MODULI OF NODAL CURVES ON SMOOTH SURFACES OF GENERAL TYPE

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ABSTRACT. In this paper we focus on the problem of computing the *number of moduli* of the so called *Severi varieties* (denoted by $V_{|D|,\delta}$), which parametrize universal families of irreducible, δ-nodal curves in a complete linear system $|D|$, on a smooth projective surface S of general type. We determine geometrical and numerical conditions on D and numerical conditions on δ ensuring that such number coincides with $dim(V_{|D|,\delta})$. As related facts, we also determine some sharp results concerning the geometry of some Severi varieties.

INTRODUCTION

Let S be a smooth, projective surface and let $|D|$ denote a complete linear system on S, whose general element is assumed to be a smooth, irreducible curve. By the hypothesis on its general element, it makes sense to consider the subscheme of $|D|$ which parametrizes a universal family of irreducible curves having only δ nodes as singular points. Such a subscheme is functorially defined, locally closed in $|D|$ (see [34] for $S = \mathbb{P}^2$ but the proof extends to any S) and denoted by $V_{|D|,\delta}$. It is usually called the *Severi variety* of irreducible δ-nodal curves in $|D|$, since Severi was the first who studied some properties of families of plane curves of given degree and given geometric genus (see [30]).

One can be interested in studying the moduli behaviour of the elements that a Severi variety parametrizes. This means to understand how the natural functorial morphism

$$\pi_{|D|,\,\delta} : V_{|D|,\delta} \longrightarrow \mathcal{M}_g$$

behaves, for each $\delta \geq 0$, where $g = p_a(D) - \delta$, $p_a(D)$ the arithmetic genus of D and $\mathcal{M}_g$ the moduli space of smooth curves of (geometric) genus g; precisely, the problem is to determine the dimension of the image of $\pi_{|D|,\,\delta}$.

In [29], Sernesi considered the case $S = \mathbb{P}^2$. Denote by

$$\pi_{n,\,\delta} : V_{n,\delta} \to \mathcal{M}_g$$

the functorial morphism from the Severi variety of plane irreducible and δ-nodal curves of degree n to the moduli space of smooth curves of genus $g = \frac{(n-1)(n-2)}{2} - \delta$. Recall that $V_{n,\delta}$ is irreducible (see [14]).

Definition 0.1. *(see* [29]*) The* number of moduli *of* $V_{n,\delta}$ *is* $dim(\pi_{n,\,\delta}(V_{n,\delta}))$. $V_{n,\delta}$ *is said to have the* expected number of moduli *if such dimension equals*

$$min(3g-3,\ 3g-3+\rho(g,2,n)),$$

where $\rho(g,2,n)$ *is the* Brill-Noether number.

Of course, when $\rho(g,2,n) \geq 0$, $V_{n,\delta}$ has the expected number of moduli $3g-3 = dim(\mathcal{M}_g)$ when every sufficiently general curve of genus g belongs to it; in such a case, this family of plane curves has *general moduli*. When $\rho(g,2,n) < 0$, the family $V_{n,\delta}$ does not have general moduli, i.e. it has *special moduli* and the number $-\rho(g,2,n)$ determines the expected codimension of $\pi_{n,\delta}(V_{n,\delta})$ in $\mathcal{M}_g$.
With this set-up, Sernesi proved the following result:

Theorem 0.1. *For all* n, g *such that*

$$n \geq 5 \ \text{ and } \ n-2 \leq g \leq \frac{(n-1)(n-2)}{2},$$

$V_{n,\delta}$ *has the expected number of moduli.*

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Remark 0.1. Since $3g-3+\rho(g,2,n) = 3n+g-9 = dim(V_{n,\delta}) - dim(Aut(\mathbb{P}^2))$, when $\rho(g,2,n) < 0$ the fact that $V_{n,\delta}$ exactly has the expected number of moduli means that its general point parametrizes a curve X which is birationally - but not projectively - equivalent to finitely many curves of the family, i.e. the normalization C of X has only finitely many linear systems of degree n and dimension 2.

In this paper, we are interested in the case of S a smooth, projective surface of general type. In such a case, the expected number of moduli equals $dim(V_{|D|,\delta})$ (see Definition 2.3).

We determine some general conditions on D, δ and, sometimes, on the geometry of S guaranteeing that such expected number of moduli is achieved (see Theorems 3.2, 3.3, 5.4, 6.1 and 6.3). As a particular case of our more general results, we get the following:

Proposition. *Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate complete intersection of general type whose canonical divisor is $K_S \sim \alpha H$, where H denotes its hyperplane section, α a positive integer and $\sim$ the linear equivalence of divisors on S. Let m be a positive integer and let $X \sim mH$ be an irreducible curve, with only δ nodes as singular points, of geometric genus $g = p_a(X) - \delta$, $\delta \geq 0$. Suppose that $[X]$ is a regular point of the Severi variety $V_{|mH|,\delta}$ (in the sense of Definition 1.4).*
Assume that:
(1) $\delta \leq \dim(|mH|)$ if

a) $\alpha \geq 2$, $m \geq \alpha + 6$, $\delta \geq 1$ *or*
b) $\alpha \geq 1$, $m \geq \alpha + 6$, $\delta = 0$*;*

(2) $\delta < \frac{m(m-4)}{4} deg(S)$ if $\alpha \geq 1$ and $5 \leq m \leq \alpha + 5$;
(3)

a) $\delta < \frac{m(m-2)}{4} deg(S)$ *if* $\alpha \geq 2$ *and* $m = 3,\ 4$ *or*
b) $\delta = 0$ *if* $\alpha = 1$ *and* $m = 3,\ 4$*;*

(4) $\delta < deg(S)(2+\alpha) + \frac{(r+3-4deg(S))}{2}\chi + \frac{(r-1)}{2}\chi^2$ if $\alpha \geq 1$ and $m = 2$, where χ is a non-negative integer in $[\frac{2deg(S)-1}{r-1} - 1,\ \frac{2deg(S)-1}{r-1})$;
(5) $\delta < \frac{deg(S)}{2}(1+\alpha) + \frac{(r-2deg(S)+2)}{2}\chi + \frac{(r-2)}{2}\chi^2$ if $\alpha \geq 1$ and $m = 1$, where χ is a non-negative integer in $[\frac{deg(S)-1}{r-2} - 1,\ \frac{deg(S)-1}{r-2})$;

Then the morphism

$$\pi_{|mH|,\delta} : V_{|mH|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. In particular, it has finite fibres on each generically regular component of $V_{|mH|,\delta}$, so each such component parametrizes a family having the expected number of moduli.

In particular, we have the following:

Corollary. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$ and let $[X] \in V_{|mH|,\delta}$ be a regular point.*
Assume that:
(1) $\delta \leq \dim(|mH|)$ if

a) $d \geq 6$, $m \geq d+2$, $\delta \geq 1$ *or*
b) $d \geq 5$, $m \geq d+2$, $\delta = 0$*;*

(2) $\delta < \frac{m(m-4)}{4} d$ if $d \geq 5$ and $5 \leq m \leq d+1$;
(3)

a) $\delta < \frac{m(m-2)}{4} d$ *if* $d \geq 6$ *and* $m = 3,\ 4$ *or*
b) $\delta = 0$ *if* $d = 5$ *and* $m = 3,\ 4$*;*

(4) $\delta < d-2$ if $d \geq 5$ and $m = 2$;
(5) $\delta < d-3$ if $d \geq 5$ and $m = 1$.

Then the morphism

$$\pi_{|mH|,\delta} : V_{|mH|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. In particular, it has finite fibres on each generically regular component of $V_{|mH|,\delta}$, so each such component parametrizes a family having the expected number of moduli.

The paper consists of seven sections. In Section 1, we recall some terminology and notation. Section 2 contains fundamental definitions and technical details which are used for our proofs. Section 3 contains the main results of the paper (Theorems 3.2, 3.3). In Section 4 we consider a fundamental proposition, which is the key point to determine the results of Sections 5 and 6. Such theorems focus on cases to which the results of Section 3 cannot apply. For simplicity, in Section 7 we sum up our results in the particular cases of Severi varieties of the form $V_{|mH|,\delta}$ on smooth complete intersection surfaces of general type or on smooth surfaces in $\mathbb{P}^3$ of degree $d \geq 5$.

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1. Notation and Preliminaries

We work in the category of $\mathbb{C}$-schemes. Y is a m*- fold* if it is a reduced, irreducible and non-singular scheme of finite type over $\mathbb{C}$ and of dimension m. If $m = 1$, then Y is a (smooth) *curve*; $m = 2$ is the case of a (non-singular) *surface*. If Z is a closed subscheme of a scheme Y, $\mathcal{I}_{Z/Y}$ (or $\mathcal{I}_Z$) denotes the *ideal sheaf* of Z in Y whereas $\mathcal{N}_{Z/Y}$ is the *normal sheaf* of Z in Y. When Y is a smooth variety, K_Y denotes a canonical divisor whereas $\mathcal{T}_Y$ denotes its tangent bundle.

Let Y be a m-fold and let $\mathcal{E}$ be a rank r vector bundle on Y; $c_i(\mathcal{E})$ denotes the i^{th}*-Chern class* of $\mathcal{E}$, $1 \leq i \leq r$. The symbol $\sim$ will always denote linear equivalence of divisors on Y. As usual, $h^i(Y, -) :=$ dim $H^i(Y, -)$.

If D is a reduced curve, $p_a(D) = h^1(\mathcal{O}_D)$ denotes its *arithmetic genus*, whereas $g(D) = p_g(D)$ denotes its *geometric genus*, the arithmetic genus of its normalization. For a smooth curve D, ω_D denotes its canonical sheaf, i.e. $\omega_D \cong \mathcal{O}_D(K_D)$.

Definition 1.1. *Let S be a smooth, projective surface and $Div(S)$ be the set of divisors on S. An element $B \in Div(S)$ is said to be* nef, *if $B \cdot D \geq 0$ for each irreducible curve D on S (where $\cdot$ denotes the intersection form on S; in the sequel we will omit $\cdot$). A nef divisor B is said to be* big *if $B^2 > 0$.*

Remark 1.1. We recall that, given a smooth surface S, $N(S)^+$ is the set of divisor classes with positive intersection numbers with itself and with an ample class. By Kleiman's criterion (see, for example, [16]), a nef divisor B is in the closure of $N(S)^+$.

Definition 1.2. Let $S \subset \mathbb{P}^r$ be a smooth surface, H its hyperplane section and $D \in Div(S)$. We denote by $\nu(D, H)$ the *Hodge number* of D and H,

$$\nu(D, H) := (DH)^2 - D^2H^2.$$

By the Index Theorem (see, for example, [3] or [12]) this is non-negative since H is a very ample divisor.

Definition 1.3. Let S be a smooth, projective surface. A rank 2 vector bundle $\mathcal{E}$ on S is said to be *Bogomolov-unstable* if there exist $M, \ B \in Div(S)$ and a 0-dimensional scheme Z (possibly empty) fitting in the exact sequence

$$0 \to \mathcal{O}_S(M) \to \mathcal{E} \to \mathcal{I}_Z(B) \to 0 \qquad (1)$$

such that $(M - B) \in N(S)^+$.

Remark 1.2. Recall that $\mathcal{E}$ is Bogomolov- unstable when $c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) > 0$ (see [4] or [26]).

It is also useful to remind some standard terminology and techniques on Severi varieties. Consider S a smooth, projective surface and assume that, for given $D \in Div(S)$ and δ positive integer, $V_{|D|,\,\delta} \neq \emptyset$. If $[X] \in V_{|D|,\,\delta}$, N will always denote the scheme of nodes of X, which is a closed zero-dimensional subscheme of S of degree δ. From now on, denote by

$$\varphi : C \to X \subset S \tag{2}$$

the normalization map of X. Thus, on C we have the exact sequence of vector bundles

$$0 \to \mathcal{T}_C \to \varphi^*(\mathcal{T}_S) \to \mathcal{N}_\varphi \to 0, \tag{3}$$

where $\mathcal{N}_\varphi$ is the *normal bundle* of φ. Observe that, if $\tilde{N}$ denotes the pull-back of N to C, by (3) we get that $\mathcal{N}_\varphi \cong \mathcal{O}_C(\varphi^*(D) - \tilde{N})$, so we have

$$H^i(\mathcal{N}_\varphi) \cong H^i(\mathcal{O}_C(\varphi^*(D) - \tilde{N})),\ i \geq 0. \tag{4}$$

From Horikawa's theory (see [18]), $H^0(\mathcal{N}_\varphi)$ parametrizes all first-order equisingular deformations of X in S. Therefore, one gets

$$T_{[X]}(V_{|D|,\,\delta}) \cong H^0(\mathcal{I}_{N/S}(D))/ < X >, \tag{5}$$

which is the subspace of $H^0(\mathcal{N}_\varphi)$ contained in $T_{[X]}(|D|) \cong H^0(\mathcal{O}_S(D))/ < X >$.

Remark 1.3. When S is assumed to be a regular surface, then

$$(H^0(S, \mathcal{I}_{N/S}(D))/ < X >) \ \cong\ H^0(C, \mathcal{O}_C(\varphi^*(D) - \tilde{N})),$$

which means that all first-order equisingular deformations of X in S are in $|D|$, i.e.

$$T_{[X]}(V_{|D|,\,\delta}) \cong H^0(C,\, \mathcal{N}_\varphi) \tag{6}$$

Remark 1.4. From the exact sequence

$$0 \to \mathcal{I}_{N/S}(D) \to \mathcal{O}_S(D) \to \mathcal{O}_N(D) \to 0 \tag{7}$$

and from (5), we get

$$dim(T_{[X]}(V_{|D|,\,\delta})) \geq h^0(\mathcal{O}_S(D)) - \delta - 1 = dim(|D|) - \delta;$$

the above inequality is an equality if and only if the surjection $H^1(\mathcal{I}_{N/S}(D)) \to H^1(\mathcal{O}_S(D))$ is an isomorphism, i.e. if and only if N imposes independent conditions to the linear system $|D|$. In such a case $V_{|D|,\,\delta}$ is smooth at $[X]$ of codimension δ in $|D|$.

We recall the following:

Definition 1.4. *$V_{|D|,\,\delta}$ is said to be* regular *at the point $[X]$ if it is smooth at $[X]$ of dimension $dim(|D|) - \delta$. Otherwise, the component of $V_{|D|,\,\delta}$ containing $[X]$ is said to be a* superabundant component. *A component of a Severi variety is said to be* regular *if it is regular at each point,* generically regular *if it is regular at its general point.*

We recall that the regularity is a very strong condition, indeed it implies that the nodes of X can be independently smoothed in S (see, for example, [7] and [29]).

2. Basic definitions and technical tools

In this section we introduce fundamental definitions and remarks which are used to compute the number of moduli of some Severi varieties.

From now on, S will denote a smooth, projective surface of general type, unless otherwise specified. Let $|D|$ be a complete linear system on S, whose general element is supposed to be a smooth, irreducible curve. Denote by X an irreducible curve in $|D|$ having only $\delta \geq 0$ nodes as singularities. As in (2), the map $\varphi : C \to X \subset S$ denotes its normalization, where C is a smooth curve of geometric genus $g = p_a(D) - \delta$.

We shall always assume that $g \geq 2$, for each $\delta \geq 0$. This assumption is not so restrictive for the problems we are interested in.

With this setup, for each $\delta \geq 0$ one can consider the morphisms:

$$\pi_{|D|,\,\delta} : V_{|D|,\delta} \longrightarrow \mathcal{M}_g, \tag{8}$$

where $\mathcal{M}_g$ denotes the moduli space of smooth curves of (geometric) genus g. Indeed, if $F_\delta : \mathcal{X}_\delta \to V_{|D|,\delta}$ denotes the universal family of δ-nodal curves in S parametrized by $V_{|D|,\delta}$, the fibres of F_δ can be simultaneously desingularized, so there exists a diagram of proper morphisms

$$\begin{array}{ccccc} \mathcal{C}_\delta & \xrightarrow{\Phi_\delta} & \mathcal{X}_\delta & \subset S \times V_{|D|,\delta} \\ & \searrow^{f_\delta} & \downarrow^{F_\delta} & \\ & & V_{|D|,\delta} & \end{array}$$

where Φ_δ is fibrewise the normalization map. In other words Φ_δ is the blow-up of $\mathcal{X}_\delta$ along its codimension-one singular locus and, for each $\delta \geq 1$, the morphism

$$\pi_{|D|,\delta} : V_{|D|,\delta} \to \mathcal{M}_g$$

is functorially defined by f_δ. When $\delta = 0$, $V_{|D|,0}$ is the open dense subscheme of smooth curves in $|D|$, so Φ_0 is the identity map and we have $\pi_{|D|,0} : V_{|D|,0} \to \mathcal{M}_{p_a(D)}$.
The problem is to determine, for each morphism $\pi_{|D|,\delta}$, the dimension of its image.

Different from the case of $S = \mathbb{P}^2$, Severi varieties on surfaces of general type are, in general, reducible; for example, Chiantini and Ciliberto ([6]) showed that even in the most natural case of a general surface $S = S_d \subset \mathbb{P}^3$ of degree $d \geq 5$, Severi varieties on S of the form $V_{|mH|,\delta}$, $m \geq d$ and H the plane section of S, always admit at least one (generically) regular component but, sometimes, also some other superabundant components with a dimension bigger than the expected one. On the other hand, there are also some results which give upper-bounds on m and δ ensuring that all the components of such a Severi variety are regular (see [7] and [9]). Thus, to precisely approach the problem, we make the following:

Definition 2.1. *Let S be a smooth, projective surface of general type and let D be a smooth, irreducible curve on S. Let $\delta \geq 0$ be such that $V_{|D|,\,\delta} \neq \emptyset$. If $V \subseteq V_{|D|,\,\delta}$ is an irreducible component, then the* number of moduli *of the family of curves parametrized by V is*

$$\nu_{D,\delta}|_V := dim(\pi_{|D|,\delta}(V)).$$

Since the behaviour of superabundant components is difficult to predict, we focus on generically regular components of $V_{|D|,\,\delta}$. For this reason, we have to introduce the following condition:

$$\delta \leq dim(|D|). \tag{9}$$

Indeed on such a surface, in general, we have $dim(|D|) < p_a(D)$ (e.g. if D is a very ample divisor, it directly follows from the fact that the characteristic linear system on D is special); therefore $V_{|D|,\,\delta}$ cannot have the expected dimension if $\delta >> 0$, i.e. if δ is near $p_a(D)$.

Definition 2.2. *The integer δ will be called* admissible *if δ is as in (9) and such that $g = p_a(D) - \delta \geq 2$.*

From Theorem 0.1 and Remark 0.1 one can eurhystically give the following:

Definition 2.3. *Let $V \subseteq V_{|D|,\,\delta}$ be an irreducible and generically regular component, with δ admissible. Then, the* expected number of moduli *of V is*

$$expmod(V) := dim(V).$$

Thus, what is expected is that V parametrizes a family having special moduli and, moreover, that its number of moduli is the biggest possible; in other words, a regular point $[X] \in V \subseteq V_{|D|,\,\delta}$ is expected to be birationally isomorphic to finitely many curves in V.

By using vector bundle theory on regular surfaces S with effective canonical divisor, one can easily determine some examples of regular components of Severi varieties of the form $V_{|K_S|,\delta}$ having the expected number of moduli (see [9] for details). On the other hand, there are also some examples which show that such expected number of moduli is not always achieved. Indeed, one can consider particular smooth, projective and regular surfaces of general type which belong to a class of surfaces that Catanese has recently studied (see [5]), called *Beauville's surfaces* or *fake quadrics* (see [31], page 195). Such a surface is of the form $S := (C \times C)/G$, where C is a smooth curve of genus $g \geq 2$, G is a finite group acting on each factor C and freely acting on the product $C \times C$ so that the quotient is a smooth surface and the projection $p : C \times C \to S$ is a topological covering. Moreover, if $|G| = (g-1)^2$ and if the action of G on C is such that $C/G \cong \mathbb{P}^1$, then one determines in S an isotrivial rational pencil of smooth curves $\overline{C}$ of genus g, parametrized by an open dense subset of $\mathbb{P}^1$. From the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(\overline{C}) \to \mathcal{O}_{\overline{C}}(\overline{C}) \to 0,$$

the regularity of S and the fact that $deg(\mathcal{O}_{\overline{C}}(\overline{C})) = 0$, we get that $dim(|\mathcal{O}_S(\overline{C})|) = 1$, so the complete linear system coincides with the isotrivial family. Therefore, the morphism

$$\pi_{|\overline{C}|,0} : V_{|\overline{C}|,0} \to \mathcal{M}_{p_a(\overline{C})}$$

is constant.

Remark. The previous example shows that we cannot expect to have, "tout court", the expected number of moduli, even in the case of families of smooth curves on smooth, projective, minimal and regular surfaces of general type.

From what observed, it is natural to give the following:

Definition 2.4. *With the same conditions as in Definition 2.3, the* moduli problem *consists in determining for which kind of divisor classes $D \in Div(S)$, the number of moduli of generically regular components $V \subseteq V_{|D|,\delta}$ coincides with the expected one, i.e. when*

$$\nu_{D,\delta}|_V = expmod(V)$$

holds.

Our approach to the moduli problem is analogous to that of Sernesi in [29], where he applied infinitesimal deformation theory to families of plane nodal curves. This uses the exact sequence (3).

When, in particular, $V_{|D|,0}$ is considered, if we denote always by X the general (smooth) element of $|D|$, then $N = \emptyset$ and the Zariski tangent space to $V_{|D|,0}$ at $[X]$ coincides with $H^0(\mathcal{O}_S(D))/ < X >$, reflecting the fact that $V_{|D|,0}$ is an open dense subscheme of $|D|$. Moreover, the exact sequence (3) reduces to the standard normal sequence of X in S. Therefore, if X is a smooth element in $|D|$, we get

$$0 \to H^0(\mathcal{T}_S|_X) \to H^0(\mathcal{N}_{X/S}) \xrightarrow{\partial} H^1(\mathcal{T}_X) \to \cdots , \tag{10}$$

where $h^1(\mathcal{T}_X) = 3p_a(X) - 3 = dim(\mathcal{M}_{p_a(X)})$, by assumption on $p_a(X) = p_g(X)$.

On the other hand, if $[X] \in V_{|D|,\delta}$, $\delta \geq 1$, from (3) we get

$$0 \to H^0(\varphi^*(\mathcal{T}_S)) \to H^0(\mathcal{N}_\varphi) \xrightarrow{\partial} H^1(\mathcal{T}_C) \to \cdots , \tag{11}$$

where $h^1(\mathcal{T}_C) = 3g - 3 = 3(p_a(X) - \delta - 1) = dim(\mathcal{M}_g)$, with $g \geq 2$ by assumption.

Therefore, when $[X] \in V_{|D|,\delta}$, $\delta \geq 0$, is a regular point, the compositions

$$T_{[X]}(V_{|D|,0}) \hookrightarrow H^0(\mathcal{N}_{X/S}) \xrightarrow{\partial} H^1(\mathcal{T}_X)$$

and

$$T_{[X]}(V_{|D|,\delta}) \hookrightarrow H^0(\mathcal{N}_\varphi) \xrightarrow{\partial} H^1(\mathcal{T}_C), \quad \delta \geq 1,$$

can be identified with the differentials of the morphisms $\pi_{|D|,\delta}$, $\delta \geq 0$, at the points $[X]$ and $[C \to X \subset S]$, respectively.

Remark 2.1. If $[X] \in V_{|D|,\delta}$, $\delta \geq 0$, is a regular point and if we further assume that $[X]$ is a general point of an irreducible component V of $V_{|D|,\delta}$, to give positive answers to the moduli problem of Definition 2.4 we need to show that the differential $(\pi_{|D|,\delta})_{*,[X]}$ is injective. From (10) and (11) this reduces to finding, for which divisor classes D, $h^0(D, \mathcal{T}_S|_D) = 0$ and $h^0(C, \varphi^*(\mathcal{T}_S)) = 0$ hold, respectively.

3. The main result

From what observed in Remark 2.1, we start by proving the following general result.

Theorem 3.1. *Let S be a smooth, projective surface of general type. Let $X \sim D$ be an irreducible, δ-nodal curve, $\delta \geq 0$, whose set of nodes is denoted by N. Then,*

$$h^1(S,\ \mathcal{I}_{N/S} \otimes \Omega^1_S(D+K_S)) = 0 \Rightarrow h^0(C,\ \varphi^*(\mathcal{T}_S)) = 0. \tag{12}$$

In particular, when $\delta = 0$,

$$h^1(S,\ \Omega^1_S(D+K_S)) = 0 \Rightarrow h^0(D,\ \mathcal{T}_S|_D) = 0 \tag{13}$$

Proof. If $N \neq \emptyset$, denote by $\mu : \tilde{S} \to S$ the blow-up of S along N, so that one can consider the following diagram of morphisms:

$$\begin{array}{ccc} C & \subset & \tilde{S} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mu} \\ X & \subset & S \end{array} .$$

Thus,

$$H^0(\varphi^*(\mathcal{T}_S)) = H^0(\mu^*(\mathcal{T}_S)|_C).$$

If we tensor the exact sequence defining C in $\tilde{S}$ with $\mu^*(\mathcal{T}_S)$, we get

$$0 \to \mu^*(\mathcal{T}_S)(-C) \to \mu^*(\mathcal{T}_S) \to \mu^*(\mathcal{T}_S)|_C \to 0. \tag{14}$$

Observe that

$$H^0(\mu^*(\mathcal{T}_S)) \cong H^0(\mathcal{T}_S) = (0),$$

since $H^0(\mathcal{T}_S)$ is isomorphic to the Lie algebra of the Lie group $Aut(S)$, which is finite by assumption on S (see [21]); thus, the cohomology sequence associated to (14) reduces to

$$0 \to H^0(\mu^*(\mathcal{T}_S)|_C) \to H^1(\mu^*(\mathcal{T}_S)(-C)) \to \cdots .$$

A sufficient condition for $h^0(\varphi^*(\mathcal{T}_S)) = 0$ is therefore $h^1(\mu^*(\mathcal{T}_S)(-C)) = 0$. By Serre duality on $\tilde{S}$, we have

$$h^1(\mu^*(\mathcal{T}_S)(-C)) = h^1((\mu^*(\mathcal{T}_S))^\vee \otimes \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + C)). \tag{15}$$

Since $\mathcal{T}_S$ is locally free, then $\mu^*(\mathcal{T}_S)^\vee = \mu^*(\mathcal{T}_S^\vee) = \mu^*(\Omega^1_S)$, so (15) becomes

$$h^1(\mu^*(\mathcal{T}_S)(-C)) = h^1(\mu^*(\Omega^1_S)(K_{\tilde{S}} + C)). \tag{16}$$

Denote by B the μ-exceptional divisor in $\tilde{S}$ such that $B = \Sigma_{i=1}^{\delta} E_i$. From standard computations with blow-ups, we get $K_{\tilde{S}} + C = \mu^*(K_S + X) - B$. Therefore, the right-hand side of (16) becomes $h^1(\mu^*(\Omega^1_S(K_S + X)) \otimes \mathcal{O}_{\tilde{S}}(-B))$. Since we have

$$H^1(\mu^*(\Omega^1_S(K_S + X)) \otimes \mathcal{O}_{\tilde{S}}(-B)) \cong H^1(\mathcal{I}_{N/S}(X + K_S) \otimes \Omega^1_S),$$

from the fact that $X \sim D$ on S, we get (12).
For (13), i.e. $\delta = 0$, one can directly use the exact sequence

$$0 \to \mathcal{T}_S(-D) \to \mathcal{T}_S \to \mathcal{T}_S|_D \to 0.$$

□

As an application of Remark 2.1 and Theorem 3.1, the moduli problem of Definition 2.4 reduces to finding for which divisors D on $S \subset \mathbb{P}^r$ the conditions

$$H^1(S,\ \Omega^1_S(K_S + D)) = (0) \tag{17}$$

and

$$H^1(S,\ \mathcal{I}_{N/S} \otimes \Omega^1_S(K_S + D)) = (0) \tag{18}$$

hold. The main results of this section (Theorems 3.2 and 3.3) determine sufficient conditions on D implying (17) and (18).

Remark To prove the basic Lemma 3.1 and Theorem 3.2, we shall use some projective-bundle arguments by following the approach of [17], Sect. II.7. Thus, in the following two results, if $\mathcal{E}$ is a vector bundle on a smooth, projective variety Y, $\mathbb{P}_Y(\mathcal{E})$ denotes the *projective space bundle* on Y, defined as $Proj(Sym(\mathcal{E}))$. We have a surjection $\pi^*(\mathcal{E}) \to \mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$, where $\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$ is the *tautological line bundle* on $\mathbb{P}_Y(\mathcal{E})$ and where $\pi : \mathbb{P}_Y(\mathcal{E}) \to Y$ is the natural projection morphism.

Lemma 3.1. *Let $S \subset \mathbb{P}^r$ be a smooth surface and let $\mathcal{E}$ be a rank 2 vector bundle on S. Assume that $\mathcal{E}$ is big and nef on S (i.e. the tautological line bundle $\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ is big and nef on $\mathbb{P}_S(\mathcal{E})$). Then*

$$H^i(S, \omega_S \otimes \mathcal{E} \otimes det(\mathcal{E}) \otimes \mathcal{O}_S(L)) = (0),$$

for $i > 0$ and for each nef divisor L.

Proof. By definition, $\mathbb{P}_S(\mathcal{E})$ is a smooth projective variety. From the assumptions on $\mathcal{E}$ and L and from the Kawamata-Viehweg vanishing theorem (see, for example, [22], page 146), it follows that

$$H^i(\mathbb{P}_S(\mathcal{E}), \omega_{\mathbb{P}_S(\mathcal{E})} \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(m) \otimes \pi^*(\mathcal{O}_S(L))) = (0), \ \text{ for } i, \ m > 0. \tag{19}$$

Consider the natural projection morphism $\pi : \mathbb{P}_S(\mathcal{E}) \to S$ and recall that

$$\pi_*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(m)) \cong Sym^m(\mathcal{E}), \ m \geq 1, \text{ and } \pi_*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}) \cong \mathcal{O}_S,$$

(see [17], Prop. II.7.11). From the relative Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})} \to \pi^*(\mathcal{E}^\vee) \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1) \to \mathcal{T}_{\mathbb{P}_S(\mathcal{E})/S} \to 0$$

and from the exact sequence

$$0 \to \mathcal{T}_{\mathbb{P}_S(\mathcal{E})/S} \to \mathcal{T}_{\mathbb{P}_S(\mathcal{E})} \to \pi^*(\mathcal{T}_S) \to 0,$$

we get that

$$\omega_{\mathbb{P}_S(\mathcal{E})} \cong \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(-2) \otimes \pi^*(\omega_S \otimes det(\mathcal{E})).$$

Therefore, if we consider $m = 3$ in (19), we get

$$H^i(\mathbb{P}_S(\mathcal{E}), \pi^*(\omega_S \otimes det(\mathcal{E}) \otimes \mathcal{O}_S(L)) \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)) = (0), \ \text{for } i > 0. \tag{20}$$

By projection formula,

$$R^i\pi_*(\pi^*(\omega_S \otimes det(\mathcal{E}) \otimes \mathcal{O}_S(L)) \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)) \cong \omega_S \otimes det(\mathcal{E}) \otimes \mathcal{O}_S(L) \otimes R^i\pi_*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)), \tag{21}$$

for each $i > 0$. Since the fibres of π are isomorphic to $\mathbb{P}^1$ and since $\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ is relatively ample, all the higher direct image sheaves in (21) are zero; thus, by Leray spectral sequence and by (20), we get the statement. □

Now, we can prove our main result.

Theorem 3.2. *Let $S \subset \mathbb{P}^r$ be a smooth surface of general type with hyperplane divisor H. Suppose that the linear system $|D|$ on S has general element which is a smooth, irreducible curve. Let $X \sim D$ be an irreducible, δ-nodal curve of geometric genus $g = p_a(D) - \delta$, where $\delta \geq 0$ admissible (as in Definition 2.2). Assume that:*

(i) *$\Omega^1_S(K_S)$ is globally generated;*
(ii) *$D \sim K_S + 6H + L$, where L is a nef divisor;*
(iii) *the Severi variety $V_{|D|,\delta}$ is regular at $[X]$ (in the sense of Definition 1.4).*

Then, the morphism

$$\pi_{|D|,\delta} : V_{|D|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. In particular, $\pi_{|D|,\delta}$ has finite fibres on each generically regular component of $V_{|D|,\delta}$, so each such component parametrizes a family having the expected number of moduli.

Proof. First of all, we want to show that hypothesis (ii) implies (17). To prove this, we will use Lemma 3.1. Therefore, the first step of our analysis is to apply such vanishing result to the vector bundle

$$\mathcal{E} = \Omega^1_S(aH),$$

where a is a positive integer. The problem reduces to finding which "twists" of Ω^1_S are big and nef on $S \subset \mathbb{P}^r$. In the sequel we shall write for short $\Omega^1_S(a)$ instead of $\Omega^1_S(aH)$. From the exact sequence

$$0 \to Con_{S/\mathbb{P}^r}(a) \to \Omega^1_{\mathbb{P}^r}(a)|_S \to \Omega^1_S(a) \to 0,$$

it is useful compute for which positive integers a the vector bundle $\Omega^1_{\mathbb{P}^r}(a)$ is ample or globally generated (see [16]). From the Euler sequence of $\mathbb{P}^r$ one deduces the exact sequence

$$0 \to \Omega^2_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}^{\oplus \frac{r(r+1)}{2}}(-2) \to \Omega^1_{\mathbb{P}^r} \to 0$$

(see [24], page 6, and [32], page 73); therefore, one trivially has

$$0 \to \Omega^2_{\mathbb{P}^r}(2) \to \mathcal{O}_{\mathbb{P}^r}^{\oplus \frac{r(r+1)}{2}} \to \Omega^1_{\mathbb{P}^r}(2) \to 0,$$

i.e. $\Omega^1_{\mathbb{P}^r}(2)$ and, so, $\Omega^1_S(2)$ are globally generated whereas $\Omega^1_{\mathbb{P}^r}(a)$ and $\Omega^1_S(a)$ are ample, for $a \geq 3$.

Recall now that $\mathbb{P}_{\mathbb{P}^r}(\Omega^1_{\mathbb{P}^r}(1))$ is the universal line over the Grassmannian $\mathbb{G}(1, r)$ of lines in $\mathbb{P}^r$ (see, for example, [19], app. B and C, or [20], page 369). By standard properties of projective bundles, $\mathbb{P}_{\mathbb{P}^r}(\Omega^1_{\mathbb{P}^r}(1)) \cong \mathbb{P}_{\mathbb{P}^r}(\Omega^1_{\mathbb{P}^r}(2))$, thus we have

$$\mathbb{F} := \mathbb{P}_{\mathbb{P}^r}(\Omega^1_{\mathbb{P}^r}(2)) \subset \mathbb{G}(1, r) \times \mathbb{P}^r$$

with the natural projection p_i on the i-th factor, $1 \leq i \leq 2$. If γ denotes the Plücker embedding of $\mathbb{G}(1, r)$ in $\mathbb{P}^{\oplus \frac{r(r+1)}{2} - 1}$, one determines the map

$$f := \gamma \circ p_1 : \mathbb{F} \to \mathbb{P}^{\oplus \frac{r(r+1)}{2} - 1}.$$

On the other hand, we can consider the complete tautological linear system $|\mathcal{O}_{\mathbb{F}}(1)|$, which is free since $\Omega^1_{\mathbb{P}^r}(2)$ is globally generated. From the Leray spectral sequence, the Euler sequence and the Bott formula (see [24], page 8), we get that

$$H^0(\mathbb{F}, \mathcal{O}_{\mathbb{F}}(1)) \cong H^0(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r}(2)) \cong \bigwedge^2 V,$$

where here $\mathbb{P}^r = \mathbb{P}(V) = Proj(Sym(V))$. Therefore, the complete linear system $|\mathcal{O}_{\mathbb{F}}(1)|$ defines a morphism

$$\Phi : \mathbb{F} \to \mathbb{P}(\bigwedge^2 V^*) \cong \mathbb{P}^{\oplus \frac{r(r+1)}{2} - 1}.$$

One easily sees that Φ and f coincide, so the global sections of $\mathcal{O}_{\mathbb{F}}(1)$ contract the p_1-fibres of $\mathbb{G}(1, r)$ in $\mathbb{F}$, which are lines in $\mathbb{P}^r$.

From the fact that $\mathbb{P}_S(\Omega^1_S(2)) \subset \mathbb{P}_S(\Omega^1_{\mathbb{P}^r}|_S(2))$, the restriction of Φ to $\mathbb{P}_S(\Omega^1_S(2))$ is generically finite since S, being of general type, is not filled by lines. Thus the rank 2 vector bundle $\Omega^1_S(2)$ is globally generated and big and nef. By Lemma 3.1, $H^1(S,\ \omega_S \otimes \Omega^1_S(2) \otimes det(\Omega^1_S(2)) \otimes \mathcal{O}_S(L)) = (0)$, for each nef divisor L. Since $det(\Omega^1_S(2)) = \mathcal{O}_S(K_S + 4H)$, we have that $H^1(S, \Omega^1_S(2K_S + 6H + L)) = (0)$, for each nef divisor L.

Therefore, if $D \sim K_S + 6H + L$, with L nef, then

$$(*) \quad H^1(S,\ \Omega^1_S(K_S + D)) = (0).^*$$

The vanishing result $(*)$ is a fundamental tool for the following second part of the proof. On S we can consider the exact sequence

$$0 \to \mathcal{I}_{N/S}(D) \to \mathcal{O}_S(D) \to \mathcal{O}_N(D) \to 0 \tag{22}$$

---

*Observe that if one directly applies Griffiths vanishing results, i.e. Theorem (5.52), Theorem (5.64) and Corollary (5.65) in [32], to the vector bundle $\Omega^1_S(a)$, $a \geq 2$, one determines stronger conditions on D. Precisely, L must be ample instead of nef. Therefore, the approach above determines more general conditions on D.

which determines the restriction map ρ_D:

$$0 \to H^0(\mathcal{I}_{N/S}(D)) \to H^0(\mathcal{O}_S(D)) \stackrel{\rho_D}{\to} H^0(\mathcal{O}_N(D)) \to H^1(\mathcal{I}_{N/S}(D)) \to \cdots .$$

By hypothesis (iii), ρ_D is surjective. Next, by tensoring the exact sequence (22) with $\Omega^1_S(K_S)$, we get

$$0 \to H^0(\mathcal{I}_{N/S}(D) \otimes \Omega^1_S(K_S)) \to H^0(\Omega^1_S(K_S+D)) \stackrel{\rho_{\Omega^1_S(K_S+D)}}{\longrightarrow}$$

$$\stackrel{\rho_{\Omega^1_S(K_S+D)}}{\longrightarrow} H^0(\mathcal{O}_N(\Omega^1_S(K_S+D))) \cong \mathbb{C}^{2\delta} \to H^1(\mathcal{I}_{N/S}(D) \otimes \Omega^1_S(K_S)) \to$$

$$\to H^1(\Omega^1_S(K_S+D)) \to 0.$$

Thus, the map $\rho_{\Omega^1_S(K_S+D)}$ is surjective if and only if $H^1(\mathcal{I}_{N/S}(D) \otimes \Omega^1_S(K_S)) \cong H^1(\Omega^1_S(K_S+D))$. From the first part of this proof, hypothesis (ii) implies that $H^1(\Omega^1_S(K_S+D)) = (0)$, so we have

$$h^1(\mathcal{I}_{N/S}(D) \otimes \Omega^1_S(K_S)) = 0 \Leftrightarrow \rho_{\Omega^1_S(K_S+D)} \text{ surjective.}$$

By (12) of Theorem 3.1, the surjectivity of $\rho_{\Omega^1_S(K_S+D)}$ implies therefore that $h^0(\varphi^*(\mathcal{T}_S)) = 0$ and so the statement.

The last step is to determine if, with the given hypotheses, the map $\rho_{\Omega^1_S(K_S+D)}$ is surjective. Consider the map

$$H^0(\Omega^1_S(K_S+D)) \stackrel{\rho_{\Omega^1_S(K_S+D)}}{\longrightarrow} H^0(\mathcal{O}_N(\Omega^1_S(K_S+D)) \cong \mathbb{C}^{2\delta} \cong \bigoplus_{i=1}^{\delta} \mathbb{C}^2_{(i)}. \tag{23}$$

By hypothesis (i), for each $p \in S$, the sheaf morphism

$$H^0(\Omega^1_S(K_S)) \otimes \mathcal{O}_{S,p} \to \Omega^1_S(K_S)|_p \cong \mathcal{O}^{\oplus 2}_{S,p}$$

is surjective; thus, for each $p \in S$ there exist two global sections $s^p_1,\ s^p_2 \in H^0(\Omega^1_S(K_S))$ which generate the stalk $\Omega^1_S(K_S)|_p$ as an $\mathcal{O}_S$-module, i.e.

$$s^p_1(p) = (1,0) \text{ and } s^p_2(p) = (0,1) \in \mathcal{O}^{\oplus 2}_{S,p} .$$

If $N = \{p_1,\ p_2,\ \ldots,\ p_\delta\}$ is the set of nodes of X, then $H^0(\mathcal{O}_N(D)) \cong \mathbb{C}^\delta \cong \mathbb{C}_{(1)} \oplus \mathbb{C}_{(2)} \oplus \cdots \oplus \mathbb{C}_{(\delta)}$. The surjectivity of ρ_D implies there exist global sections $\sigma_i \in H^0(\mathcal{O}_S(D))$ such that

$$\sigma_i(p_j) = (0,\ 0,\ \ldots,\ 0),\ if\ 1 \leq i \neq j \leq \delta,$$

$$\sigma_i(p_i) = (0,\ \ldots,\ 0,\ 1,\ 0,\ \ldots,\ 0),\ 1 \in \mathbb{C}_{(i)},\ 1 \leq i \leq \delta.$$

Therefore, $s^{p_i}_1 \otimes \sigma_i,\ s^{p_i}_2 \otimes \sigma_i \in H^0(\Omega^1_S(D+K_S))$ and

$$s^{p_i}_1 \otimes \sigma_i(p_j) = s^{p_i}_2 \otimes \sigma_i(p_j) = (0,\ \ldots,\ 0) \in \mathbb{C}^2_{(1)} \oplus \cdots \oplus \mathbb{C}^2_{(\delta)} \cong \mathbb{C}^{2\delta},\ 1 \leq i \neq j \leq \delta,$$

$$s^{p_i}_1 \otimes \sigma_i(p_i) = ((0,0),\ \ldots,\ (1,0),\ \ldots,\ (0,0)) = (0,\ \ldots,\ 1,\ 0,\ \ldots,\ 0) \in \mathbb{C}^{2\delta},$$

where $(1,0) \in \mathbb{C}^2_{(i)}$ and

$$s^{p_i}_2 \otimes \sigma_i(p_i) = ((0,0),\ \ldots,\ (0,1),\ \ldots,\ (0,0)) = (0,\ \ldots,\ 0,\ 1,\ \ldots,\ 0) \in \mathbb{C}^{2\delta},$$

where $(0,\ 1) \in \mathbb{C}^2_{(i)}$, for $1 \leq i \leq \delta$. This means that the map (23) is surjective. Moreover, since the condition for a point $[X] \in V_{|D|,\delta}$ to be regular is an open condition in the family, it follows that the component of $V_{|D|,\delta}$ containing $[X]$ has the expected number of moduli. □

From the first part of the proof of Theorem 3.2 we observe that in the case of familes of smooth curves one can eliminate hypotheses (i) and (iii). Indeed, we have the following result.

Theorem 3.3. *Let $S \subset \mathbb{P}^r$ be a smooth surface of general type and let D be an effective divisor on S. Denote by H the hyperplane section of S. Assume that*

$$D \sim K_S + 6H + L,$$

where L is a nef divisor on S. Then, $H^1(S,\ \Omega^1_S(K_S+D)) = (0)$.
If, moreover, $|D|$ contains smooth, irreducible elements, the family of smooth curves $V_{|D|,0}$ has the expected number of moduli.

Proof. For the first part of the statement, one can repeat the procedure at the beginning of the proof of Theorem 3.2. From (13) we get the second part of the statement. □

Let $S = S_d \subset \mathbb{P}^3$ be a smooth surface of degree d; in view of the fact that $K_S \sim (d-4)H$, as a corollary of Theorems 3.2 and 3.3 we get:

Corollary 3.1. *If $S_d \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 6$, the generically regular components of $V_{|mH|,\delta}$ have the expected number of moduli, when $m \geq d+2$ and $\delta \geq 1$ admissible. The same conclusion holds for the family of smooth curves $V_{|mH|,0}$, when $d \geq 5$ and $m \geq d+2$.*

Remark 3.1. More generally, if $S \subset \mathbb{P}^r$ is of general type with $K_S \sim \alpha H$, then we have positive answers to the moduli problem for all generically regular components of $V_{|mH|,\delta}$, with $m \geq \alpha + 6$, when $\alpha \geq 2$ and $\delta \geq 1$ admissible, and with $m \geq \alpha + 6$, when $\delta = 0$ and $\alpha \geq 1$.

Remark 3.2. The conditions $\Omega^1_S(2)$ globally generated and big and nef on S play a crucial role in the proof of Theorem 3.2. Thus with this approach this result is, in a certain sense, sharp. For example, if we focus on regular surfaces, Ω^1_S cannot be globally generated, since $\overline{H^{1,0}}(S) = H^{0,1}(S)$ and $H^{1,0}(S) \cong H^0(S, \Omega^1_S)$ whereas $H^{0,1}(S) \cong H^1(S, \mathcal{O}_S) = (0)$. If S is also a non-degenerate complete intersection in $\mathbb{P}^r$, then $\Omega^1_S(1)$ cannot be globally generated. Furthermore, we have some results of Schneider (see [27]) which state that, even in the most natural case of smooth surfaces $S_d \subset \mathbb{P}^3$ of degree $d \geq 5$, $\Omega^1_{S_d}$ and $\Omega^1_{S_d}(1)$ are not ample.

4. A FUNDAMENTAL PROPOSITION

The aim of this and the following two sections is to find other results giving positive answers to the moduli problem, posed in Definition 2.4, for some other classes of divisors on S which are not covered by Theorems 3.2 and 3.3.

From now on we shall focus on the case of regular surfaces; therefore S will always denote a smooth, regular surface of general type, unless otherwise specified. In such a case, we are able, in particular, to complete Remark 3.1 by also including divisors $D \sim mH$ with $1 \leq m \leq \alpha + 5$ and with some further conditions on δ.

The first step of our analysis is based on a key proposition concerning first-order deformations of the normalization morphism $\varphi : C \to X \subset S$. Then we conclude, in some cases, by using a detailed analysis of the Brill-Noether map of the line bundle $\mathcal{O}_C(\varphi^*(H))$, in some other cases, by using uniqueness results of certain linear systems on C.

The core of this section is to prove such a fundamental proposition. Before doing this, we need to remind some general facts.

Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate surface (not necessarily regular and of general type). As in (2), the normalization morphism φ is a map from C to S such that $Im(\varphi) = X \subset S$. If $i : S \hookrightarrow \mathbb{P}^r$ is the natural embedding, we have the following diagram of morphisms:

$$\begin{array}{ccc} C & & \\ \downarrow^{\varphi} & \searrow^{\psi} & \\ S & \overset{i}{\hookrightarrow} & \mathbb{P}^r \end{array}$$

where $\psi = i \circ \varphi : C \to \mathbb{P}^r$. By pulling back to C the normal sequence of S in $\mathbb{P}^r$, we get the exact sequence of vector bundles on C

$$0 \to \varphi^*(\mathcal{T}_S) \to \psi^*(\mathcal{T}_{\mathbb{P}^r}) \to \varphi^*(\mathcal{N}_{S/\mathbb{P}^r}) \to 0.$$

Thus,

$$0 \to H^0(\varphi^*(\mathcal{T}_S)) \to H^0(\psi^*(\mathcal{T}_{\mathbb{P}^r})) \to H^0(\varphi^*(\mathcal{N}_{S/\mathbb{P}^r})) \to \cdots \tag{24}$$

holds, where $H^0(\varphi^*(\mathcal{T}_S))$ parametrizes first-order deformations of the map $\varphi : C \to S$, with C and S both fixed, as well as $H^0(\psi^*(\mathcal{T}_{\mathbb{P}^r}))$ parametrizes first-order deformations of the map $\psi : C \to \mathbb{P}^r$, with C and $\mathbb{P}^r$ both fixed (see [15]).

We also recall the following useful definition (see [7]).

Definition 4.1. *Let X be any reduced, irreducible curve in $\mathbb{P}^r$. X is said to be* geometrically linearly normal *(*g.l.n. *for short) if the normalization map $\varphi : C \to X \subset \mathbb{P}^r$ cannot be factored into a non-degenerate morphism $C \to \mathbb{P}^N$, with $N > r$, followed by a projection.*

In other words, if H is the hyperplane section of X, $|\mathcal{O}_C(\varphi^*(H))|$ must be complete.

We are now able to give the following:

Proposition 4.1. *Let $S \subset \mathbb{P}^r$ be a smooth, regular, non-degenerate and linearly normal surface of general type. Let $[X] \in V_{|D|,\delta}$ be a regular point of the Severi variety $V_{|D|,\delta}$ on S.*
(i) Assume that X is non-degenerate in $\mathbb{P}^r$ and geometrically linearly normal. If $h^0(C, \psi^*(\mathcal{T}_{\mathbb{P}^r})) = (r+1)^2 - 1$, then all first-order deformations of the map $\psi : C \to \mathbb{P}^r$, with C fixed, are induced by first-order projectivities (i.e. by elements of $H^0(\mathcal{T}_{\mathbb{P}^r})$). Moreover, $h^0(\varphi^*(\mathcal{T}_S)) = 0$.
(ii) Assume that $D \sim H$ on S and that $X \subset H \cong \mathbb{P}^{r-1}$ is non-degenerate and g.l.n. as a curve in $\mathbb{P}^{r-1}$. Suppose also that S is such that $h^1(\mathcal{O}_S(H)) = 0$ and $|K_S| \neq \emptyset$. If $h^0(\psi^*(\mathcal{T}_{\mathbb{P}^r})) = r^2 + r - 1$, then all first-order deformations of the map $\psi : C \to \mathbb{P}^r$, with C fixed, are induced by first-order projectivities not fixing pointwise the hyperplane $H \subset \mathbb{P}^r$. Moreover, $h^0(\varphi^*(\mathcal{T}_S)) = 0$.

Proof. (i) The first part of the statement is a straightforward computation. We shall briefly recall the fundamental steps of its proof. If $\mu : \tilde{S} \to S$ is the blow-up of S along $N = Sing(X)$, by the hypotheses on S and by the pull-back to $\tilde{S}$ of the Euler sequence, we get

$$H^0(\mathcal{T}_{\mathbb{P}^r}) \cong H^0(\mathcal{T}_{\mathbb{P}^r}|_S) \cong H^0(\mu^*(\mathcal{T}_{\mathbb{P}^r})).$$

Since X is g.l.n. and non-degenerate in $\mathbb{P}^r$, by Serre duality and by the pull-back on C of the Euler sequence we have

$$(*) \quad 0 \to K \to H^0(\psi^*(\mathcal{T}_{\mathbb{P}^r})) \to (coker(\mu_{0,C}))^\vee \to 0,$$

where $K = (H^0(\mathcal{O}_C(\varphi^*(H)))^\vee \otimes H^0(\mathcal{O}_C(\varphi^*(H))))/H^0(\mathcal{O}_C) \cong H^0(\mathcal{T}_{\mathbb{P}^r})$ and where

$$\mu_{0,C} : H^0(\mathcal{O}_C(\varphi^*(H)) \otimes H^0(\omega_C(-\varphi^*(H))) \to H^0(\omega_C)$$

is the Brill-Noether map of $\mathcal{O}_C(\varphi^*(H))$. Since $h^0(\mathcal{T}_{\mathbb{P}^r}) = dim(PGL(r+1, \mathbb{C})) = (r+1)^2 - 1$, from $(*)$ it follows that $h^0(\psi^*(\mathcal{T}_{\mathbb{P}^r})) = (r+1)^2 - 1$ iff $dim(coker(\mu_{0,C})) = 0$. In this case, by standard Brill-Noether theory (see [2], Proposition 4.1, page 187), there is no first-order deformation of $\psi : C \to \mathbb{P}^r$, with C fixed, induced by first-order deformations of the linear system $|\mathcal{O}_C(\varphi^*(H))|$; so all such deformations are induced by elements of $H^0(\mathcal{T}_{\mathbb{P}^r})$.

To get the second part of statement (i) observe that, by the regularity of S and by (6), $H^0(\mathcal{N}_\varphi) \cong T_{[X]}(V_{|D|,\delta})$. Assume, by contradiction, that $h^0(\varphi^*(\mathcal{T}_S)) \neq 0$ and let $v \in H^0(\varphi^*(\mathcal{T}_S))$ be a non-zero vector. Such v corresponds to a tangential direction $v \in T_{[X]}(V_{|D|,\delta})$ since, by (11), we have $H^0(\varphi^*(\mathcal{T}_S)) \subseteq H^0(\mathcal{N}_\varphi)$.

By the regularity assumption of $[X] \in V_{|D|,\delta}$, all directions in $T_{[X]}(V_{|D|,\delta})$ are unobstructed. This means there exist a one-dimensional base scheme Δ, smooth at the central point $o \in \Delta$, and a family $\mathcal{X} \to \Delta$ such that

$$\mathcal{X} = \{X_t\}_{t \in \Delta} \subset S \times \Delta$$

where

$$[X_t] \in V_{|D|,\delta}, \ \forall\, t \in \Delta, \ [X_o] = [X], \ \text{and } T_{[o]}(\Delta) = < v > .$$

Since $< v > \subset H^0(\varphi^*(\mathcal{T}_S))$, the family $\mathcal{X} \to \Delta$ corresponds to a family of maps $\Phi : C \times \Delta \to S \times \Delta$, for which

$$\Phi = \{\varphi_t\}_{t \in \Delta}, \ \varphi_o = \varphi, \ \varphi_t = \Phi|_t : C \times \{t\} \to S \times \{t\}, \ \varphi_t(C) = X_t \subset S.$$

By composing Φ with the map $i \times id_\Delta$, where $i : S \hookrightarrow \mathbb{P}^r$, we get a family of maps $\Psi : C \times \Delta \to \mathbb{P}^r \times \Delta$ for which

$$\Psi = \{\psi_t\}_{t \in \Delta}, \ \psi_o = \psi, \ \psi_t = \Psi|_t : C \times \{t\} \to \mathbb{P}^r \times \{t\}, \ \psi_t(C) = X_t \subset S \subset \mathbb{P}^r.$$

From (24), we know that $H^0(\varphi^*(\mathcal{T}_S)) \subseteq H^0(\psi^*(\mathcal{T}_{\mathbb{P}^r}))$ and, from the above computations, we have $H^0(\psi^*(\mathcal{T}_{\mathbb{P}^r})) \cong H^0(\mathcal{T}_{\mathbb{P}^r})$. Therefore, the element $v \in H^0(\psi^*(\mathcal{T}_{\mathbb{P}^r}))$ is induced by first-order projectivities, so the family $\Psi \to \Delta$ is determined by a family $\Omega \to \Delta$, where

$$\Omega \subset PGL(r+1, \mathbb{C}),\ \Omega : X \times \Delta \to \mathbb{P}^r \times \Delta,\ \Omega = \{\omega_t\}_{t \in \Delta}$$

such that $\psi_t = \omega_t \circ \psi$ and $[\omega_t(\psi(C))] = [X_t] \in V_{|D|,\delta}$, for each $t \in \Delta$ whereas $[\omega_o(\psi(C))] = [X]$.

Since S is of general type, then $\Omega \subset PGL(r+1, \mathbb{C}) \setminus Aut(S)$. Therefore, if

$$X_t = \omega_t(X) \subset S,\ \forall\, t \in \Delta,$$

then

$$X \subset \omega_t^{-1}(S) = S_t,\ \forall\, t \in \Delta,$$

where $S_t \subset \mathbb{P}^r$ is a smooth surface projectively equivalent to S, for each $t \in \Delta$, and $S_o = S$. We therefore obtain a family of maps $\Lambda : S \times \Delta \to \mathbb{P}^r \times \Delta$ such that $\Lambda|_t = \omega_t^{-1}$, for each $t \in \Delta$. By composing such family of maps with $\mu \times id_\Delta$, $\mu : \tilde{S} \to S$, we thus get a family of maps

$$\Theta : \tilde{S} \times \Delta \to \mathbb{P}^r \times \Delta$$

where

$$\Theta|_t = \omega_t^{-1} \circ \mu : \tilde{S} \times \{t\} \to \mathbb{P}^r \times \{t\},\ \Theta|_t(\tilde{S}) = \omega_t^{-1}(\mu(\tilde{S})) = \omega_t^{-1}(S) = S_t \subset \mathbb{P}^r.$$

Since $\Theta|_o = id_{\mathbb{P}^r} \circ \mu$ and $T_o(\Delta) =< v >$, the element $v \in H^0(\varphi^*(\mathcal{T}_S)) \subset H^0(\psi^*(\mathcal{T}_{\mathbb{P}^r}))$ is also an element of $H^0(\mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C))$.

This leads to a contradiction; indeed, by tensoring the exact sequence defining C in $\tilde{S}$ with $\mu^*(\mathcal{T}_{\mathbb{P}^r})$, we get

$$0 \to \mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C) \to \mu^*(\mathcal{T}_{\mathbb{P}^r}) \to \mu^*(\mathcal{T}_{\mathbb{P}^r})|_C \cong \psi^*(\mathcal{T}_{\mathbb{P}^r}) \to 0.$$

From the above computations, we know that $H^0(\mu^*(\mathcal{T}_{\mathbb{P}^r})) \cong H^0(\psi^*(\mathcal{T}_{\mathbb{P}^r}))$, which implies $h^0(\mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C)) = 0$.

(ii) In this case $X \sim H$ on S and $X \subset H \cong \mathbb{P}^{r-1}$ is non-degenerate in $\mathbb{P}^{r-1}$, then

$$\psi^*(\mathcal{T}_{\mathbb{P}^r}) \cong \psi^*(\mathcal{T}_{\mathbb{P}^{r-1}}) \oplus \mathcal{O}_C(\psi^*(H))$$

(with abuse of notation, we denote always by ψ the map $\psi : C \to X \subset H \cong \mathbb{P}^{r-1}$). From the hypotheses on X, we get

$$h^0(\psi^*(\mathcal{T}_{\mathbb{P}^r})) = h^0(\psi^*(\mathcal{T}_{\mathbb{P}^{r-1}})) + r.$$

By using the same computations of (i), we get

$$0 \to H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_C(\psi^*(H))^\vee \otimes H^0(\mathcal{O}_C(\psi^*(H)) \to H^0(\psi^*(\mathcal{T}_{\mathbb{P}^{r-1}})) \to (coker(\mu_{0,C}))^\vee \to 0,$$

where

$$(coker(\mu_{0,C}))^\vee \cong T_{[|\mathcal{O}_C(\psi^*(H))|]}(G^{r-1}_{deg(X)}(C)).$$

Thus, as in (i), $h^0(\psi^*(\mathcal{T}_{\mathbb{P}^{r-1}})) = r^2 - 1$ if and only if $dim(coker(\mu_{0,C})) = 0$.

Note that

$$\frac{H^0(\tilde{S}, \mu^*(\mathcal{T}_{\mathbb{P}^r}))}{H^0(\tilde{S}, \mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C)))} \overset{\beta}{\hookrightarrow} H^0(C, \psi^*(\mathcal{T}_{\mathbb{P}^r})). \tag{25}$$

From the pull-back of the Euler sequence and from the hypotheses on S, we get

$$0 \to \mathcal{O}_{\tilde{S}}(-C) \to H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H)))^\vee \otimes \mathcal{O}_{\tilde{S}}(\mu^*(H) - C) \to \mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C) \to 0. \tag{26}$$

Observe that $h^0(\mathcal{O}_{\tilde{S}}(-C)) = h^1(\mathcal{O}_{\tilde{S}}(-C)) = 0$: indeed, the first vanishing trivially holds whereas, by Leray's isomorphism and by Serre duality, we have $h^1(\mathcal{O}_{\tilde{S}}(-C)) = h^1(\mathcal{I}_{N/S}(K_S + H))$; from the regularity of $[X] \in V_{|H|,\delta}$, Remark 1.4 and the hypothesis $h^1(\mathcal{O}_S(H)) = 0$, we get $h^1(\mathcal{I}_{N/S}(H)) = 0$. Since K_S is effective by assumption, N also imposes independent conditions to $|K_S + H|$. By standard Mumford's vanishing theorem, we have $h^1(\mathcal{O}_S(K_S + H)) = 0$, so $h^1(\mathcal{I}_{N/S}(K_S + H)) = 0$.

We therefore obtain

$$\begin{aligned} H^0(\mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C)) &\cong H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H)))^\vee \otimes H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H) - C)) \\ &= H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H)))^\vee \otimes H^0(\mathcal{O}_{\tilde{S}}(2B)), \end{aligned}$$

where $B = \Sigma_{i=1}^{\delta} E_i$ is the μ-exceptional divisor. Since $2B$ is a fixed divisor, $h^0(\mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C)) = h^0(\mathcal{O}_{\tilde{S}}(\mu^*(H)) = r + 1$. Moreover, since $H^0(\mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C)) \subset H^0(\mu^*(\mathcal{T}_{\mathbb{P}^r})) \cong H^0(\mathcal{T}_{\mathbb{P}^r})$, the elements of such a vector space correspond to first-order projectivities fixing pointwise the hyperplane $H \subset \mathbb{P}^r$. Turning back to (25), $h^0(\psi^*(\mathcal{T}_{\mathbb{P}^r})) = r^2 + r - 1$ if and only if β is an isomorphism. In such a case, all first-order deformations of $\psi : C \to \mathbb{P}^r$, with C fixed, are induced up to first-order by projectivities not fixing pointwise the curve $X \subset H$.

For the second statement in (ii), one can follow the same procedure in (i). By supposing there exists a non-zero vector $v \in H^0(\varphi^*(\mathcal{T}_S))$, one determines a family $\Omega \to \Delta$, where $\Omega \subset PGL(r+1, \mathbb{C}) \setminus Aut(S)$, such that

$$\Omega = \{\omega_t\}_{t \in \Delta},\ \omega_t(X) = X_t \subset S, \text{ and } T_o(\Delta) =< v > .$$

As before, one obtains $v \in H^0(\mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C))$, so the family Ω is contained in the sugroup $\Gamma < PGL(r + 1, \mathbb{C})$, whose elements pointwise fix the curve X. Therefore, we have $\omega_t(X) = X$, for each $t \in \Delta$, contradicting the existence of the non-trivial, one-dimensional family $\mathcal{X} = \{X_t\}_{t \in \Delta}$. ☐

From Remark 2.1, in the sequel we will be concerned in finding conditions which imply the hypotheses of Proposition 4.1. These will give further affirmative answers to the moduli problem posed in Definition 2.4 for Severi varieties on smooth, regular and non-degenerate surfaces $S \subset \mathbb{P}^r$ of general type.

5. Number of moduli for families of non-degenerate, nodal curves on linearly normal surfaces of general type.

As remarked at the beginning of Section 4, we want to find some other conditions establishing positive answers to the moduli problem for those Severi varieties which do not satisfy the hypotheses of Theorems 3.2 and 3.3.

Here we shall focus on the case of $S \subset \mathbb{P}^r$ a smooth surface of general type which is regular, non-degenerate, linearly normal and such that $h^1(\mathcal{O}_S(H)) = 0$, H the hyperplane section of S. Observe that, in this case, one can obviously apply the results in Section 3, since they are more generally valid.

The results we obtain here apply, for example, to some cases which are not covered by Corollary 3.1 and Remark 3.1 even though their statement gives some restrictions to the admissible number of nodes δ with respect to (9).

In this section, we consider $[X] \in V_{|D|,\delta}$ on S such that X is non-degenerate in $\mathbb{P}^r$. From Proposition 4.1 (i), we want to find conditions on D in order that $h^0(\psi^*(\mathcal{T}_{\mathbb{P}^r})) = (r+1)^2 - 1 = dim(PGL(r+1, \mathbb{C}))$. To this aim, put

$$\mathcal{O}_C(\psi^*(H)) = \mathcal{O}_C(\tilde{H}), \tag{27}$$

then X is geometrically linearly normal (see Definition 4.1) if and only if $|\mathcal{O}_C(\tilde{H})|$ is complete of dimension r. In such a case, we consider the *Brill-Noether map* of the line bundle $\mathcal{O}_C(\tilde{H})$, i.e.

$$\mu_{0,C} : H^0(\mathcal{O}_C(\tilde{H})) \otimes H^0(\omega_C(-\tilde{H})) \to H^0(\omega_C). \tag{28}$$

Remark 5.1. Similarly to Definition 1.1.2 in [25], if X is g.l.n. and if the map $\mu_{0,C}$ is surjective, then $|\mathcal{O}_C(\tilde{H})|$ is called an *isolated linear system* on C. The surjectivity of $\mu_{0,C}$ implies the injectivity of the dual map $\mu_{0,C}^\vee$ so the Euler exact sequence on C,

$$0 \to \mathcal{O}_C \to H^0(\mathcal{O}_C(\tilde{H}))^\vee \otimes \mathcal{O}_C(\tilde{H}) \to \psi^*(\mathcal{T}_{\mathbb{P}^r}) \to 0, \tag{29}$$

gives

$$h^0(\psi^*(\mathcal{T}_{\mathbb{P}^r})) = (r + 1)^2 - 1 = dim(PGL(r + 1, \mathbb{C})). \tag{30}$$

Therefore, from Remark 2.1 and from Proposition 4.1 (i), we deduce the following:

Proposition 5.1. *Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate and regular surface of general type and let $[X] \in V_{|D|,\delta}$ be a regular point corresponding to a non-degenerate and g.l.n. curve in $\mathbb{P}^r$ for which δ is admissible and the Brill-Noether map $\mu_{0,C}$ of $\mathcal{O}_C(\psi^*(H))$ is surjective. Then, the morphism $\pi_{|D|,\delta}$ has injective differential at $[X]$. In particular, if $[X]$ is the general point of an irreducible component V of $V_{|D|,\delta}$, then V has the expected number of moduli.*

Our next aim is to find conditions guaranteeing that X is g.l.n. with Brill-Noether map $\mu_{0,C}$ surjective. We start by considering the following crucial remark.

Remark 5.2. Suppose that $|D|$ is a complete linear system on S whose general element is a smooth, irreducible and non-degenerate curve (so that $|H - D| \neq \emptyset$). Assume that $[X] \in V_{|D|,\delta}$ corresponds to a g.l.n. curve on S. Denote by $\mu : \tilde{S} \to S$ the blow-up of S along $N = Sing(X)$, so that $\mu|_C = \varphi$, and consider $B = \sum_{i=1}^{\delta} E_i$ the μ-exceptional divisor.
(a) By the hypotheses on S and X, we have

$$H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H))) \cong H^0(\mathcal{O}_S(H)) \cong H^0(\mathcal{O}_C(\tilde{H})).$$

(b) From the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(K_{\tilde{S}}) \to \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + C) \to \omega_C \to 0,$$

we get that $H^0(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + C)) \to H^0(\omega_C)$ is surjective since, by Serre duality and by hypothesis on S, $h^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}})) = h^1(\mathcal{O}_{\tilde{S}}) = h^1(\mathcal{O}_S) = 0$. Therefore, by linear equivalence,

$$H^0(\mathcal{O}_{\tilde{S}}(\mu^*(K_S + D) - B)) \to H^0(\omega_C)$$

is surjective.
(c) As in (b), since $h^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} - \mu^*(H))) = h^1(\mathcal{O}_{\tilde{S}}(\mu^*(H))) = h^1(\mathcal{O}_S(H)) = 0$ by hypothesis on S, we get the surjective map

$$H^0(\mathcal{O}_{\tilde{S}}(\mu^*(K_S + D - H) - B)) \to H^0(\omega_C(-\tilde{H})).$$

Thus, we can consider the following diagram:

$$\begin{array}{ccc} H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H))) \otimes H^0(\mathcal{O}_{\tilde{S}}(\mu^*(K_S + D - H) - B)) & \stackrel{\mu_{0,\tilde{S}}}{\longrightarrow} & H^0(\mathcal{O}_{\tilde{S}}(\mu^*(K_S + D) - B)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}_C(\tilde{H})) \otimes H^0(\omega_C(-\tilde{H})) & \stackrel{\mu_{0,C}}{\longrightarrow} & H^0(\omega_C), \end{array}$$

where the vertical maps are surjective by (a), (b) and (c). On the other hand, we have

$$\begin{array}{ccc} H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H))) \otimes H^0(\mathcal{O}_{\tilde{S}}(\mu^*(K_S + D - H) - B)) & \stackrel{\mu_{0,\tilde{S}}}{\longrightarrow} & H^0(\mathcal{O}_{\tilde{S}}(\mu^*(K_S + D) - B)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{I}_{N/S}(K_S + D - H)) & \stackrel{\mu_{0,S}}{\longrightarrow} & H^0(\mathcal{I}_{N/S}(K_S + D)), \end{array}$$

where the vertical maps are isomorphisms. Thus, $\mu_{0,C}$ is surjective if $\mu_{0,S}$ is.

Recall that, if $\mathcal{I}_{N/S}(K_S + D - H)$ is a 0-*regular* coherent sheaf on S, the maps

$$H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{I}_{N/S}(K_S + D + (\alpha - 1)H)) \to H^0(\mathcal{I}_{N/S}(K_S + D + \alpha H))$$

are surjective, for all $\alpha \geq 0$ (for terminology and results on m-regularity see, for example, [23]). Therefore, the 0-regularity of $\mathcal{I}_{N/S}(K_S + D - H)$ is a sufficient condition for the surjectivity of $\mu_{0,S}$ (and so of $\mu_{0,C}$). By definition, the given sheaf is 0-regular iff

$$H^1(\mathcal{I}_{N/S}(K_S + D - 2H)) = H^2(\mathcal{I}_{N/S}(K_S + D - 3H)) = (0). \tag{31}$$

Our next result determines numerical conditions on the divisor class D and an upper-bound on the number of nodes δ implying (31).

Theorem 5.1. *Let $S \subset \mathbb{P}^r$ be a smooth surface and let $|D|$ be a a complete linear system on S whose general element is a smooth, irreducible divisor. Suppose that:*

i) $(D - 3H)H > 0$;

ii) $(D-4H)^2 > 0$ *and* $D(D-4H) > 0$*;*

iii) $\nu(D,H) < D(D-4H)-4$*, where* $\nu(D,H)$ *is the Hodge number of* D *and* H *(see Def. 1.2);*

iv) $\delta < \frac{D(D-4H)+\sqrt{D^2(D-4H)^2}}{8}$.

If $X \sim D$ *is a reduced, irreducible curve with only* δ *nodes as singular points and if* $N = Sing(X)$*, then*

$$h^1(\mathcal{I}_{N/S}(K_S+D-2H)) = h^2(\mathcal{I}_{N/S}(K_S+D-3H)) = 0;$$

in other words, $\mathcal{I}_{N/S}(K_S+D-H)$ *is* 0*-regular on* S*.*

Proof. We start by considering the vanishing $h^1(\mathcal{I}_{N/S}(K_S+D-2H)) = 0$. By contradiction, assume that N does not impose independent conditions to $|K_S+D-2H|$. Let $N_0 \subset N$ be a minimal 0-dimensional subscheme of N for which this property holds and let $\delta_0 = |N_0|$. This means that $h^1(S, \mathcal{I}_{N_0}(D+K_S-2H)) \neq 0$ and that N_0 satisfies the Cayley-Bacharach condition (see, for example [13]). Therefore, a non-zero element of $H^1(\mathcal{I}_{N_0}(D+K_S-2H))$ gives rise to a non-trivial rank 2 vector bundle $\mathcal{E} \in Ext^1(\mathcal{I}_{N_0}(D-2H), \mathcal{O}_S)$ fitting in the following exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_{N_0}(D-2H) \to 0, \tag{32}$$

with $c_1(\mathcal{E}) = D-2H$ and $c_2(\mathcal{E}) = \delta_0 \geq 0$. Hence

$$c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = (D-2H)^2 - 4\delta_0. \tag{33}$$

Since D is effective and irreducible with $D^2 > 4HD > 0$, from $ii)$ it follows that D is a big and nef divisor (see Def. 1.1). By applying the Index theorem to the divisor pair $(D, D-4H)$ and by $iv)$, we get

$$2D(D-4H) - 8\delta \geq D(D-4H) + \sqrt{D^2(D-4H)^2} - 8\delta > 0.$$

Therefore,

$$c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) \geq (D-2H)^2 - 4\delta = D(D-4H) - 4\delta + 4H^2 > 0,$$

which means that $\mathcal{E}$ is Bogomolov-unstable (see Definition 1.3 and Remark 1.2), hence $h^0(\mathcal{E}(-M)) \neq 0$. Twisting (32) by $\mathcal{O}_S(-M)$, we obtain

$$0 \to \mathcal{O}_S(-M) \to \mathcal{E}(-M) \to \mathcal{I}_{N_0}(D-2H-M) \to 0. \tag{34}$$

We claim that $h^0(\mathcal{O}_S(-M)) = 0$; otherwise, $-M$ would be an effective divisor, therefore $-MA > 0$, for each ample divisor A. From (1), it follows that $c_1(\mathcal{E}) = M+B$, so, by (1) and (32),

$$M - B = 2M - D + 2H \in N(S)^+. \tag{35}$$

Thus $MH > \frac{(D-2H)H}{2}$; next by $i)$ it follows that $H(D-2H) > 0$, hence $-MH < 0$.

The cohomology sequence associated to (34) ensures there exists a divisor $\Delta \sim D-2H-M$ s.t. $N_0 \subset \Delta$ and s.t. the irreducible nodal curve $X \sim D$, whose set of nodes is N, is not a component of Δ (otherwise, $-M-2H$ would be an effective divisor, which contradicts the non-effectiveness of $-M$).

Next, by Bezout's theorem, we get

$$X\Delta = X(D-2H-M) \geq 2\delta_0. \tag{36}$$

On the other hand, taking M maximal, we may further assume that the general section of $\mathcal{E}(-M)$ vanishes in codimension 2. Denote by Z this vanishing-locus, thus, $c_2(\mathcal{E}(-M)) = deg(Z) \geq 0$; moreover, $c_2(\mathcal{E}(-M)) = c_2(\mathcal{E}) + M^2 + c_1(\mathcal{E})(-M) = \delta_0 + M^2 - M(D-2H)$, which implies

$$\delta_0 \geq M(D-2H-M). \tag{37}$$

(Note that $M^2 \geq 0$ since $2M-(D-2H) \in N^+(S)$ and $(D-2H)$ is effective).

By applying the Index theorem to the divisor pair $(D,\ 2M-D+2H)$, we get

$$D^2(2M-D+2H)^2 \leq (D(D-H) - 2D(D-2H-M))^2. \tag{38}$$

Note now that, from hypotheses $i)$ and $ii)$ it follows that $D(D-2H) > 0$, since $D(D-4H) > 0$ hence $D^2 - 2HD > 2HD > 0$. From (36) and from the positivity of $D(D-2H)$, it follows

$$D(D-2H) - 2D(D-2H-M) \leq D(D-2H) - 4\delta_0. \tag{39}$$

We observe that the left side member of (39) is non-negative, since $D(D-2H)-2D(D-2H-M) = D(2M-D+2H)$, where D is effective and, by (35), $2M-D+2H \in N(S)^+$. Squaring both sides of (39), together with (38), we find

$$D^2(2M-D+2H)^2 \leq (D(D-2H)-4\delta_0)^2. \tag{40}$$

On the other hand, by (37), we get

$$(2M-D+2H)^2 = 4(M-\frac{(D-2H)}{2})^2 = (D-2H)^2-4(D-2H-M)M \geq (D-2H)^2-4\delta_0,$$

i.e.

$$(2M-D+2H)^2 \geq (D-2H)^2-4\delta_0. \tag{41}$$

Next, we define

$$F(\delta_0) := 4\delta_0^2 - 4D(D-4H)\delta_0 + (DH)^2 - D^2H^2. \tag{42}$$

Putting together (40) and (41), it follows that $F(\delta_0) \geq 0$. We will show that, with our numerical hypotheses, one has $F(\delta_0) < 0$, proving the statement.

Indeed, the discriminant of the equation $F(\delta_0) = 0$ is $D^2(D-4H)^2$, and it is a positive number, since $(D-4H)^2 > 0$, by $ii)$, and $D^2 > 0$. We remark that $F(\delta_0) < 0$ iff $\delta_0 \in (\alpha(D,\ H),\ \beta(D,\ H))$, where

$$\alpha(D,H) = \frac{D(D-4H) - \sqrt{D^2(D-4H)^2}}{8}$$

and

$$\beta(D,H) = \frac{D(D-4H) + \sqrt{D^2(D-4H)^2}}{8};$$

so we have to show that, $\delta_0 \in (\alpha(D,\ H),\ \beta(D,\ H))$.

From $iv)$, it follows that $\delta_0 < \beta(D,H)$. Note that $\alpha(D,H) \geq 0$: indeed, if $\alpha(D,H) < 0$ then $D(D-4H) < \sqrt{D^2(D-4H)^2}$, which contradicts the Index Theorem, since $D(D-4H) > 0$ and D nef. Moreover, we have $\alpha(D,H) < 1$: for simplicity, put $t = D(D-4H)$; thus $\alpha(D,H) < 1$ iff

$$(*)\ \ t-8 < \sqrt{D^2(D-4H)^2} = \sqrt{t^2-16((DH)^2-D^2H^2)}.$$

If $t-8 < 0$, $(*)$ trivially holds; on the other hand, if $t-8 \geq 0$, by squaring both sides of $(*)$ we get $t^2-16t+64 < t^2-16\nu(D,H)$ which means $\nu(D,H) < t-4 = D(D-4H)-4$, i.e. hypothesis $iii)$. With analogous computations we get that $\beta(D,H) > 1$, which ensures there exists at least a positive integral value for the number of nodes.

In conclusion, our numerical hypotheses contradict $F(\delta_0) \geq 0$, therefore the assumption $h^1(\mathcal{I}_N(D-2H+K_S)) \neq 0$ leads to a contradiction.

For what concerns the other vanishing, i.e. $h^2(\mathcal{I}_{N/S}(K_S+D-3H)) = 0$, if we consider the exact sequence

$$0 \to \mathcal{I}_{N/S}(K_S+D-3H) \to \mathcal{O}_S(K_S+D-3H) \to \mathcal{O}_N(K_S+D-3H) \to 0,$$

by Serre duality we get $h^2(\mathcal{I}_{N/S}(K_S+D-3H)) = h^2(\mathcal{O}_S(K_S+D-3H)) = h^0(\mathcal{O}_S(-D+3H)) = 0$ since, by $i)$, $3H-D$ cannot be effective. □

Corollary 5.1. *If $D \sim mH$ on S, with $m \geq 5$, and if $[X] \in V_{|mH|,\delta}$ is such that*

$$\delta < \frac{m(m-4)}{4}deg(S), \tag{43}$$

then $\mathcal{I}_{N/S}(K_S+(m-1)H)$ is 0-regular on S.

We may observe that Theorem 5.1 also implies the geometric linear normality of the curve X. To do this, we have to recall the following results from [11], which are a generalization of what Chiantini and Sernesi proved in [7] for surfaces in $\mathbb{P}^3$:

Theorem 5.2. *Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate and linearly normal surface (not necessarily of general type) such that $h^1(\mathcal{O}_S(H)) = 0$. Let $|D|$ be a complete linear system on S whose general element is supposed to be smooth, irreducible and linearly normal in $\mathbb{P}^r$. Then, X is g.l.n. if and only if N imposes independent conditions to the linear system $|D + K_S - H|$*

Theorem 5.3. *Let $S \subset \mathbb{P}^r$ be a smooth surface and let $|D|$ be a complete linear system, whose general element is a smooth, irreducible divisor. Suppose that:*

i) $(D - H)H > 0$;
ii) $(D - 2H)^2 > 0$ *and* $D(D - 2H) > 0$;
iii) $\nu(D, H) < 4(D(D - 2H) - 4)$, *where* $\nu(D, H)$ *is the Hodge number of* D *and* H;
iv) $\delta < \frac{D(D-2H)+\sqrt{D^2(D-2H)^2}}{8}$.

If $X \sim D$ is a reduced, irreducible curve with only δ nodes as singular points and if $N = Sing(X)$, then $h^1(\mathcal{I}_{N/S}(D + K_S - H)) = 0$ so N imposes independent conditions to $|D + K_S - H|$. In particular, if S is also assumed to be non-degenerate, linearly normal and such that $h^1(S,\ \mathcal{O}_S(H)) = 0$ and if the general element of $|D|$ is also linearly normal in $\mathbb{P}^r$, then X is geometrically linearly normal.

If $D \sim mH$ then, when $m \geq 3$, all numerical conditions in Theorem 5.3 hold and $iv)$ becomes

$$\delta < \frac{m(m-2)}{4} deg(S). \tag{44}$$

Remark 5.3. It is a straightforward computation to verify that numerical conditions in Theorem 5.1 imply the ones in Theorem 5.3. Thus, if $S \subset \mathbb{P}^r$ is a smooth, non-degenerate, regular and linearly normal surface of general type such that $h^1(\mathcal{O}_S(H)) = 0$ and if $|D|$ is a complete linear system, whose general element is a smooth, irreducible and linearly normal curve satisfying numerical hypotheses in Theorem 5.1, then X is g.l.n and the map $\mu_{0,C}$ is surjective (see Remark 5.2).

By summarizing, we have the following result:

Theorem 5.4. *Let $S \subset \mathbb{P}^r$ be a smooth, regular, non-degenerate and linearly normal surface of general type, such that $h^1(\mathcal{O}_S(H)) = 0$. Denote by $|D|$ a complete linear system, whose general element is assumed to be a smooth, irreducible and linearly normal curve satisfying numerical hypotheses in Theorem 5.1. Let $[X] \in V_{|D|,\delta}$ be a regular point of the Severi variety (in the sense of Definition 1.4), with δ as in $iv)$ of Theorem 5.1, i.e. $\delta < \frac{D(D-4H)+\sqrt{D^2(D-4H)^2}}{8}$. Then, the morphism*

$$\pi_{|D|,\delta} : V_{|D|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. In particular, $\pi_{|D|,\delta}$ has finite fibres on each generically regular component of $V_{|D|,\delta}$, so each such component parametrizes a family having the expected number of moduli.

Proof. See Remark 5.1, Remark 5.2, Theorem 5.1 and Remark 5.3. □

Corollary 5.2. *Let S be as in Theorem 5.4 and let $D \sim mH$ on S, with $m \geq 5$, and assume that $[X] \in V_{|D|,\delta}$ is a regular point of the Severi variety, with δ as in (43), i.e.*

$$\delta < \frac{m(m-4)}{4} deg(S).$$

Then, the morphism $\pi_{|mH|,\delta}$ has injective differential at $[X]$. In particular, $\pi_{|mH|,\delta}$ has finite fibres on each generically regular component of $V_{|mH|,\delta}$, so each such component parametrizes a family having the expected number of moduli.

Remark 5.4. A particular case of the corollary above is when S is a complete intersection in $\mathbb{P}^r$ of type $(a_1, \ldots, a_{r-2})$; as already observed the upper-bound on δ, ensuring that X is g.l.n., becomes $\delta < \frac{m(m-2)}{4} deg(S)$, as in (44), whereas the bound on δ ensuring that all components of $V_{|D|,\delta}$ are regular is

$$\delta < \frac{m(m - 2((\Sigma_{i=1}^{r-2} a_i) - r - 1)}{4} deg(S) \tag{45}$$

(see [7] and [10]). This shows that, in general, the strongest restriction on δ is given by asking the regularity property of the point $[X]$ in the sense of Severi variety theory, then the 0-regularity property of the sheaf $\mathcal{I}_{N/S}(D+K_S-H)$ on S and, finally, the geometric linear normality property for the curve X.

Remark 5.5. As an interesting related result, we may observe that the bound on δ in Theorem 5.1 ensuring the 0-regularity of the sheaf $\mathcal{I}_{N/S}(K_S+D-H)$ is sharp. The following example was inspired by Corollary C in [33].

Example: Let $S \subset \mathbb{P}^3$ be a smooth sextic. We want to show there exist irreducible nodal curves X, such that $[X] \in V_{|8H|,48}$, for which $\mathcal{I}_{N/S}(K_S+D-H) = \mathcal{I}_{N/S}(9)$ is not 0-regular. Since $X \sim 8H$, one trivially has

$$h^2(\mathcal{I}_{N/S}(K_S+D-3H)) = h^2(\mathcal{I}_{N/S}(7)) = h^2(\mathcal{O}_S(7)) = h^0(\mathcal{O}_S(-5)) = 0;$$

thus the condition of 0-regularity fails as soon as $h^1(\mathcal{I}_{N/S}(8)) \neq 0$. We will show that, for such a curve X, its set of nodes N imposes one condition less to $|8H|$ proving the sharpness of (43) in Corollary 5.1 (observe in fact that $48 = \frac{6}{4}8(8-4)$).
As a preliminary count, observe that the family of curves in $|8H|$ with nodes in 48 given points has, at least, dimension 10. To construct an explicit example, let N be a 0-dimensional complete intersection subscheme of S obtained by the intersection of a general element C_2 of $|2H|$ and of a general element C_4 of $|4H|$; thus N is supported on 48 reduced points. By using the Koszul sequence of N in S, we immediately find

$$h^1(\mathcal{I}_{N/S}(9)) = 0 \text{ and } h^1(\mathcal{I}_{N/S}(8)) = 1.$$

Observe that

$$dim(|\mathcal{I}_{N/S}(6)|) = 43,\ dim(|\mathcal{I}_{N/S}(4)|) = 10,\ dim(|\mathcal{I}_{N/S}(2)|) = 0;$$

let $\Gamma_4,\ \Lambda_4 \in |\mathcal{I}_{N/S}(4)|$, $\Delta_6 \in |\mathcal{I}_{N/S}(6)|$ and $\Delta_2 \in |\mathcal{I}_{N/S}(2)|$ be general elements in such linear systems, which are smooth curves simply passing through N. Put

$$Y_1 = \Gamma_4 + \Lambda_4 \text{ and } Y_2 = \Delta_2 + \Delta_6;$$

thus Y_1 and Y_2 are reducible nodal curves on S, linearly equivalent to $8H$ and having nodes in N. Let

$$\mathcal{F}_{\lambda,\mu} = \{\lambda Y_1 + \mu Y_2 | [\lambda, \mu] \in \mathbb{P}^1\}$$

be the pencil of curves generated by Y_1 and Y_2. Its general element $X_{\lambda,\mu}$ is an irreducible curve linearly equivalent to $8H$ on S passing doubly through N. To conclude, we have to show that $X_{\lambda,\mu}$ has only nodes in N. To prove this, observe that

$$\Gamma_4\Delta_2 = \Lambda_4\Delta_2 = 48;$$

thus, among the points $Y_1Y_2 = (\Gamma_4+\Lambda_4)(\Delta_2+\Delta_6)$, those which are nodes for both Y_1 and Y_2 are only the points of N. Therefore, $X_{\lambda,\mu}$ has only nodes in N.
On the other hand, observe that such curves are geometrically linearly normal, since 48 is strictly less than the bound in (44) which is $\frac{6}{4}8(8-2) = 72$.

Remark 5.6. Note that the example above also determines non-regular points of the Severi variety $V_{|8H|,48}$. We recall that Chiantini and Sernesi constructed in [7] some examples of non-regular points of the Severi varieties $V_{|mH|,\frac{5}{4}m(m-4)}$, $m \geq 5$, on a general quintic surface $S \subset \mathbb{P}^3$, proving the sharpness of (45). These examples were generalized in [11] to Severi varieties on general canonical (i.e. $K_S \sim H$) and non-degenerate complete intersection surfaces in $\mathbb{P}^r$. The key point to construct such examples was that on a canonical surface the condition for a nodal curve $X \subset S$ to be g.l.n. is equivalent to the fact that $[X]$ is a regular point; in particular, (44) and (45) coincide.
In the same way, when S is 2-canonical (as it particularly happens in the example above) the 0-regularity of the sheaf $\mathcal{I}_{N/S}(K_S+D-H)$ is equivalent to the fact that $[X]$ is a regular point; in particular, (43) and (45) coincide.

6. Number of moduli for families of nodal curves on complete intersection surfaces of general type

To complete the overview on positive answers to the moduli problem for divisors of the form mH on $S \subset \mathbb{P}^r$, the cases $1 \leq m \leq 4$, which are not covered by Corollary 5.2, must be still considered.
From now on, we shall focus on the case of $S \subset \mathbb{P}^r$ a smooth, non-degenerate complete intersection surface of general type; thus,

$$K_S \sim \alpha H,$$

for some positive integer α.

We first consider the cases $m = 3$ and 4.

Theorem 6.1. *Let $[X] \in V_{|mH|,\delta}$ on S be a regular point, with $m \geq 3$, and assume that $K_S \sim \alpha H$, with $\alpha \geq 2$. If δ is as in (44), i.e.*

$$\delta < \frac{m(m-2)}{4} deg(S),$$

then the morphism

$$\pi_{|mH|,\delta} : V_{|mH|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. In particular, $\pi_{|mH|,\delta}$ has finite fibres on each generically regular component of $V_{|mH|,\delta}$, so each such component parametrizes a family having the expected number of moduli. The same conclusion holds for the family of smooth curves $V_{|mH|,0}$ also with $\alpha = 1$.

Proof. By the hypothesis on S and by the facts that $m \geq 3$ and δ is as in (44), we get that X is g.l.n. (see Theorem 5.3). Therefore, as in (a) of Remark 5.2,

$$h^0(\mathcal{O}_{\tilde{S}}(\mu^*(H))) = h^0(\mathcal{O}_S(H)) = h^0(\mathcal{O}_C(\tilde{H})) = r + 1.$$

By combining the pull-back to $\tilde{S}$ of the Euler sequence in $\mathbb{P}^r$ and the exact sequence defining C in $\tilde{S}$, we get the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathcal{O}_{\tilde{S}}(-C) & \to & H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H)))^{\vee} \otimes \mathcal{O}_{\tilde{S}}(\mu^*(H) - C) & \to & \mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C) & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathcal{O}_{\tilde{S}} & \to & H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H)))^{\vee} \otimes \mathcal{O}_{\tilde{S}}(\mu^*(H)) & \to & \mu^*(\mathcal{T}_{\mathbb{P}^r}) & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathcal{O}_C & \to & H^0(\mathcal{O}_C(\tilde{H}))^{\vee} \otimes \mathcal{O}_C(\tilde{H}) & \to & \varphi^*(\mathcal{T}_{\mathbb{P}^r}) & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

From the regularity of S, we get

$$\begin{array}{ccccc}
 & \downarrow & & \downarrow & \\
\cdots \to & H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H)))^{\vee} \otimes H^0(\mathcal{O}_{\tilde{S}}(\mu^*(H))) & \xrightarrow{g} & H^0(\mu^*(\mathcal{T}_{\mathbb{P}^r})) & \to 0 \\
 & \downarrow^{h} & & \downarrow^{h'} & \\
\cdots \to & H^0(\mathcal{O}_C(\tilde{H}))^{\vee} \otimes H^0(\mathcal{O}_C(\tilde{H})) & \xrightarrow{g'} & H^0(\varphi^*(\mathcal{T}_{\mathbb{P}^r})) & \to H^1(\mathcal{O}_C) \\
 & \downarrow & & \downarrow & \\
 & & & H^1(\mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C)) & .
\end{array}$$

From the second row, $\mu_{0,C}$ is surjective if and only if g' is. Since $h' \circ g = g' \circ h$ and since g is surjective, it suffices to prove that h' is surjective. With the given hypotheses, we shall prove that

$$(*) \qquad h^1(\mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C)) = 0$$

holds. By Serre duality and by Leray's isomorphism, $h^1(\mu^*(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_{\tilde{S}}(-C)) = h^1(\mathcal{I}_{N/S} \otimes \Omega^1_{\mathbb{P}^r}|_S \otimes \mathcal{O}_S(K_S + mH))$. The regularity of $V_{|mH|,\delta}$ at $[X]$ implies that the restriction map

$$H^0(\mathcal{O}_S(mH)) \xrightarrow{\rho_m} H^0(\mathcal{O}_N(mH))$$

is surjective. By tensoring the exact sequence

$$0 \to \mathcal{I}_{N/S} \to \mathcal{O}_S \to \mathcal{O}_N \to 0$$

with the vector bundle $\Omega^1_{\mathbb{P}^r}|_S \otimes \mathcal{O}_S(K_S + mH)$, we get

$$\cdots \to H^0(\Omega^1_{\mathbb{P}^r}|_S \otimes \mathcal{O}_S(K_S+mH)) \stackrel{\rho_{\Omega^1_{\mathbb{P}^r}|_S \otimes \mathcal{O}_S(K_S+mH)}}{\longrightarrow} H^0(\Omega^1_{\mathbb{P}^r}|_N \otimes \mathcal{O}_S(K_S+mH)) \to$$
$$H^1(\mathcal{I}_{N/S} \otimes \Omega^1_{\mathbb{P}^r}|_S \otimes \mathcal{O}_S(K_S+mH)) \to H^1(\Omega^1_{\mathbb{P}^r}|_S \otimes \mathcal{O}_S(K_S+mH)) \to \cdots .$$

Since S is a non-degenerate c.i. (in particular projectively normal), from standard computations involving the Euler sequence restricted to S we find $h^1(\Omega^1_{\mathbb{P}^r}|_S \otimes \mathcal{O}_S(K_S + mH)) = 0$ (for details, see [9]). Therefore, the vanishing $(*)$ holds if and only if the map $\rho_{\Omega^1_{\mathbb{P}^r}|_S \otimes \mathcal{O}_S(K_S+mH)}$ is surjective. By the assumption $K_S \sim \alpha H$, with $\alpha \geq 2$, the vector bundle $\Omega^1_{\mathbb{P}^r}|_S \otimes \mathcal{O}_S(\alpha)$ is globally generated; then one concludes as in Theorem 3.2. In the same way, one concludes also in the case $\alpha = 1$ and $\delta = 0$.
Since we have proven that X is g.l.n and that the map $\mu_{0,C}$, by Propositions 4.1 (i) and 5.1 we get the statement. □

The above result gives new positive answers to the moduli problem for Severi varieties of the form $V_{|mH|,\delta}$, for $m = 3$ and 4 on smooth, complete intersection surfaces of general type. These cases are covered neither by the results in Section 3 nor by those in Section 5.

For what concerns the cases $m = 1$ and 2, we cannot apply Theorem 6.1, since by hypothesis, m must be bigger than 2. In such cases, we shall make use of the following theorem in [8] (which generalizes a result of Accola in [1]):

Theorem 6.2. *(see* [8]*, Teorema 2.11) Let $\Gamma \subset \mathbb{P}^r$ be an irreducible, non-degenerate curve of degree n and let $\pi : \tilde{\Gamma} \to \Gamma$ be its normalization. Let $n \geq r \geq 2$ and let $\chi(n, r)$ be the* Castelnuovo number, *which is a non-negative integer such that*

(46) $$\frac{n-1}{r-1} - 1 \leq \chi(n,r) < \frac{n-1}{r-1},$$

where $\chi(n, r) = 0$ iff Γ is a smooth, rational normal curve. Put

(47) $$g(n,r) = \chi(n,r)[n - r - \frac{\chi(n,r)-1}{2}(r-1)].$$

Assume there exists on $\tilde{\Gamma}$ a linear system g^s_m with $m \leq n$ and $s \geq r$. Then, either
(i) $g^s_m = g^r_n$, (where g^r_n is the birational linear system on $\tilde{\Gamma}$ related to π)
or
(ii) $g(\tilde{\Gamma}) \leq \Phi(n, r) := g(n, r) - \chi(n, r) + 1$.

Remark 6.1. In our cases, we have that $\Gamma = X$ is a nodal curve which is linearly equivalent to mH on a smooth, complete intersection surface $S \subset \mathbb{P}^r$ of degree d, $\tilde{\Gamma} = C$ and $\pi = \varphi$.
(a) When $m = 2$, $X \subset \mathbb{P}^r$ is a non-degenerate, irreducible, nodal curve of degree $2d$ on S and φ is related to a linear system g^r_{2d} mapping C birationally onto X. By adjunction on S,

$$g(C) = p_a(X) - \delta = \frac{(2H + K_S)2H}{2} + 1 - \delta = (2+\alpha)d + 1 - \delta.$$

From Theorem 6.2, if $\delta < p_a(X) - \Phi(2d, r)$, i.e.

(48) $$\delta < d(2+\alpha) + \frac{(r+3-4d)}{2}\chi(2d,r) + \frac{(r-1)}{2}(\chi(2d,r))^2,$$

with $\alpha \geq 1$ and $\chi(2d, r) \in \mathbb{Z}_{\geq 0} \cap [\frac{2d-1}{r-1} - 1, \frac{2d-1}{r-1})$, then the g^r_{2d} on C is uniquely determined.
(b) If $m = 1$, we have a nodal curve $X \sim H$ on S, which is a hyperplane section of a non-degenerate surface, so $X \subset \mathbb{P}^{r-1} \cong H$ is non-degenerate in H. Thus, we have a g^{r-1}_d on C. As before, if $\delta < p_a(H) - \Phi(d, r-1)$, i.e.

(49) $$\delta < \frac{d(1+\alpha)}{2} + \frac{(r-2d+2)}{2}\chi(d,r-1) + \frac{(r-2)}{2}(\chi(d,r-1))^2,$$

with $\alpha \geq 1$ and $\chi(d, r-1) \in \mathbb{Z}_{\geq 0} \cap [\frac{d-1}{r-2} - 1, \frac{d-1}{r-2})$, the g^{r-1}_d on C is unique.

By using Remark 6.1, we can conclude with the following

Theorem 6.3. *Let $D \sim mH$ on S, with $1 \leq m \leq 2$, and assume that $[X] \in V_{|D|,\delta}$ is a regular point of the Severi variety. Suppose that*

$$\delta < d(2+\alpha) + \frac{(r+3-4d)}{2}\chi(2d,r) + \frac{(r-1)}{2}(\chi(2d,r))^2 \text{ and } \alpha \geq 1, \text{ if } \ m=2,$$

where $\chi(2d,r) \in \mathbb{Z}_{\geq 0} \cap [\frac{2d-1}{r-1} - 1, \frac{2d-1}{r-1})$, and

$$\delta < \frac{d(1+\alpha)}{2} + \frac{(r-2d+2)}{2}\chi(d,r-1) + \frac{(r-2)}{2}(\chi(d,r-1))^2 \text{ and } \alpha \geq 1, \text{ if } m=1,$$

where $\chi(d,r-1) \in \mathbb{Z}_{\geq 0} \cap [\frac{d-1}{r-2} - 1, \frac{d-1}{r-2})$. Then, the morphism $\pi_{|mH|,\delta}$ has injective differential at $[X]$. In particular, $\pi_{|mH|,\delta}$ has finite fibres on each generically regular component of $V_{|mH|,\delta}$, so each such component parametrizes a family having the expected number of moduli.

Proof. Suppose, by contradiction, that $h^0(\varphi^*(\mathcal{T}_S)) \neq 0$; thus, $dim(\pi^{-1}_{|mH|,\delta}(\pi_{|mH|,\delta}([X]))) > 0$. Since $[X]$ is by assumption a regular point, it corresponds to an unobstructed curve in S. Therefore, an element of $T_{[X]}(\pi^{-1}_{|mH|,\delta}(\pi_{|mH|,\delta}([X]))$ is induced by an effective algebraic deformation. From what observed in Remark 6.1, such deformations must be induced by projectivities. Then, one can conclude by using Proposition 4.1. □

Example: if we consider an irreducible, nodal plane section X on a smooth quintic $S \subset \mathbb{P}^3$, we get that $\chi(d,r-1) = \chi(5,2) = 3$; so if $[X]$ is a regular point of the corresponding Severi variety and, by (49), if $\delta < \frac{10}{2} - \frac{15}{2} + \frac{9}{2} = 2$, the component passing through $[X]$ has the expected number of moduli.

Remark 6.2. We cannot apply what observed in Remark 6.1 when $m=3$ and 4 since, in such cases, one can show that $p_a(3H) - \Phi(3d,r) < 0$ and $p_a(4H) - \Phi(4d,r) < 0$.

7. Examples and final remarks

For clarity sake, here we shall summarize what one can deduce from our more general results of Sections 3, 5 and 6 in the particular cases of Severi varieties $V_{|mH|,\delta}$ on $S \subset \mathbb{P}^r$ a smooth, non-degenerate complete intersection of general type or, in particular, on $S = S_d \subset \mathbb{P}^3$ of degree $d \geq 5$.

Proposition 7.1. *Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate complete intersection of general type whose canonical divisor is $K_S \sim \alpha H$, where H denotes its hyperplane section. Suppose that $[X]$ is a regular point of the Severi variety $V_{|mH|,\delta}$.*
Assume that:
(1) $\delta \leq \dim(|mH|)$ if

a) $\alpha \geq 2$, $m \geq \alpha + 6$, $\delta \geq 1$ *or*
b) $\alpha \geq 1$, $m \geq \alpha + 6$, $\delta = 0$*;*

(2) $\delta < \frac{m(m-4)}{4} deg(S)$ if $\alpha \geq 1$ and $5 \leq m \leq \alpha + 5$;
(3)

a) $\delta < \frac{m(m-2)}{4} deg(S)$ *if* $\alpha \geq 2$ *and* $m = 3,\ 4$ *or*
b) $\delta = 0$ *if* $\alpha = 1$ *and* $m = 3,\ 4$*;*

(4) $\delta < deg(S)(2+\alpha) + \frac{(r+3-4deg(S))}{2}\chi(2deg(S),r) + \frac{(r-1)}{2}(\chi(2deg(S),r))^2$ if $\alpha \geq 1$ and $m=2$, where $\chi(2deg(S),r)$ is a non-negative integer in $[\frac{2deg(S)-1}{r-1} - 1, \frac{2deg(S)-1}{r-1})$;
(5) $\delta < \frac{deg(S)}{2}(1+\alpha) + \frac{(r-2deg(S)+2)}{2}\chi(d,r-1) + \frac{(r-2)}{2}(\chi(d,r-1))^2$ if $\alpha \geq 1$ and $m=1$, where $\chi(deg(S),r)$ is a non-negative integer in $[\frac{deg(S)-1}{r-2} - 1, \frac{deg(S)-1}{r-2})$;

Then the morphism

$$\pi_{|mH|,\delta} : V_{|mH|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. In particular, it has finite fibres on each generically regular component of $V_{|mH|,\delta}$, so each such component parametrizes a family having the expected number of moduli.

In particular, we have:

Corollary 7.1. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$ and let $[X] \in V_{|mH|,\delta}$ be a regular point.*
Assume that:
(1) $\delta \leq \dim(|mH|)$ if

a) $d \geq 6$, $m \geq d+2$, $\delta \geq 1$ *or*
b) $d \geq 5$, $m \geq d+2$, $\delta = 0$*;*

(2) $\delta < \frac{m(m-4)}{4}d$ if $d \geq 5$ and $5 \leq m \leq d+1$;
(3)

a) $\delta < \frac{m(m-2)}{4}d$ *if* $d \geq 6$ *and* $m = 3,\ 4$ *or*
b) $\delta = 0$ *if* $d = 5$ *and* $m = 3,\ 4$*;*

(4) $\delta < d-2$ if $d \geq 5$ and $m = 2$;
(5) $\delta < d-3$ if $d \geq 5$ and $m = 1$.

Then the morphism

$$\pi_{|mH|,\delta} : V_{|mH|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. In particular, it has finite fibres on each generically regular component of $V_{|mH|,\delta}$, so each such component parametrizes a family having the expected number of moduli.

Observe that our results generalize what can be proven in the case of a general smooth, complete intersection surface $S \subset \mathbb{P}^r$ by using a recent result of Schoen, [28]. In his paper, he studies algebraic varieties which are dominated by products of varieties of smaller dimension (abbreviated *DPV*); in the case of products of curves, one writes *DPC*. The main goal of Schoen's paper is to discuss, via real algebraic group theory and Hodge theory, some obstructions to *DPC* and *DPV* properties. As a result, he shows for example that if $W \subset \mathbb{P}^N$ is a sufficiently general complete intersection variety of degree $d > N+1$ and of dimension $n \geq 2$, then W cannot satisfy the *DPC*-property. Thus, the general complete intersection surface $S \subset \mathbb{P}^r$, of degree $d \geq r+2$, cannot be dominated by a product of curves $C_1 \times C_2$. Therefore, there cannot exist isotrivial pencils of smooth or δ-nodal curves in $|mH|$, otherwise, after a suitable base change, such a surface would be *DPC*.

Thus, via Schoen's results, one can answer the moduli problem, for smooth and nodal curves in the linear system $|mH|$, $m \geq 1$, on a general complete intersection surface $S \subset \mathbb{P}^r$ of degree $d \geq r+2$. Our results are more generally valid for divisors D on S, where S is not necessarily a general complete intersection and can have a wildly complicated $Div(S)$. Moreover, our techniques involve only vanishing theorems, vector bundle theory on smooth projective surfaces and Brill-Noether theory on smooth projective curves, so they are of a more elementary nature and give simpler proofs.

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