

BRILL–NOETHER LOCI OF STABLE RANK–TWO VECTOR BUNDLES ON A GENERAL CURVE

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ABSTRACT. In this note we give an easy proof of the existence of generically smooth components of the expected dimension of certain Brill–Noether loci of stable rank 2 vector bundles on a curve with general moduli, with related applications to Hilbert scheme of scrolls.

INTRODUCTION

The Brill–Noether theory of linear series on a smooth, irreducible, complex, projective curve C of genus g was initiated in the second half of XIX century and fully developed about one century later by the brilliant work of several mathematicians (see [2] for a general reference). As a result, we have now a complete understanding of the *Brill–Noether loci* of line bundles L of degree d with $h^0(C, L) > r$ on a curve C with general moduli. They can be described as determinantal loci inside $\text{Pic}^d(C)$ and we know their Zariski tangent spaces, their dimension, their singularities, how they are contained in each other, etc.

The study of n –dimensional scrolls over curves (with $n \geq 2$) also goes back to the second half of XIX century. It is equivalent to the study of rank n vector bundles over curves, and as such it has received a lot of attention in more recent times. In order to have reasonable moduli spaces for these bundles, one has to restrict the attention to semistable ones. For them it has been set up an analogue of Brill–Noether’s theory. Unfortunately the results here are not so complete as in the rank one case, and we are still far from having a full understanding of the situation. We refer the reader to [9] (and to the references therein) for a general account on the subject.

In this paper we deal with the rank 2 case and C with general moduli and genus g . A result by M. Teixidor (see Theorem 1.6) provides examples of components of the expected dimension (see (1.2) below) of Brill–Noether loci of stable, rank 2 vector bundles \mathcal{F} of degree d with $h^0(C, \mathcal{F}) > \ell$, in suitable ranges for d, g and ℓ . Teixidor’s ingenious, but not easy, proof uses a degeneration of C to a rational g –cuspidal curve C_0 and analyses the limits of the required bundles on C_0 .

This note is devoted to prove a similar result (i.e. Theorem 2.1). The ranges for d, g and ℓ for which we prove the existence of our components of the Brill–Noether loci are slightly worse than Teixidor’s ones. On the other hand we are able to prove a bit more than Teixidor does: not only the components in questions have the expected dimension, but they are also generically smooth. In addition, our approach is quite easy and does not require degenerating C . We construct our bundles as extensions of line bundles, and we prove that their *Petri map* (see §1.2) is in general injective, which is the same as proving that the corresponding Brill–Noether loci are generically smooth and of the expected dimension.

The paper is organized as follows. In §1 we recall the basics about moduli spaces of semistable rank–two vector bundles on a curve (see §1.1), Brill–Noether loci (see §1.2) and Teixidor’s theorem (see §1.3). The full §2 is devoted to the construction of our examples. In §3 we make some applications to Hilbert schemes of scrolls in projective spaces. We show that our examples give rise to linearly normal, smooth scrolls belonging to irreducible components of the Hilbert scheme, which are generically smooth of the expected dimension (see §§3.1, 3.2). In §3.3 we show that, by contrast, their projections in \mathbb{P}^{d-2g+1} do not fill up irreducible components of the Hilbert scheme: they are in fact contained in the unique component $\mathcal{H}_{d,g}$ of the Hilbert scheme containing all linearly normal scrolls of degree d and genus g in \mathbb{P}^n (cf. [3, Theorem 1.2] and [4, Theorem 1]).

Acknowledgements: It is a pleasure to dedicate this paper to our friend and colleague Gerard van der Geer on the occasion of his 60th birthday.

We thank A. Verra for useful discussions on the subject of this paper.

2000 *Mathematics Subject Classification.* Primary 14J26, 14C05, 14H60; Secondary 14D06, 14D20.
Key words and phrases. Brill–Noether theory of vector bundles, Hilbert schemes of scrolls, Moduli.

1. PRELIMINARIES

1.1. Moduli spaces of semistable rank–two vector bundles. For any integer d , we denote by $U_C(d)$ the moduli space of rank 2, semistable vector bundles of degree d on C . Recall that a rank 2 vector bundle \mathcal{F} of degree d is *semistable* [resp. *stable*] if for all quotient line bundles $\mathcal{F} \twoheadrightarrow L$ of degree d_1 one has $d \leq 2d_1$ [resp. $d < 2d_1$]. $U_C(d)$ is a projective variety and we let $U_C^s(d)$ be its open subset whose points correspond to stable vector bundles. If \mathcal{F} is a semistable rank–two vector bundle on C , we denote by $[\mathcal{F}]$ its class in $U_C(d)$.

The cases $0 \leq g \leq 1$ are quite classical and well known (see, e.g., [10, Chapt. V, §2], [9, 11]). In general we have (cf. [11, Sect. 5]):

Proposition 1.1. *If $g \geq 2$, then:*

- (i) $U_C(d)$ is irreducible, normal, of dimension $4g - 3$ and $U_C^s(d)$ is the set of smooth points of $U_C(d)$;
- (ii) if d is odd, then $U_C(d) = U_C^s(d)$ whereas if d is even, the inclusion $U_C^s(d) \subset U_C(d)$ is strict.

1.2. Speciality and Brill–Noether loci. If $[\mathcal{F}] \in U_C(d)$, we denote by $i(\mathcal{F})$, or simply by i if there is no danger of confusion, the integer $h^1(C, \mathcal{F})$, which is called the *speciality* of \mathcal{F} . Similarly we set $\ell(\mathcal{F}) = h^0(C, \mathcal{F})$, and $r(\mathcal{F}) = \ell(\mathcal{F}) - 1$, and we may often write ℓ, r rather than $\ell(\mathcal{F}), r(\mathcal{F})$. By Riemann–Roch theorem, we have

$$\ell(\mathcal{F}) = d - 2g + 2 + i(\mathcal{F}).$$

Fix positive integers d and i . Set $\ell = d - 2g + 2 + i$. One can consider the subset $B_C^\ell(d)$ of all classes $[\mathcal{F}] \in U_C(d)$ such that $i(\mathcal{F}) \geq i$ and accordingly $\ell(\mathcal{F}) \geq \ell$. This is called the ℓ^{th} -*Brill–Noether locus* and it has a natural determinantal scheme structure (see, e.g. [9]). A lower bound for the dimension of $B_C^\ell(d)$ as a determinantal locus is its *expected dimension*, given by the *Brill–Noether number*

$$\rho_d^\ell := 4g - 3 - i\ell. \tag{1.2}$$

The infinitesimal deformations of \mathcal{F} along which all sections in $H^0(C, \mathcal{F})$ deform, fill up the vector subspace of $H^1(C, \mathcal{F} \times \mathcal{F}^*) \cong H^0(C, \omega_C \otimes \mathcal{F} \times \mathcal{F}^*)^*$ which is the annihilator of the image of the cup–product map

$$P_{\mathcal{F}} : H^0(C, \mathcal{F}) \otimes H^0(C, \omega_C \otimes \mathcal{F}^*) \longrightarrow H^0(C, \omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*),$$

called the *Petri map* of \mathcal{F} (see, e.g. [12]). In other words $\text{Ann}(\text{Im}(P_{\mathcal{F}}))$ is the Zariski tangent space of $B_C^\ell(d)$ at $[\mathcal{F}]$, where $\deg(\mathcal{F}) = d$ and $\ell = h^0(C, \mathcal{F})$. In this case, by Riemann–Roch theorem, one has

$$\rho_d^\ell = h^1(C, \mathcal{F} \otimes \mathcal{F}^*) - h^0(C, \mathcal{F})h^1(C, \mathcal{F}).$$

Hence:

Lemma 1.3. *In the above setting, $B_C^\ell(d)$ is non–singular, of dimension ρ_d^ℓ at $[\mathcal{F}]$ if and only if $P_{\mathcal{F}}$ is injective.*

We finish this section by recalling two results. For the first, see [4, Proposition 3]:

Proposition 1.4. *Let C be a smooth, irreducible, projective curve of genus $g \geq 2$. If $d \geq 2g$ then $i(\mathcal{F}) = 0$ for $[\mathcal{F}] \in U_C(d)$ general.*

Indeed, we will be interested in the case $d \geq 2g$ in the rest of this paper. As for the next result, which will somehow justify our construction in §2, see [8, Corollary 7.3]:

Proposition 1.5. *Let C be a smooth, irreducible, projective curve of genus $g \geq 1$ and let \mathcal{F} be a special rank 2 vector bundle on C . Then there is a quotient $\mathcal{F} \twoheadrightarrow L$ with L a special line bundle.*

1.3. A result by M. Teixidor. If $d \geq 2g$, any rank–two vector bundle \mathcal{F} on C has $\ell(\mathcal{F}) \geq 2$ by Riemann–Roch theorem. Hence $B_C^2(d) = U_C(d)$ in this case (cf. [13, Note, p. 123]). Then, if $d \geq 2g$, it is no restriction to consider Brill–Noether loci $B_C^\ell(d)$, with $\ell \geq 2$. We record here the main result of [14]:

Theorem 1.6. *If $\ell \geq 2$, $i \geq 2$, C has general moduli, and either $\rho_d^\ell \geq 1$ and d is odd, or $\rho_d^\ell \geq 5$ and d is even, then $B_C^\ell(d)$ is not empty and of the expected dimension.*

Remark 1.7. It is useful to express the numerical conditions in Theorem 1.6 in terms of the speciality. Since $\ell = d - 2g + 2 + i \geq 2$, then $d \geq 2g - i$. In addition, when d is odd, one has $\rho_d^\ell \geq 1$, which reads

$$d \leq \frac{i+2}{i}(2g-2) - i; \tag{1.8}$$

when d is even one has $\rho_d^\ell \geq 5$, i.e.

$$d \leq \frac{i+2}{i}(2g-2) - i - \frac{4}{i}. \tag{1.9}$$

2. EXAMPLES OF BRILL-NOETHER LOCI

In this section we give examples of generically smooth components of the expected dimension of Brill-Noether loci of speciality $i \geq 1$ in $U_C(d)$, with C a curve of genus g with general moduli.

Theorem 2.1. *Let g, i be a integers such that*

$$i < \sqrt{g+4} - 2. \quad (2.2)$$

Let then d and d_1 be integers such that

$$g+4 \leq d_1 \leq (g-i) \frac{(i+1)}{i} \quad (2.3)$$

and

$$d_1 + g + 3 \leq d < 2d_1 \quad (2.4)$$

Set $\ell = d - 2g + 2 + i$.

If C is a curve of genus g with general moduli, there is an irreducible component of $B_C^\ell(d)$ which is generically smooth, of the expected dimension, containing points corresponding to stable, very-ample vector bundles \mathcal{F} , with $i(\mathcal{F}) = i$, whose minimal degree line bundle quotients have degree d_1 and speciality i .

The proof of Theorem 2.1 will follow from a series of remarks and lemmas presented below.

Remark 2.5. (i) Note that (2.2) implies $g \geq 6$ if $i = 1$ and $g \geq 13$ if $i \geq 2$. Moreover $i < \frac{g}{4}$ since $\sqrt{g+4} - 2 \leq \frac{g}{4}$ for any g .

(ii) The interval for the integer d_1 in (2.3) is in general not empty, since

$$(g+3)i < (g-i)(i+1) \quad \text{for any } i \geq 1. \quad (2.6)$$

If $i = 1$, this follows from $g \geq 6$. If $i \geq 2$, then (2.6) is equivalent to $i^2 + 4i - g < 0$, which follows from (2.2).

(iii) The inequalities in (2.4) are necessary for the stability of \mathcal{F} and for the very-ampleness of the line bundle N appearing in (2.8) below (cf. Lemmas 2.7 (i), 2.12 and 2.9).

(iv) The bound

$$d < 2(g-i) \frac{(i+1)}{i}$$

following from (2.3) and (2.4), is in general slightly worse than (1.8) and (1.9), but the difference, for i close to the upper bound in (2.2), is of the order of \sqrt{g} .

(v) The upper-bound in (2.3) implies the following inequality for the *Brill-Noether number*

$$\rho(g, d_1, d_1 - g + i) \geq 0. \quad (2.7)$$

Now we are going to produce the components of $B_C^\ell(d)$ announced in the statement of Theorem 2.1. From (2.7) we have $\dim(W_{d_1}^{d_1-g+i}(C)) = \rho(g, d_1, d_1 - g + i) \geq 0$, because C has general moduli. Consider extensions

$$0 \rightarrow N \rightarrow \mathcal{F} \rightarrow L \rightarrow 0, \quad (2.8)$$

with $N \in \text{Pic}^{d-d_1}(C)$ general and $L \in W_{d_1}^{d_1-g+i}(C)$ general (or any L if $\rho(g, d_1, d_1 - g + i) = 0$), so that $h^1(C, L) = i$. By (2.4), one has $d - d_1 \geq g + 3$; since $N \in \text{Pic}^{d-d_1}(C)$ is general, one has $h^1(C, N) = 0$. Therefore \mathcal{F} is a rank-two vector bundle of degree d and speciality $i = i(\mathcal{F})$, i.e. $[\mathcal{F}] \in B_C^\ell(d)$, with $\ell = d - 2g + 2 + i$, and we can look at it as an element of $\text{Ext}^1(L, N)$.

Lemma 2.9. *In the above setting, any $\mathcal{F} \in \text{Ext}^1(L, N)$ is very ample on C .*

Proof. A sufficient condition for \mathcal{F} to be very-ample is that both L and N are. By [2, (1.8) Theorem, p. 216], a sufficient condition for both L and N to be very ample on C with general moduli is

$$h^0(C, N) = d - d_1 - g + 1 \geq 4, \quad h^0(C, L) = d_1 - g + 1 + i \geq 4.$$

The first inequality holds by (2.4), the second by (2.3). \square

Remark 2.10. Note that (2.7) and the proof of Lemma 2.9 yield $g - 4i = \rho(g, d_1, 3) \geq \rho(g, d_1, d_1 - g + i) \geq 0$, so $i \leq \frac{g}{4}$ is a necessary condition for the ampleness of L (see Remark 2.5, (i)).

A general bundle $\mathcal{F} \in \text{Ext}^1(L, N)$ as above gives rise to the projective bundle $\mathbb{P}(\mathcal{F}) \xrightarrow{f} C$, which is embedded, via $|\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)|$, as a smooth scroll S of degree d and sectional genus g in \mathbb{P}^r , with $r = r(\mathcal{F}) = d - 2g + 1 + i$. The quotient $\mathcal{F} \twoheadrightarrow L$ corresponds to a section $C \xrightarrow{s} \mathbb{P}(\mathcal{F})$ of $\mathbb{P}(\mathcal{F}) \xrightarrow{f} C$, whose image is a unisecant, irreducible curve $\Gamma \cong C$ (cf. [10, § V, Prop. 2.6 and 2.9]). Since $h^1(C, N) = 0$, then $\Gamma \subset S \subset \mathbb{P}^r$ is linearly normally embedded in a linear subspace of dimension $d_1 - g + i$, as a curve of degree d_1 and speciality i .

Given $L \in W_{d_1}^{d_1-g+i}(C)$, $N \in \text{Pic}^{d-d_1}(C)$ and \mathcal{F} in $\mathbb{P}(\text{Ext}^1(L, N))$, the embedding $\mathbb{P}(\mathcal{F}) \rightarrow \mathbb{P}^r$ varies by projective automorphisms of \mathbb{P}^r . Thus the surface S varies, describing an irreducible locally closed subset $\mathcal{H}_C(d, i)$ of the Hilbert scheme.

Remark 2.11. It is useful to describe $\mathcal{H}_C(d, i)$ in a different way.

Let $L \in W_{d_1}^{d_1-g+i}(C)$ be general as above. Let $M \in \text{Pic}^\delta(C)$ be any line bundle of degree $\delta \gg 0$. Consider the projective bundle $\mathbb{P}(L \oplus M)$, which embeds as a smooth scroll $\Sigma = \Sigma_{L, M}$ of degree $d_1 + \delta$ and sectional genus g in $\mathbb{P}^{r'}$, $r' = r(L \oplus M) = d_1 + \delta - 2g + 1 + i = r + \delta$, via $|\mathcal{O}_{\mathbb{P}(L \oplus M)}(1)|$. By [5, Thm. 3.11], Σ contains a unique special section E of degree d_1 and speciality i , corresponding to the quotient $L \oplus M \twoheadrightarrow L$. One has $E^2 = d_1 - \delta \ll 0$ (cf. [10, § 5]).

One can choose M in such a way that there are $\delta + d_1 - d$ linearly independent points on Σ such that, projecting Σ from their span, the image of this projection is the surface $S \subset \mathbb{P}^r$ as above. In this projection E is isomorphically mapped to Γ .

Thus, $\mathcal{H}_C(d, i)$ can be thought of as the family of all projections of scrolls of the form $\Sigma_{L, M}$, with L, M as above, from the span of $\delta + d_1 - d$ general points on $\Sigma_{L, M}$.

Lemma 2.12. *In the above setting, Γ is the unique section of minimal degree of the scroll S . Hence the bundle \mathcal{F} is stable.*

Proof. Assume, by contradiction, there is a section $\Gamma' \subset S$ of degree $d_1 - \nu$, with $\nu \geq 0$.

Let $\Sigma = \Sigma_{L, M}$ be a scroll as in Remark 2.11 and let $\varphi : \Sigma \dashrightarrow S$ be the projection of Σ to S from suitable $\delta + d_1 - d$ general points on it. Let X be the set of these points. Then Γ' is the image via φ of a section $\Delta \neq E$ of degree $d_1 - \nu + h$ passing through a subset $Y \subseteq X$ of h points, for some $h \geq 0$. If we denote by F the ruling of Σ , then $\Delta \equiv E + (h - \nu)F$. Then:

- (i) $\Delta \cdot E \geq 0$ implies $E^2 + h - \nu \geq 0$, therefore $\Delta^2 = E^2 + 2(h - \nu) \geq -E^2 \gg 0$;
- (ii) hence $h^1(\Delta, \mathcal{N}_{\Delta|\Sigma}) = 0$ so $h^0(\Delta, \mathcal{N}_{\Delta|\Sigma}) = d_1 - \delta + 2(h - \nu) - g + 1$. Since Y consists of h general points, we must have $d_1 - \delta + 2(h - \nu) - g + 1 \geq h$, i.e. $h - 2\nu \geq \delta - d_1 + g - 1$.

Putting the above inequalities together, we have

$$\delta + d_1 - d \geq h \geq h - 2\nu \geq \delta - d_1 + g - 1 \quad \text{hence} \quad d \leq 2d_1 - g + 1$$

which, by the first inequality in (2.4), implies $d_1 \geq 2g + 2$, contrary to the fact that L is special. This proves the first assertion. Then the stability of \mathcal{F} follows from $d < 2d_1$ in (2.4). \square

Remark 2.13. Let us compute $y := \dim(\mathcal{H}_C(d, i)) - \dim(\text{PGL}(r + 1, \mathbb{C}))$. A scroll S corresponding to a point of $\mathcal{H}_C(d, i)$ is of the type $\mathbb{P}(\mathcal{F})$, with \mathcal{F} an extension as in (2.8). By Lemma 2.12, this extension is *essentially unique*, i.e. two of them correspond to the same point of $\mathbb{P}(\text{Ext}^1(L, N))$ (cf. [7, p. 31]). Therefore y is the sum of the following quantities:

- $\rho(g, d_1, d_1 - g + i)$, i.e. the number of parameters for the line bundle L ;
- g , i.e. the number of parameters for the line bundle N ;
- $\dim(\mathbb{P}(\text{Ext}^1(L, N))) = 2d_1 - d + g - 2$: indeed, $\deg(N - L) = d - 2d_1 < 0$, thus $h^1(C, N \otimes L^*) = 2d_1 - d + g - 1$.

Consider the *modular map* $\mu : \mathcal{H}_C(d, i) \dashrightarrow B_d^\ell(C)$ sending the point corresponding to $S \cong \mathbb{P}(\mathcal{F})$ to $[\mathcal{F}]$. This is well defined since $S \cong \mathbb{P}(\mathcal{F}) \cong \mathbb{P}(\mathcal{F}')$ implies $\mathcal{F} \cong \mathcal{F}'$. The fibres of μ are orbits by the $\text{PGL}(r + 1, \mathbb{C})$ -action on $\mathcal{H}_C(d, i)$. Therefore y is the dimension of the image of μ , hence $\dim(B_d^\ell(C)) \geq y$.

The next lemma shows that the image of μ lies in a component of $B_d^\ell(C)$ which is generically smooth and of the expected dimension, thus concluding the proof of Theorem 2.1.

Lemma 2.14. *Let \mathcal{F} be a bundle appearing in (2.8) with $L \in W_{d_1}^{d_1-g+i}(C)$ and $N \in \text{Pic}^{d-d_1}(C)$ general. Then the Petri map $P_{\mathcal{F}}$ is injective.*

Proof. For all $\mathcal{F} \in \text{Ext}^1(L, N)$, one has $h^1(C, \mathcal{F}) = i$, hence the domain of $P_{\mathcal{F}}$ has constant dimension $i(d - 2g + 2 + i)$. Therefore, by semicontinuity, it suffices to prove the assertion for a particular such \mathcal{F} , even if the dimension of the target of $P_{\mathcal{F}}$ jumps up. We will specialize to $\mathcal{F}_0 = L \oplus N$. We have

$$H^0(C, \mathcal{F}_0) = H^0(C, L) \oplus H^0(C, N) \quad \text{and} \quad H^0(C, \omega_C \otimes \mathcal{F}_0^*) = H^0(C, \omega_C \otimes L^*).$$

So the domain of $P_{\mathcal{F}_0}$ is

$$H^0(C, \mathcal{F}_0) \otimes H^0(C, \omega_C \otimes \mathcal{F}_0^*) = (H^0(C, L) \otimes H^0(C, \omega_C \otimes L^*)) \oplus (H^0(C, N) \otimes H^0(C, \omega_C \otimes L^*)),$$

whereas the target is

$$H^0(C, \omega_C \otimes \mathcal{F}_0 \otimes \mathcal{F}_0^*) = H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes L \otimes N^*) \oplus H^0(C, \omega_C \otimes N \otimes L^*) \oplus H^0(C, \omega_C).$$

The map $P_{\mathcal{F}_0}$ can be written on decomposable tensors as

$$(a \otimes b, \alpha \otimes \beta) \xrightarrow{P_{\mathcal{F}_0}} (ab, 0, \alpha\beta, 0),$$

for $a \otimes b \in H^0(C, L) \otimes H^0(C, \omega_C \otimes L^*)$ and for $\alpha \otimes \beta \in H^0(C, N) \otimes H^0(C, \omega_C \otimes L^*)$. In other words,

$$P_{\mathcal{F}_0} = \mu_L \oplus \mu_{L, N}$$

where $\mu_L : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^*) \rightarrow H^0(C, \omega_C)$ is the Petri map for L and $\mu_{L, N} : H^0(C, N) \otimes H^0(C, \omega_C \otimes L^*) \rightarrow H^0(C, \omega_C \otimes N \otimes L^*)$ is the multiplication map.

Since C has general moduli, the map μ_L is injective. We need to prove that $\mu_{L, N}$ is also injective. To do this, it suffices to show that $\mu_{L, N}$ is injective for some particular line bundle N_0 of degree $d - d_1$, even if N_0 becomes special and therefore $h^0(C, N_0) > h^0(C, N) = d - d_1 - g + 1$. Indeed, when a general N flatly tends to N_0 , the vector spaces $H^0(C, N)$ and $H^0(C, \omega_C \otimes N \otimes L^*)$ will respectively tend to subspaces $V \subseteq H^0(C, N_0)$ and $W \subseteq H^0(C, \omega_C \otimes N_0 \otimes L^*)$ of the same dimensions, and the limit of $\mu_{L, N}$ will be the multiplication map $\mu_{L, V} : V \otimes H^0(C, \omega_C \otimes L^*) \rightarrow W$. Hence $\mu_{L, V}$ (hence, by semicontinuity, $\mu_{L, N}$) is injective if μ_{L, N_0} is.

Let $\Delta \in \text{Div}^{2d_1 - d}(C)$ be an effective divisor. Let $N_0 = L(-\Delta) \in \text{Pic}^{d - d_1}(C)$ and set $\mu_0 = \mu_{L, N_0}$. If we tensor the exact sequence

$$0 \rightarrow L(-\Delta) \cong N_0 \rightarrow L \rightarrow L|_{\Delta} \rightarrow 0$$

by $H^0(C, \omega_C \otimes L^*)$, we get the commutative diagram with exact rows

$$\begin{array}{ccc} 0 \rightarrow & H^0(C, N_0) \otimes H^0(C, \omega_C \otimes L^*) & \longrightarrow & H^0(C, L) \otimes H^0(C, \omega_C \otimes L^*) \\ & \downarrow \mu_0 & & \downarrow \mu_L \\ 0 \rightarrow & H^0(C, \omega_C(-\Delta)) & \longrightarrow & H^0(C, \omega_C) \end{array}$$

Since μ_L is injective, μ_0 is also injective, which ends our proof. \square

Remark 2.15. Except in the case $i = 1$ and $d_1 = 2g - 2$, the general point of a component of $B_d^\ell(C)$ we constructed does not lie in the image of μ . Indeed, by recalling (1.2), one has

$$y - \rho_d^\ell = d(i - 1) - d_1(i - 2) - (g - 1)(i + 1).$$

(i) When $i = 1$, one has $y - \rho_d^{d-2g+3} = d_1 - 2(g - 1)$, which is zero if and only if $d_1 = 2g - 2$. If $i = 1$ consider a general vector bundle \mathcal{F} in our component. By Proposition 1.5, there is an exact sequence of the form (2.8) with $h^1(C, L) > 0$, hence $h^1(C, L) = 1$. Then the above argument shows that $L \cong \omega_C$.

(iii) When $i \geq 2$, by $d < 2d_1$ and (2.3) one has

$$y - \rho_d^\ell < i(d_1 - g + 1) + 1 - g \leq 1 - i^2 < 0.$$

The problem of describing the general element of a component of $B_d^\ell(C)$ we constructed when $i > 1$ (the case $i = 1$ is treated in (i)) looks interesting and we plan to come back to it in a future research.

Corollary 2.16. *Let C a general curve of genus $g \geq 6$. For any $3g + 1 \leq d \leq 4g - 5$, there is a component of $B_d^{d-2g+3}(C)$, which is generically smooth and of the expected dimension, whose general point corresponds to a very ample, stable vector bundle \mathcal{F} of speciality 1, fitting in an exact sequence*

$$0 \rightarrow N \rightarrow \mathcal{F} \rightarrow \omega_C \rightarrow 0$$

where ω_C is the minimal degree quotient line bundle of \mathcal{F} and $N \in \text{Pic}^{d-2g+2}(C)$ is general.

3. APPLICATIONS TO HILBERT SCHEMES OF SCROLLS

In this section we use Theorem 2.1 to study some components of Hilbert schemes of special scrolls.

3.1. Normal bundle cohomology. Here we prove the following:

Proposition 3.1. *Assumptions as in Theorem 2.1. Let $r = d - 2g + 1 + i$ and $S \subset \mathbb{P}^r$ be a smooth, linearly normal, special scroll of degree d , genus g , speciality i , with general moduli, which corresponds to a general point of $\mathcal{H}_C(d, i)$ as in Remark 2.11. If $\mathcal{N}_{S|\mathbb{P}^r}$ is the normal bundle of S in \mathbb{P}^r , then:*

- (i) $h^0(S, \mathcal{N}_{S|\mathbb{P}^r}) = 7(g - 1) + (r + 1)(r + 1 - i)$;
- (ii) $h^1(S, \mathcal{N}_{S|\mathbb{P}^r}) = 0$;
- (iii) $h^2(S, \mathcal{N}_{S|\mathbb{P}^r}) = 0$.

Proof of Proposition 3.1. First, we prove (iii). Since S is linearly normal, from Euler's sequence we get:

$$\cdots \rightarrow H^0(S, \mathcal{O}_S(H))^* \otimes H^2(S, \mathcal{O}_S(H)) \rightarrow H^2(S, \mathcal{T}_{\mathbb{P}^r|_S}) \rightarrow 0$$

where H is a hyperplane section of S . Since S is a scroll, then $h^2(S, \mathcal{O}_S(H)) = 0$, which implies $h^2(S, \mathcal{T}_{\mathbb{P}^r|_S}) = 0$. Thus (iii) follows by the normal bundle sequence

$$0 \rightarrow \mathcal{T}_S \rightarrow \mathcal{T}_{\mathbb{P}^r|_S} \rightarrow \mathcal{N}_{S|\mathbb{P}^r} \rightarrow 0. \quad (3.2)$$

Since S is a scroll of genus g , we have

$$\chi(\mathcal{O}_S) = 1 - g, \quad \chi(\mathcal{T}_S) = 6 - 6g. \quad (3.3)$$

Since S is linearly normal, from Euler's sequence we then get

$$\chi(\mathcal{T}_{\mathbb{P}^r|_S}) = (r + 1)(r + 1 - i) + g - 1. \quad (3.4)$$

Thus, from (iii) and (3.3), (3.4) we get

$$\chi(\mathcal{N}_{S|\mathbb{P}^r}) = h^0(S, \mathcal{N}_{S|\mathbb{P}^r}) - h^1(S, \mathcal{N}_{S|\mathbb{P}^r}) = 7(g - 1) + (r + 1)(r + 1 - i). \quad (3.5)$$

The rest of the proof is concentrated on computing $h^1(S, \mathcal{N}_{S|\mathbb{P}^r})$.

Since $S = \mathbb{P}(\mathcal{F})$ is a scroll corresponding to a general point $[\mathcal{F}] \in \mathcal{H}_C(d, i)$, let Γ be the unisecant of S of degree d_1 corresponding to the special quotient line bundle $\mathcal{F} \twoheadrightarrow L$.

Claim 3.6. *One has $h^1(S, \mathcal{N}_{S|\mathbb{P}^r}(-\Gamma)) = h^2(S, \mathcal{N}_{S|\mathbb{P}^r}(-\Gamma)) = 0$, hence*

$$h^1(S, \mathcal{N}_{S|\mathbb{P}^r}) = h^1(\Gamma, \mathcal{N}_{S|\mathbb{P}^r}|_\Gamma). \quad (3.7)$$

Proof of Claim 3.6. Look at the exact sequence

$$0 \rightarrow \mathcal{N}_{S|\mathbb{P}^r}(-\Gamma) \rightarrow \mathcal{N}_{S|\mathbb{P}^r} \rightarrow \mathcal{N}_{S|\mathbb{P}^r}|_\Gamma \rightarrow 0.$$

From (3.2) tensored by $\mathcal{O}_S(-\Gamma)$ we see that $h^2(S, \mathcal{N}_{S|\mathbb{P}^r}(-\Gamma)) = 0$ follows from $h^2(S, \mathcal{T}_{\mathbb{P}^r|_S}(-\Gamma)) = 0$ which, by Euler's sequence, follows from $h^2(S, \mathcal{O}_S(H - \Gamma)) = h^0(S, \mathcal{O}_S(K_S - H + \Gamma)) = 0$, since $K_S - H + \Gamma$ intersects the ruling of S negatively.

As for $h^1(S, \mathcal{N}_{S|\mathbb{P}^r}(-\Gamma)) = 0$, this follows from $h^1(S, \mathcal{T}_{\mathbb{P}^r|_S}(-\Gamma)) = h^2(S, \mathcal{T}_S(-\Gamma)) = 0$. By Euler's sequence, the first vanishing follows from $h^2(S, \mathcal{O}_S(-\Gamma)) = h^1(S, \mathcal{O}_S(H - \Gamma)) = 0$. Since $K_S + \Gamma$ meets the ruling negatively, one has $h^0(S, \mathcal{O}_S(K_S + \Gamma)) = h^2(S, \mathcal{O}_S(-\Gamma)) = 0$. Moreover $h^1(S, \mathcal{O}_S(H - \Gamma)) = h^1(C, N) = 0$.

In order to prove $h^2(S, \mathcal{T}_S(-\Gamma)) = 0$, consider the exact sequence

$$0 \rightarrow \mathcal{T}_{rel} \rightarrow \mathcal{T}_S \rightarrow \rho^*(\mathcal{T}_C) \rightarrow 0$$

arising from the structure morphism $S = \mathbb{P}(\mathcal{F}) \xrightarrow{\rho} C$. The vanishing we need follows from $h^2(S, \mathcal{T}_{rel} \otimes \mathcal{O}_S(-\Gamma)) = h^2(S, \mathcal{O}_S(-\Gamma) \otimes \rho^*(\mathcal{T}_C)) = 0$. The first vanishing holds since $\mathcal{T}_{rel} \cong \mathcal{O}_S(2H - dF)$, where F is a ruling of S , and therefore, $\mathcal{O}_S(K_S + \Gamma) \otimes \mathcal{T}_{rel}^*$ restricts negatively to the ruling. Similar considerations yield the second vanishing. \square

Consider the exact sequence

$$0 \rightarrow \mathcal{N}_{\Gamma|_S} \rightarrow \mathcal{N}_{\Gamma|\mathbb{P}^r} \rightarrow \mathcal{N}_{S|\mathbb{P}^r}|_\Gamma \rightarrow 0. \quad (3.8)$$

Claim 3.9. *The map $H^1(\Gamma, \mathcal{N}_{\Gamma|_S}) \xrightarrow{\alpha} H^1(\Gamma, \mathcal{N}_{\Gamma|\mathbb{P}^r})$ arising from (3.8) is surjective, hence $h^1(\Gamma, \mathcal{N}_{S|\mathbb{P}^r}|_\Gamma) = 0$.*

Proof of Claim 3.9. Equivalently, we show the injectivity of the dual map

$$H^0(\Gamma, \omega_\Gamma \otimes \mathcal{N}_{\Gamma, \Gamma|\mathbb{P}^R}^*) \xrightarrow{\alpha^*} H^0(\Gamma, \omega_\Gamma \otimes \mathcal{N}_{\Gamma|S}^*) \cong H^0(C, \omega_C \otimes N \otimes L^*). \quad (3.10)$$

Consider $\Gamma \subset \mathbb{P}^h$, where $h = d_1 - g + i$, and the Euler sequence of \mathbb{P}^h restricted to Γ . By taking cohomology and dualizing, we get

$$0 \rightarrow H^1(\Gamma, \mathcal{T}_{\mathbb{P}^h|_\Gamma})^* \rightarrow H^0(\Gamma, \mathcal{O}_\Gamma(H)) \otimes H^0(\Gamma, \omega_\Gamma(-H)) \xrightarrow{\mu_0} H^0(\Gamma, \omega_\Gamma),$$

where μ_0 is the Brill-Noether map of $\mathcal{O}_\Gamma(H)$. Since $\Gamma \cong C$ has general moduli, then μ_0 is injective by Gieseker-Petri's theorem (cf. [2]) so $h^1(\Gamma, \mathcal{T}_{\mathbb{P}^h|_\Gamma}) = 0$. From the exact sequence

$$0 \rightarrow \mathcal{T}_\Gamma \rightarrow \mathcal{T}_{\mathbb{P}^h|_\Gamma} \rightarrow \mathcal{N}_{\Gamma|\mathbb{P}^h} \rightarrow 0$$

we get $h^1(\Gamma, \mathcal{N}_{\Gamma|\mathbb{P}^h}) = 0$. From the inclusions $\Gamma \subset \mathbb{P}^h \subset \mathbb{P}^R$ we have the sequence

$$0 \rightarrow \mathcal{N}_{\Gamma|\mathbb{P}^h} \rightarrow \mathcal{N}_{\Gamma|\mathbb{P}^R} \rightarrow \mathcal{N}_{\mathbb{P}^h|\mathbb{P}^R}|_\Gamma \rightarrow 0,$$

which shows that $H^1(\Gamma, \mathcal{N}_{\Gamma|\mathbb{P}^R}) \cong H^1(\Gamma, \mathcal{N}_{\mathbb{P}^h|\mathbb{P}^R}|_\Gamma)$, i.e.

$$H^0(\Gamma, \omega_\Gamma \otimes \mathcal{N}_{\Gamma|\mathbb{P}^R}^*) \cong H^0(\Gamma, \omega_\Gamma \otimes \mathcal{N}_{\mathbb{P}^h|\mathbb{P}^R}|_\Gamma^*). \quad (3.11)$$

On the other hand, from (2.8) and the non-speciality of N , we get

$$0 \rightarrow H^0(C, L)^* \rightarrow H^0(C, \mathcal{F})^* \rightarrow H^0(C, N)^* \rightarrow 0.$$

Since $H^0(S, \mathcal{O}_S(H)) \cong H^0(C, \mathcal{F})$ and $\mathcal{O}_\Gamma(H) \cong L$, the Euler sequences restricted to Γ give the following commutative diagram

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 & \rightarrow & \mathcal{O}_\Gamma & \rightarrow & H^0(C, L)^* \otimes L & \rightarrow & \mathcal{T}_{\mathbb{P}^h|_\Gamma} & \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow & \\ 0 & \rightarrow & \mathcal{O}_\Gamma & \rightarrow & H^0(C, \mathcal{F})^* \otimes L & \rightarrow & \mathcal{T}_{\mathbb{P}^r|_\Gamma} & \rightarrow 0 \\ & & & & \downarrow & & \downarrow & \\ & & & & H^0(C, N)^* \otimes L & \cong & \mathcal{N}_{\mathbb{P}^h|\mathbb{P}^r}|_\Gamma & \\ & & & & \downarrow & & \downarrow & \\ & & & & 0 & & 0 & \end{array}$$

This gives

$$H^0(\Gamma, \omega_\Gamma \otimes \mathcal{N}_{\mathbb{P}^h|\mathbb{P}^R}|_\Gamma^*) \cong H^0(C, N) \otimes H^0(C, \omega \otimes L^*). \quad (3.12)$$

By (3.10), (3.11) and (3.12), we see that $\alpha^* = \mu_{L, N}$, whose injectivity has been shown in Lemma 2.14. \square

From Claim 3.9, (3.5) and (3.7), both (i) and (ii) follow. \square

3.2. Components of the Hilbert scheme of linearly normal, special scrolls. We denote by $\text{Hilb}(d, g, i)$ the open subset of the Hilbert scheme parametrizing smooth scrolls in \mathbb{P}^r of genus g , degree d and speciality i , with $r = d - 2g + 1 + i$.

Theorem 3.13. *Numerical assumptions as in Theorem 2.1. Then $\text{Hilb}(d, g, i)$ has an irreducible component \mathcal{H} which contains all $\mathcal{H}_C(d, i)$ with C a general curve of genus g . The general point $[S] \in \mathcal{H}$ is a smooth scroll of degree d , genus g and speciality i , which is linearly normal in \mathbb{P}^r . Moreover:*

- (i) \mathcal{H} is generically smooth of dimension $\dim(\mathcal{H}) = 7g - 7 + (r + 1)(r + 1 - i)$;
- (ii) $[S] \in \mathcal{H}$ general corresponds to a pair (\mathcal{F}, C) , where C has general moduli and \mathcal{F} is stable of speciality i on C .

When $i = 1$ and $3g + 1 \leq d \leq 4g - 5$, the union of all $\mathcal{H}_C(d, i)$ with C a general curve of genus g is dense in \mathcal{H} and the general scroll $[S] \in \mathcal{H}$ has a canonical curve as the unique special section of minimal degree.

Proof. The construction of \mathcal{H} is clear. Its generic smoothness and the dimension count follow from Proposition 3.1. The last part of the statement follows from Corollary 2.16. \square

Remark 3.14. As we saw in Remark 2.15, the union of $\mathcal{H}_C(d, i)$ with C a general curve of genus g is never dense in \mathcal{H} unless $i = 1$ and $d_1 = 2g - 2$.

Remark 3.15. In [5] we constructed components of Hilbert schemes parametrizing smooth, linearly normal, special scrolls $S \subset \mathbb{P}^r$, of degree d , genus g having the base curve with general moduli. Such components were constructed for any $g \geq 3$, $i \geq 1$ and for any $d \geq \frac{7g-\epsilon}{2} - 2i + 2$, $0 \leq \epsilon \leq 1$, $\epsilon \equiv g \pmod{2}$, unless $i = 2$ where $d \geq 4g - 3$ (cf. [5, Thm. 6.1]). The general point of any such component corresponds to an unstable vector bundle on C (cf. [5, Rem. 6.3]).

3.3. Non-linearly normal, special scrolls. Let $n = d - 2g + 1$. Recall that there is a unique component $\mathcal{H}_{d,g}$ of the Hilbert scheme containing all linearly normal, non-special scrolls of degree d and genus g in \mathbb{P}^n (cf. [3, Theorem 1.2] and [4, Theorem 1]).

Consider now the family \mathcal{Y}_i whose general element is a general projection to \mathbb{P}^n of the scroll $S \subset \mathbb{P}^r$, $r = n + i$, with $[S] \in \mathcal{H}$ general as in Theorem 3.13. The following proposition shows that the families \mathcal{Y}_i never fill up components of the Hilbert scheme of \mathbb{P}^n .

Proposition 3.16. *In the above setting, for d, d_1, g and i as in Theorem 2.1, \mathcal{Y}_i is a generically smooth subset of $\mathcal{H}_{d,g}$ of codimension i^2 whose general point is smooth for $\mathcal{H}_{d,g}$.*

Proof. Let $[S] \in \mathcal{H}$ be general with $S \cong \mathbb{P}(\mathcal{F})$ and let $S' \subset \mathbb{P}^n$ be a general projection of S . Let $G_{S'} \subset \mathrm{PGL}(n+1, \mathbb{C})$ be the subgroup of projective transformations fixing S' . Since $G_S \subset \mathrm{Aut}(S) \cong \mathrm{Aut}(\mathbb{P}(\mathcal{F}))$, one has $\dim(G_{S'}) = 0$, because \mathcal{F} is stable.

Then $\dim(\mathcal{Y}_i)$ is:

- $3g - 3$, for the parameters on which C depends, plus
- $4g - 3 - i(r + 1)$, for the parameters on which \mathcal{F} depends, plus
- $\dim(\mathbb{G}(n, r)) = (n + 1)(r - n) = (n + 1)i$, which are the parameters for the projections, plus
- $(n + 1)^2 - 1 = \dim(\mathrm{PGL}(n + 1, \mathbb{C}))$, minus
- $\dim(G_{S'}) = 0$.

Adding up, we get $\dim(\mathcal{Y}_i) = \dim(\mathcal{H}_{d,g}) - i^2$.

Consider the *Rohn exact sequence*

$$0 \rightarrow \mathbb{C}^i \otimes \mathcal{O}_S(H) \rightarrow \mathcal{N}_{S|\mathbb{P}^r} \rightarrow \mathcal{N}_{S'|\mathbb{P}^n} \rightarrow 0$$

(see, e.g. [6], p. 358, formula (2.2)). From Proposition 3.1, (ii), we have $h^1(S, \mathcal{N}_{S|\mathbb{P}^r}) = 0$, therefore also $h^1(S', \mathcal{N}_{S'|\mathbb{P}^n}) = 0$. Hence \mathcal{Y}_i is contained in a component \mathcal{Z} of the Hilbert scheme of dimension $\chi(\mathcal{N}_{S'|\mathbb{P}^n}) = 7(g - 1) + (r + 1)^2$ and the general point of \mathcal{Y}_i is a smooth point of \mathcal{Z} .

The general point of \mathcal{Y}_i is a smooth scroll on C arising from a stable, rank-two vector bundle. The component $\mathcal{H}_{d,g}$ is the only component of the Hilbert scheme whose general point corresponds to a stable scroll (cf. the proof of [4, Theorem 2]). Therefore, $\mathcal{Z} = \mathcal{H}_{d,g}$. The map $H^0(S, \mathcal{N}_{S|\mathbb{P}^r}) \rightarrow H^0(S', \mathcal{N}_{S'|\mathbb{P}^n})$ is not surjective: its cokernel is $\mathbb{C}^i \otimes H^1(\mathcal{O}_S(H))^{\oplus i}$, which has dimension i^2 . This means that \mathcal{Y}_i is a generically smooth subset of $\mathcal{H}_{d,g}$ of codimension i^2 . \square

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