in $\mathbb{P}^5$ as a hyperplane. We can certainly find a smooth quadric $\bar{Q}$ in $\mathbb{P}^5$ containing Q' and containing a plane Π intersecting the $\mathbb{P}^4$ in Σ'. The curve Γ is now the projection of $\mathcal{C}'$ from Π. The assertion follows from Proposition 6.2.

If $k = 4$, then the vertex V of Q is a $\mathbb{P}^{r-4}$. Suppose first that Σ contains V; then by projecting from V to $\mathbb{P}^3$, the quadric Q maps to a smooth quadric Q', containing $\mathcal{C}'$, the image of $\bar{\mathcal{C}}$, and the point $p \in Q'$, the image of Σ, which does not sit on $\mathcal{C}'$. The curve Γ is the projection of $\mathcal{C}'$ from p. At this point, we can finish as in the previous case, by embedding the $\mathbb{P}^3$ in $\mathbb{P}^5$ and finding a smooth quadric $\bar{Q}$ in $\mathbb{P}^5$ containing Q' and the plane Π intersecting the $\mathbb{P}^3$ in p.

If Σ does not contain V, it intersects V in $W \cong \mathbb{P}^{r-5}$. By projecting from W to $\mathbb{P}^4$ we get a situation similar to the case $k = 5$. The only difference is that Q' is now singular at a point p, however Σ', the projection of Σ, does not contain p. So we can conclude exactly as in the case $k = 5$. ☐

Remark 6.9. We add a little remark to Theorem 6.8. Let Γ be a plane curve which is a plane section of a scroll $F \subset \mathbb{P}^3$, which is not a cone. So if one applies Theorem 6.8, the scroll which extends Γ is certainly not developable.

As Zappa does in [23], one can get an interesting consequence of Theorem 6.8 by applying duality. Recall that the *class* of an irreducible plane curve is the degree of the dual curve.

Corollary 6.10. *An irreducible, plane curve of class d and geometric genus g, such that $d \geq \max\{g+5, 2g+2\}$, is the branch curve of a projection of a scroll in $\mathbb{P}^3$ of degree d and genus g, which is not a cone.*

Proof. Let $D \subset \mathbb{P}^2$ be an irreducible plane curve of class d. Let $\Gamma \subset (\mathbb{P}^2)^*$ be the dual curve. By Theorem 6.8, Γ is the plane section of a scroll Φ which is not a cone. By standard properties of duality, D is the branch curve of the projection of $F = \Phi^*$ from the point corresponding to the plane in which Γ sits. ☐

The argument of Zappa to prove the degeneration of a scroll to a union of planes runs as follows. Zappa considers the scroll F whose hyperplane section Γ is a general member of the Severi variety $V_{d,g}$ of plane curves of degree d and geometric genus g. Then he lets Γ degenerate to a general union of d lines. From a complicated analysis involving the degeneration of Γ and the degeneration of its dual curve, which is the branch curve of the projection of the dual of the surface on the plane (see Corollary 6.10), Zappa deduces that in this degeneration of Γ, F degenerates to a union of planes. Moreover, he controls the degeneration of the linearly normal model of F deducing that it also degenerates to a union of planes with only points of type R_3 and S_4.

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