Open Problems

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A problem session has been held during the workshop. We gather here the problems and questions that have been proposed and discussed during this lively event.

We give the problems without further comments, and in the order in which they have been proposed during the conversation. We believe they are all interesting and inspiring, and reflect, also in their formulation, the tastes and interests of the participants who have proposed them.

Problem 1 *(Rita Pardini)***.** For all abelian varieties A, set

$$X_A := \{C : J(C) \twoheadrightarrow A\} \subset \mathcal{M}_g.$$

As explained in Problem 3, if $\dim X_A > 0$, then there are bounds on the dimension of A in terms of g. Can one obtain better bounds if $\dim X_A > 1$?

Problem 2 *(Rita Pardini)***.** Study the various possible notions of relative irregularity for families of higher-dimensional varieties (with appropriate conditions on the fibres).

Problem 3 *(Lidia Stoppino)***.** Let $f\colon S \to B$ be a fibred surface, and $\omega_f = \omega_S \otimes \omega_B^{-1}$ its relative canonical bundle. The Hodge bundle $f_*\omega_f$ has the two following splittings (the so-called first and second Fujita decompositions [7], [6]):

$$f_*\omega_f = \mathcal{E} \oplus \mathcal{O}^{q_f},$$

where $h^1(\mathcal{E} \otimes \omega_B) = 0$ and q_f is the relative irregularity; and

$$f_*\omega_f = \mathcal{A} \oplus \mathcal{U},$$

where $\mathcal{A}$ is ample and $\mathcal{U}$ is unitary flat.

One wants to get information about the relative irregularity q_f and the rank of the unitary factor $u_f = \operatorname{rk} \mathcal{U}$, in connection with other invariants and with geometric properties of the fibration.

(3.1) *Slope inequalities.* Let us consider a non-isotrivial fibration, and the relative invariants $\chi_f = \deg f_*\omega_f$ and $c_1^2(\omega_f)$. It is known that the ratio between $c_1^2(\omega_f)$ and χ_f (the so-called slope) is greater or equal to 4 if $q_f > 0$, and that $q_f = 1$ if $q_f > 0$ and $c_1^2(\omega_f) = 4\chi_f$. It is natural to conjecture a bound on the slope increasing with the relative irregularity. Bounds of this type are to be found in [4], [9], [3]; in this last paper, the following bound is conjectured:

$$K_f^2 \geqslant 4\frac{g-1}{g-q_f}\chi_f.$$

This bound has been proved in [10] for $q_f \leqslant g/9$, and disproved for high values of q_f (namely $q_f = (g+1)/2$). In the same paper, the following remarkable bound is proved:

$$K_f^2 \geqslant 4\frac{g-1}{g-q_f/2}\chi_f.$$

Problem: use the thechniques of myself and Barja, or that of Lu and Zuo, to prove sharper lower bounds for the slope. It is moreover a natural question whether or not there holds an analogous bound involving u_f (see [8]).

(3.2) A celebrated theorem of Beauville [5] states that $q_f \leqslant g$ and $q_f = g$ if and only if the fibration is birationally trivial. A conjecture due to Xiao states that for non-trivial fibred surfaces, one has $q_f \leqslant g/2+1$ [12]. This conjecture has been disproved by Pirola [11] and Albano–Pirola [1]. Let c_f be the Clifford index of the general fibre. In [2], the inequality $q_f \leqslant g - c_f$ is proved for non-isotrivial fibred surfaces. Moreover, a modified version of Xiao's conjecture is stated:

$$q_f \leqslant \left\lceil \frac{g+1}{2} \right\rceil .$$

See [8] for a more detailed description of the state of art. In the preprint [8] we prove the analogous inequality for u_f: $u_f \leqslant g - c_f$. Problem: what is in general the relation between u_f and the genus of the fibration? Is there a lower bound of Xiao type? Even the isotrivial case is not known.

Problem 4 *(Sönke Rollenske)***.** Stable surfaces are the two-dimensional analogue of stable curves: they are the varieties that are parametrised by a natural compactification of the Giesecker moduli space $\mathfrak{M}_{K^2,\chi}$ of canonical models of surfaces of general type. They were first considered by Kollár and Shepherd-Barron in [17] but the proof that the (coarse) moduli space $\overline{\mathfrak{M}}_{K^2,\chi}$ actually exists as a projective scheme had to wait for some more time. Because of the significant contributions of Alexeev [13] these moduli spaces are sometimes called KSBA-moduli spaces. More information can be found in [15, 16].

By definition, the canonical divisor of a stable surface X is $\mathbf{Q}$-Cartier, and we call the minimal number I such that IK_X is a Cartier divisor the global Cartier index of X. By the projectivity of the moduli space, the global Cartier index is bounded on $\overline{\mathfrak{M}}_{K^2,\chi}$ for fixed invariants, but we have not been able to give an explicit bound in any relevant case.

We thus propose to study this problem in a particular example, namely on the closure of $\mathfrak{M}_{1,3}$ in $\overline{\mathfrak{M}}_{1,3}$. Gorenstein stable surfaces with these invariants have been classified in [14], where some non-Gorenstein examples with small index have been described as well. Other examples can be found in [19], but it is still unclear if or how these two classes of examples are related.

Problem 5 *(Sönke Rollenske)***.** Seshadri constants measure the local positivity in a point of a polarised (smooth) variety (X, L), see for example [18]. It is known that on surfaces the Seshadri constant is either rational or has value $\sqrt{d}$ for a non-square integer d, but so far no example has been described where it is actually irrational. So the following might be a quite challenging problem:

Construct a polarised surface (X, L) with irrational Seshadri constant at the very general point or prove that the Seshadri constant is always rational.

It has been suggested during the problem session that further progress may be made via a degeneration argument, for example degenerating a hypersurface of degree $d \geqslant 4$ in $\mathbb{P}^3$ to a plane and a hypersurface of degree $d-1$.

Problem 6 *(Ciro Ciliberto)***.** Give conditions under which the following happens. Let $\mathfrak{M}$ be a component of $\mathfrak{M}_{K^2,\chi}$. Then there is a flat family over a disc such that the general fibre is the general surface in $\mathfrak{M}$ and the central fibre is an irreducible, reduced rational surface. See Problem 4 for a recap on the moduli space $\mathfrak{M}_{K^2,\chi}$.

Suggestion: Assume $q = 0$. If the number of moduli is big enough, then maybe we can impose enough triple points (elliptic singularities) to kill p_g, and find a degeneration which satisfies the necessary condition $p_g = q = 0$.

Problem 7 *(Adrien Dubouloz)***.** Let X_3 be a smooth cubic hypersurface in $\mathbf{P}^4$, and let S_3 be a smooth hyperplane section. Put $V_3 = X_3 \setminus S_3$, so that V_3 is an affine cubic hypersurface in $\mathbb{C}^4$. The unirationality but non rationality of X_3 implies that $\mathrm{Aut}(V_3)$ does not contain any nontrivial connected algebraic group. Is it true that $\mathrm{Aut}(V_3) = \mathrm{Aut}(X_3, S_3)$ so that, in particular, $\mathrm{Aut}(V_3)$ is finite ?

Problem 8 *(Ciro Ciliberto)***.** Let $S_3 \subset \mathbf{P}^3$ be a cubic surface. Let $\mathcal{C}_d$ be the family of curves of degree d in $\mathbf{P}^3$. Study the subfamily

$$\mathcal{D}_{S_3,d} = \{C \in \mathcal{C}_d : \mathrm{Card}(\mathrm{supp}(C \cap S_3)) = 1\}$$

(in other words, $\mathcal{D}_{S_3,d}$ is the family of curves interesecting S in one point only, with multiplicity $3d$); for instance, it is smooth? irreducible? of the expected dimension? etc.

Problem 9 *(Andreas Leopold Knutsen)***.** We use the standard notation from Brill–Noether theory. A smooth projective curve C of gonality k and genus g is said to satisfy the *linear growth condition* if

$$\dim W^1_{k+n}(C) \leqslant n \quad \text{for all} \quad 0 \leqslant n \leqslant g - 2k + 2.$$

Note that (i) clearly $W^1_{k+n}(C)$ has irreducible components of dimension $\geqslant n$ (just add base points to linear series of degree k), and (ii) $\dim W^1_{k+n}(C) \geqslant \rho(g, 1, k+n) = 2(d+n) - 2 - g > n$ whenever $n > g - 2k + 2$. Thus, there is no way of either weakening the assumptions on n or strengthening the claim on $\dim W^1_{k+n}(C)$. It was proved by Aprodu in [21] that the general k-gonal curve with $k \neq \frac{g+3}{2}$ satisfies the linear growth condition, and that, most interestingly, any curve satisfying the linear growth condition verifies both the Green [25] and the Green-Lazarsfeld [26] conjectures. It is therefore of interest to find out which curves do not satisfy the linear growth condition (apart from the obvious case of curves of odd genus with maximal gonality $k = \frac{g+3}{2}$, where the linear growth already fails in the first step, as $\dim W^1_k = 1$).

A special case of Martens' expectation (T) proposed in [30] yields the following:

(9.1) Conjecture. *A smooth projective curve C of gonality k and genus g satisfies the linear growth condition as soon as* $\dim W^1_k(C) = 0$.

Thus, one may hope that it is sufficient to check the linear growth condition in the lowest degree only.

It is well-known that any k-gonal curve satisfies $\dim W^1_k(C) \leqslant 1$. Examples for which equality is attained are (cf. [24, 30]) only:

- curves of odd genus with maximal gonality;
- plane curves;
- *exceptional curves*, as defined by Eisenbud, Lange, Martens and Schreyer in [24]: those are curves for which the Clifford index is not computed by a pencil, and are thus generalizations of plane curves. Letting r denote the minimal dimension of a linear system computing the Clifford index, it is conjectured in [24] that exceptional curves C with $r \geqslant 3$ are of a particular type; among other conditions, they should satisfy $g = 4r-2$ and $\mathrm{Cliff}\, C = 2r-3$. Examples of such, for any r, are constructed on $K3$ surfaces in [24];

- finite covers of the above examples.

The ELMS conjecture was proved for $r \leqslant 9$ in [24], and has been later proved to hold for any smooth curve on Del Pezzo, $K3$ and Enriques surfaces [27, 28, 29], but is in general still wide open.

In view of all this, the following problems were proposed:

- Study Conjecture (9.1).
- Find more k-gonal curves apart from the known ones satisfying $\dim W^1_k = 1$ or find conditions on such curves. (Suggestion: two base point free g^1_k's give a map from the curve to $\mathbb{P}^1 \times \mathbb{P}^1$ with image of type (k, k), so one should study families of such maps; for instance, one may try to use the results of Castelnuovo and Accola in [22] and [20] respectively).
- Study the ELMS conjecture. (Suggestion: as exceptional curves necessarily satisfy $\dim W^1_k =$ 1 (with k the gonality), a first step in this problem is the previous point).

Problem 10 *(Ciro Ciliberto)***.** The following is a folklore problem, attributed to D. Mumford:

Given an arbitrary irreducible algebraic surface S, is there a rational map $S \dashrightarrow \mathbf{P}^3$, birational onto its image Σ, with Σ having only isolated singularities?

A related question is:

Can one find a smooth irreducible surface S, such that if $C \subset S$ is any irreducible curve of genus $g(C) > 1$, then its normalization is not a smooth plane curve?

An affirmative answer to the former question would imply a negative answer to the latter.

Problem 11 *(Francesco Polizzi)***.** Recall the definition of a *general triple plane*: A triple cover $f : X \to \mathbf{P}^2$ with branch locus $B \subset \mathbf{P}^2$ is a general triple plane if the following three conditions hold: (i) f is unramified over $\mathbf{P}^2 \setminus B$; (ii) $f^*B = 2R + R_0$, with R smooth and irreducible, and R_0 reduced; (iii) $f|_R\, R \to B$ coincides with the normalization map of B. This implies in particular that the branch curve B has ordinary cusps as only possible singularities, and these are exactly the points where f is totally ramified. See [31] for more background.

Question: does there exist general triple planes with $p_g = q = 0$ and Kodaira dimension $\kappa(X) > 0$? Examples with Kodaira dimension $-\infty$ and 0 have been provided in [31].

Problem 12 *(Ciro Ciliberto)***.** Let V be a threefold, with $\mathcal{L}$ a linear system of positive dimension on it. Consider the following property: The general element $S \in |\mathcal{L}|$ is irreducible, has $p_g(\tilde{S}) = q(\tilde{S}) = 0$ (with $\tilde{S} \to S$ the resolution of singularities), but is not rational.

The problem is to classify all pairs $(V, \mathcal{L})$ satisfying this property. A first task is to list all known examples. Then one should try to bound the dimension of $|\mathcal{L}|$.

Problem 13 *(Pietro Pirola)***.** Let $S \subset \mathbf{P}^3$ be a very general surface of degree $d \geqslant 5$, in the sense that $[S]$ is a very general point of $\mathcal{SP}_d$ the moduli space of smooth surfaces of degree d in $\mathbf{P}^3$. We know that if $f : S \to X$ is a dominant map of degree $\geqslant 2$, then X is rational [35].

(13.1) We ask what happens for $d = 4$. (Easy step: $p_g(X) = q(X) = 0$ and the fundamental group of X is trivial).

(13.2) More generally, if S is a very general $K3$ surface with respect to some polarization, we ask if there exists another $K3$ surface S_0 not isomorphic to S and a dominant rational map $S \to S_0$.

(13.3) Let a and b positive integers such that $a+b>5$, let $\mathcal{SP}_{a,b}$ be the moduli space of smooth complete intersection of type (a,b) in $\mathbf{P}^4$. Consider S a very general complete intersection of type (a,b), that is the corresponding point $[S] \in \mathcal{SP}_{a,b}$ is a very general point. Let $f : S \to X$ be a dominant rational map of degree $\geqslant 2$. Is the surface X necessarily rational?

In the very recent paper [33] this is proven for the cases $(3,t),\ t>2$, $(4,t),\ t>3$, $(5,t),\ 4<t<10$ and $(6,6)$.

(13.4) What happens for hypersurfaces of $\mathbf{P}^4$?

Problem 14 *(Pietro Pirola)***.** The notation is the same as in Problem 13. Recall Diaz' result [34]: If $Y \subset \mathcal{M}_g$ is a complete subvariety of the moduli space of curve of genus $g \geqslant 2$, then $\dim Y \leqslant g-2$. Is there an analog result for $\mathcal{SP}_d$ or for $SP_{a,b}$ (under appropriate assumptions)?

Problem 15 *(Francesco Polizzi, [32])***.** Classify the configurations of d^2 points in $\mathbf{P}^3$, $d \geqslant 3$, such that for every general linear projection from a point $x \in \mathbf{P}^3$, the resulting configuration of d^2 points in $\mathbf{P}^2$ is the complete intersection of two curves of degree d. Suggestion: consider the limit of the projection when x tends to a point belonging to the configuration.

A first non-trivial example of such a configuration is given by the d^2 intersection points of d horizontal and d vertical lines on a smooth quadric surface $Q = \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$.

References

References for Problem 3 by L. Stoppino

[1] A. Albano, G. P. Pirola, *Dihedral monodromy and Xiao fibrations*, Ann. Mat. Pura Appl. **195**(4) (2016), 1255–1268.

[2] M. A. Barja, V. González-Alonso, J. C. Naranjo, *Xiao's conjecture for general fibred surfaces (arXiv:1401.7502)*, to appear on J. Reine Angew. Math., doi:10.1515/crelle-2015-0080.

[3] M. A. Barja, L. Stoppino, *Linear stability of projected canonical curves with applications to the slope of fibred surfaces*, J. Math. Soc. Japan **60**(1) (2008), 171–192.

[4] M. A. Barja and F. Zucconi, *On the slope of fibred surfaces*, Nagoya Math. J. **164** (2001), 103–131.

[5] A. Beauville, *L'inégalité $p_g \geqslant 2q+4$ pour les surfaces de type général*, Bull. Soc. Math. Franc. **110** (1982), 343–346.

[6] F. Catanese, M. Dettweiler, *Answer to a question by Fujita on variation of Hodge structures (arXiv:1311.3232)*, arXiv:1311.3232.

[7] T. Fujita, *On Kähler fiber spaces over curves*, J. Math. Soc. Japan **30**(4) (1978), 779–794.

[8] V. González-Alonso, L. Stoppino, S. Torelli, *On the rank of the flat unitary factor of the Hodge bundle*, preprint arXiv:1709.05670.

[9] K. Konno, *On the Irregularity of Special Non-Canonical Surfaces*, Publ. RIMS Kyoto Univ. **30** (1994), 671–688.

[10] X. Lu, K. Zuo *On the slope conjecture of Barja and Stoppino for fibred surfaces*, preprint arXiv:1504.06276.

[11] G. P. Pirola, *On a conjecture of Xiao*, J. Reine angew. Math. **431** (1992), 75–89.

[12] G. Xiao, *Fibred algebraic surfaces with low slope* Math. Ann. **276** (1987), 449–466.

References for Problems 4 and 5 by S. Rollenske

[13] V. Alexeev, *Boundedness and K^2 for log surfaces*, Internat. J. Math. **5**(6) (1994), 779–810.

[14] M. Franciosi, R. Pardini, S. Rollenske, *Gorenstein stable surfaces with $K_X^2 = 1$ and $p_g > 0$*, Math. Nachr. **290**(5-6) (2017), 794–814.

[15] J. Kollár, *Moduli of varieties of general type*, in G. Farkas and I. Morrison, editors, *Handbook of Moduli: Volume II*, volume 24 of *Advanced Lectures in Mathematics*, International Press, 2012, 131–158 (arXiv:1008.0621).

[16] J. Kollár, *Moduli of varieties of general type*, 2016, book in preparation.

[17] J .Kollár, N. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math., **91**(2) (1988), 299–338.

[18] R. Lazarsfeld, *Positivity in algebraic geometry I*, volume 48 of Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[19] J. Rana, G. Urzúa, *Optimal bounds for t-singularities in stable surfaces*, arXiv:1708.02278.

References for Problem 9 by A. L. Knutsen

[20] R. D. M. Accola, *On Castelnuovo's inequality for algebraic curves. I*, Trans. Amer. Math. Soc. **251** (1979), 357–373.

[21] M. Aprodu, *Remarks on syzygies of d-gonal curves*, Math. Res. Lett. **12** (2005), 387–400.

[22] G. Castelnuovo, *Sulle curve che posseggono una infinità continua di corrispondenze algebriche*, Scritti matematici offerti ad Enrico d'Ovidio, 164–174 (1918).

[23] M. Coppens, G. Martens, *Secant spaces and Clifford's theorem*, Compos. Math. **78** (1991), 193–212.

[24] D. Eisenbud, H. Lange, G. Martens, F.-O. Schreyer, *The Clifford dimension of a projective curve*, Compos. math. **72** (1989), 173–204.

[25] M. L. Green, *Koszul cohomology and the geometry of projective varieties*, J. Diff. Geom. **19** (1984), 125–171.

[26] M. L. Green, R. Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. **83** (1986), 73–90.

[27] A. L. Knutsen, *Exceptional curves on Del Pezzo surfaces*, Math. Nachr. **256** (2003), 58–81.

[28] A. L. Knutsen, *On two conjectures for curves on $K3$ surfaces*, Intern. J. Math. **20** (2009), 1547–1560.

[29] A. L. Knutsen, A. F. Lopez, *Brill-Noether theory of curves on Enriques surfaces, II. The Clifford index*, Manuscr. Math. **147** (2015), 193–237.

[30] G. Martens, *On dimension theorems of the varieties of special divisors on a curve*, Math. Ann. **267** (1984), 279–288.

References for Problems 11 and 15 by F. Polizzi

[31] D. Faenzi, F. Polizzi, and J. Vallès, *Triple planes with $p_g = q = 0$*, arXiv:1605.02102, to appear in Trans. Amer. Math. Soc.

[32] *When is a general projection of d^2 points in $\mathbb{P}^3$ a complete intersection?* `MathOverflow` question 67265, asked by F. Polizzi (2011).

References for Problems 13 and 14 by G. Pirola

[33] F. Caucci, Y. Cho, L. Rizzi, *On dominant maps from a very general complete intersection surface in* $\mathbb{P}^4$, Le Matematiche **72**(2) (2017), 183–194.

[34] S. Diaz, *A bound on the dimensions of complete subvarieties of* M_g, Duke Math. J. **51** (1984), no. 2, 405–408.

[35] Y. Lee and G. Pirola, *On subfields of the function field of a general surface in* $\mathbb{P}^3$, Int. Math. Res. Not. **24** (2015), 13245–13259.