Kubo-Martin-Schwinger, Non-Equilibrium Thermal states, and Conformal Field Theory

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Based on a joint work with S. Hollands

and previous works with Bischoff, Kawahigashi, Rehren and Camassa, Tanimoto, Weiner

Thermal equilibrium states

Thermodynamics concerns heat and temperature and their relation to energy and work. A primary role is played by the equilibrium distribution.

Gibbs states

Finite quantum system: $\mathfrak{A}$ matrix algebra with Hamiltonian H and evolution $\tau_t = \mathrm{Ad}e^{itH}$. Equilibrium state φ at inverse temperature β is given by the Gibbs property

$$\varphi(X) = \frac{\mathrm{Tr}(e^{-\beta H}X)}{\mathrm{Tr}(e^{-\beta H})}$$

What are the equilibrium states at infinite volume where there is no trace, no inner Hamiltonian?

The Kubo-Martin-Schwinger condition

The fundamental KMS equilibrium condition originated by the study of the analytic properties of the Green functions in two papers in the 50's, one by Kubo, one by Paul Martin and J. Schwinger.

R. Kubo (1957), "Statistical-Mechanical Theory of Irreversible Processes. I. General Theory and Simple Applications to Magnetic and Conduction Problems", Journal of the Physical Society of Japan 12, 570-586

Paul C. Martin, Julian Schwinger (1959), "Theory of Many-Particle Systems. I", Physical Review 115, 1342-1373,

The final form was presented by Haag, Hugenholtz and Winnink at the 1967 Baton Rouge conference on Operator Algebras.

KMS states

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$ $\exists$ $F_{XY} \in A(S_\beta)$ s.t.

(a) $F_{XY}(t) = \varphi\big(X\tau_t(Y)\big)$

(b) $F_{XY}(t + i\beta) = \varphi\big(\tau_t(Y)X\big)$

where $A(S_\beta)$ is the algebra of functions analytic in the strip $S_\beta = \{0 < \Im z < \beta\}$, bounded and continuous on the closure $\bar{S}_\beta$.

(Note: it is sufficient to check (a) and (b) for X, Y in a dense *-subalgebra $\mathfrak{B}$.)

KMS states have been so far the central objects in the Quantum Statistical Mechanics, for example in the analysis of phase transition.

Modular theory

Let $\mathcal{M}$ be a von Neumann algebra and φ a normal faithful state on $\mathcal{M}$. The fundamental Tomita-Takesaki theorem states that there exists a canonical one-parameter automorphism group σ^φ of $\mathcal{M}$ associated with φ. So, von Neumann algebras are intrinsically dynamical objects.

In the GNS representation $\varphi = (\Omega, \cdot\, \Omega)$; if S is the closure of $x\Omega \mapsto x^*\Omega$, with polar decomposition $S = J\Delta^{\frac{1}{2}}$, we have $\sigma_t^\varphi = \mathrm{Ad}\Delta^{it}$:

$$\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}, \quad JMJ = M'$$

For a remarkable historical coincidence, Tomita announced the theorem at the 1967 Baton Rouge conference. *Soon later Takesaki charcterised the modular group by the KMS condition*.

Some highlights concerning KMS/modular theory

Modular/KMS theory is fundamental in Operator Algebras. In particular, it allowed the classification of type III factors by Alain Connes, leading him to the Fields Medal.

Later crucial use (among others):

- KMS and stabilty (Haag-Kastler-Trych Polhmeyer)
- Passivity (Putz-Woronowicz)
- QFT, Bisognano-Wichmann theorem and Hawking-Unruh effect (Sewell)
- Positive cones - noncommutative measure theory (Araki, Connes, Haagerup)
- Non-commutative Chern character in cyclic cohomology (Jaffe-Lesniewski-Osterwalder)
- Jones index in the type III setting (Kosaki, L.)

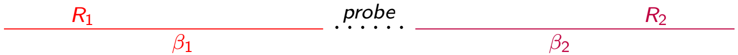
Non-equilibrium thermodynamics

Non-equilibrium thermodynamics: study physical systems not in thermodynamic equilibrium but basically described by thermal equilibrium variables. Systems, in a sense, near equilibrium; but, in general, non-uniform in space and time.

Non-equilibrium thermodynamics has been effectively studied for decades with important achievements, yet the general theory is still missing. The framework is even more incomplete in the quantum case, *non-equilibrium quantum statistical mechanics*.

We aim provide a general, model independent scheme for the above situation in the context of quantum, two dimensional *Conformal Quantum Field Theory*. As we shall see, we provide the general picture for the evolution towards a *non-equilibrium steady state*.

A typical frame described by Non-Equilibrium Thermodynamics:



Two infinite reservoirs R_1, R_2 in equilibrium at their own temperatures $T_1 = \beta_1^{-1}$, $T_2 = \beta_2^{-1}$, and possibly chemical potentials μ_1, μ_2, are set in contact, possibly inserting a probe.

As time evolves, the system should reach a non-equilibrium steady state.

This is the situation we want to analyse. As we shall see the *Operator Algebraic approach to CFT* provides a model independent description, in particular of the asymptotic steady state, and exact computation of the expectation values of the main physical quantities.

Non-equilibrium steady states

A *non-equilibrium steady state* NESS φ of $\mathfrak{A}$ satisfies property (a) in the KMS condition, for all X, Y in a dense *-subalgebra of $\mathfrak{B}$, but not necessarily property (b).

For any X, Y in $\mathfrak{B}$ the function

$$F_{XY}(t) = \varphi\big(X\tau_t(Y)\big)$$

is the boundary value of a function holomorphic in S_β. (Ruelle)

Example: the tensor product of two KMS states at temperatures β_1, β_2 is a NESS with $\beta = \min(\beta_1, \beta_2)$.

Problem: describe the NESS state ω and show that the initial state ψ evolves towards ω

$$\lim_{t\to+\infty} \psi \cdot \tau_t = \omega$$

Möbius covariant nets (Haag-Kastler nets on S^1)

A local Möbius covariant net $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

- **A.** *Isotony*. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **B.** *Locality*. $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- **C.** *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \;=\; \mathcal{A}(gI), \quad g \in \text{Möb},\; I \in \mathcal{I}.$$

- **D.** *Positivity of the energy*. Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.
- **E.** *Existence of the vacuum*. $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

Consequences

- *Irreducibility*: $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I) = B(H)$.
- *Reeh-Schlieder theorem*: Ω is cyclic and separating for each $\mathcal{A}(I)$.
- *Bisognano-Wichmann property* (KMS property of $\omega|_{\mathcal{A}(I)}$): The modular operator/conjugation Δ_I and J_I of $(\mathcal{A}(I), \Omega)$ are

$$U(\delta_I(2\pi t)) = \Delta_I^{-it},\ t \in \mathbb{R}, \qquad \text{dilations}$$
$$U(r_I) = J_I \qquad \text{reflection}$$

  (Fröhlich-Gabbiani, Guido-L.)
- *Haag duality*: $\mathcal{A}(I)' = \mathcal{A}(I')$
- *Factoriality*: $\mathcal{A}(I)$ is III_1-factor (in Connes classification)
- *Additivity*: $I \subset \cup_i I_i \implies \mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$ (Fredenhagen, Jorss).

Local conformal nets

$\mathrm{Diff}(S^1) \equiv$ group of orientation-preserving smooth diffeomorphisms of S^1

$\mathrm{Diff}_I(S^1) \equiv \{g \in \mathrm{Diff}(S^1) : g(t) = t \ \forall t \in I'\}$.

A local conformal net $\mathcal{A}$ is a Möbius covariant net s.t.

F. *Conformal covariance*. $\exists$ a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of Möb s.t.

$$
\begin{aligned}
U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\
U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}_{I'}(S^1),
\end{aligned}
$$

$\longrightarrow$ unitary representation of the *Virasoro algebra*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,n}$$

$\longrightarrow$ stress-energy tensor:

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

Representations

A (DHR) *representation* ρ of local conformal net $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \rho_I$, with ρ_I a normal rep. of $\mathcal{A}(I)$ on $\mathcal{H}$ s.t.

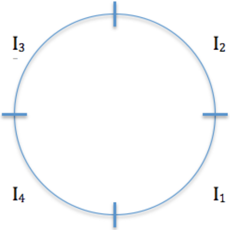
$$\rho_{\tilde{I}} \restriction \mathcal{A}(I) = \rho_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} \ .$$

Index-statistics relation (L.):

$$d(\rho) = \left[\rho_{I'}\big(\mathcal{A}(I')\big)' : \rho_I\big(\mathcal{A}(I)\big)\right]^{\frac{1}{2}}$$

$$\text{DHR dimension} = \sqrt{\text{Jones index}}$$

Complete rationality (Kawahigashi, Müger, L.)



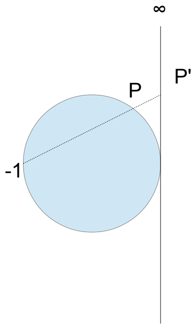
$$\mu_{\mathcal{A}} \equiv \left[\left(\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \right)' : \left(\mathcal{A}(I_2) \vee \mathcal{A}(I_4) \right) \right] < \infty$$

$\Longrightarrow$

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2$$

$\mathcal{A}$ *is modular*. (Later work by Xu, L. and Morinelli, Tanimoto, Weiner removed extra requirements.)

Circle and real line picture



$$z \mapsto i\frac{z-1}{z+1}$$

We shall frequently switch between the two pictures.

KMS and Jones index

Kac-Wakimoto formula (conjecture)

Let $\mathcal{A}$ be a conformal net, ρ representations of $\mathcal{A}$, then

$$\lim_{t\to 0^+} \frac{\mathrm{Tr}(e^{-tL_{0,\rho}})}{\mathrm{Tr}(e^{-tL_0})} = d(\rho)$$

Analog of the Kac-Wakimoto formula (theorem)

ρ a representation of $\mathcal{A}$:

$$(\xi, e^{-2\pi K_\rho}\xi) = d(\rho)$$

where K_ρ is the generator of the dilations δ_I and ξ is any vector cyclic for $\rho(\mathcal{A}(I'))$ such that $(\xi, \rho(\cdot)\xi)$ is the vacuum state on $\mathcal{A}(I')$.

Basic conformal nets

$U(1)$-**current net**

Let $\mathcal{A}$ be the local conformal net on S^1 associated with the $U(1)$-current algebra. In the real line picture $\mathcal{A}$ is given by

$$\mathcal{A}(I) \equiv \{W(f) : f \in C^\infty_{\mathbb{R}}(\mathbb{R}),\ \mathrm{supp}\, f \subset I\}''$$

where W is the representation of the Weyl commutation relations

$$W(f)W(g) = e^{-i\int fg'} W(f+g)$$

associated with the vacuum state ω

$$\omega(W(f)) \equiv e^{-||f||^2}, \quad ||f||^2 \equiv \int_0^\infty p|\tilde{f}(p)|^2 \mathrm{d}p$$

where $\tilde{f}$ is the Fourier transform of f.

$$W(f) = \exp\Big(-i\int f(x)j(x)dx\Big), \quad \big[j(f), j(g)\big] = i\int fg'dx$$

with $j(x)$ the $U(1)$-current.

There is a one parameter family $\{\gamma_q, q \in \mathbb{R}\}$ *of irreducible sectors* and all have index 1 (Buchholz, Mack, Todorov)

$$\gamma_q(W(f)) \equiv e^{i\int Ff}W(f), \quad F \in C^\infty, \quad \frac{1}{2\pi}\int F = q\ .$$

q is the called the charge of the sector.

Virasoro nets

For every possible value of the central charge c, let U_c the irreducible rep. of Diff(S^1) with lowest weight zero:

$$Vir_c(I) \equiv U_c(Diff_I(S^1))''$$

Vir$_c$ *is contained in every conformal net.*

A classification of KMS states (Camassa, Tanimoto, Weiner, L.)

How many KMS states do there exist?

Completely rational case

$\mathcal{A}$ completely rational: only one KMS state (geometrically constructed) $\beta = 2\pi$
exp: net on $\mathbb{R}$ $\mathcal{A} \to$ restriction of $\mathcal{A}$ to $\mathbb{R}^+$

$$\text{exp} \restriction \mathcal{A}(I) = \mathrm{Ad}\, U(\eta)$$

η diffeomorphism, $\eta \restriction I =$ exponential

$$\text{geometric KMS state on } \mathcal{A}(\mathbb{R}) = \text{vacuum state on } \mathcal{A}(\mathbb{R}^+) \ \circ \ \text{exp}$$

$$\varphi_{\text{geo}} = \omega \circ \text{exp}$$

Note: Scaling with dilation, we get the geometric KMS state at any give $\beta > 0$.

Non-rational case: $U(1)$-current model

The primary (locally normal) KMS states of the $U(1)$-current net are in one-to-one correspondence with real numbers $q \in \mathbb{R}$;

Geometric KMS state: $\varphi_{\rm geo} = \varphi^0$

Any primary KMS state:

$$\varphi^q = \varphi_{\rm geo} \circ \gamma_q.$$

where

$$\gamma_q(W(f)) = e^{iq \int f(x)dx} W(f)$$

γ_q is equivalent to the BMT q-sector.

Virasoro net: $c = 1$

(With $c < 1$ there is only one KMS state: the net is completely rational)

Primary KMS states of the Vir_1 net are in one-to-one correspondence with positive real numbers $|q| \in \mathbb{R}^+$; each state $\varphi^{|q|}$ is uniquely determined by its value on the stress-energy tensor T:

$$\varphi^{|q|}(T(f)) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right) \int f\,dx.$$

The geometric KMS state corresponds to $q = 0$, and the corresponding value of the 'energy density' $\frac{\pi}{12\beta^2} + \frac{q^2}{2}$ is the lowest one in the set of the KMS states.

(We construct these KMS states by composing the geometric state with automorphisms on the larger $U(1)$-current net.)

Virasoro net: $c > 1$

There is a set of primary (locally normal) KMS states of the Vir_c net with $c > 1$ w.r.t. translations in one-to-one correspondence with positive real numbers $|q| \in \mathbb{R}^+$; each state $\varphi^{|q|}$ can be evaluated on the stress-energy tensor

$$\varphi^{|q|}\left(T\left(f\right)\right) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right) \int f \, dx$$

and *the geometric KMS state corresponds to* $q = \frac{1}{\beta}\sqrt{\frac{\pi(c-1)}{6}}$ and energy density $\frac{\pi c}{12\beta^2}$.

Are they all? Probably yes...

Chemical potential

$\mathcal{A}$ a local conformal net on $\mathbb{R}$ (or on M) and φ an extremal β-KMS state on $\mathfrak{A}$ w.r.t. the time translation group τ and ρ an irreducible DHR localized endomorphism of $\mathfrak{A} \equiv \overline{\cup_{I\subset\mathbb{R}}\mathcal{A}(I)}$ with finite dimension $d(\rho)$. Assume that ρ is normal, namely it extends the weak closure $\mathcal{M}$ of $\mathfrak{A}$; automatic e.g. if φ satisfies essential duality $\pi_\varphi\big(\mathfrak{A}(I_\pm)\big)' \cap \mathcal{M} = \pi_\varphi\big((\mathfrak{A}(I_\mp)\big)''$, $I_\pm$ the $\pm$half-line.

U time translation unitary covariance cocycle in $\mathfrak{A}$:

$$\mathrm{Ad}U(t)\cdot\tau_t\cdot\rho = \rho\cdot\tau_t\ , \quad t\in\mathbb{R}\ ,$$

with $U(t+s) = U(t)\tau_t\big(U(s)\big)$ (cocycle relation) (unique by a phase, canonical choice by Möb covariance).

U is equal up to a phase to a Connes Radon-Nikodym cocycle:

$$U(t) = e^{-i2\pi\mu_\rho(\varphi)t}d(\rho)^{-i\beta^{-1}t}\big(D\varphi\cdot\Phi_\rho : D\varphi\big)_{-\beta^{-1}t}\ .$$

$\mu_\rho(\varphi)\in\mathbb{R}$ is the *chemical potential* of φ w.r.t. the charge ρ.

Here Φ_ρ is the left inverse of ρ, $\Phi_\rho \cdot \rho = \mathrm{id}$, so $\varphi \cdot \Phi_\rho$ is a KMS state in the sector ρ.

The geometric β-KMS state φ_0 has zero chemical potential.

By the holomorphic property of the Connes Radon-Nikodym cocycle:

$$e^{2\pi\beta\mu_\rho(\varphi)} = \underset{t \longrightarrow i\beta}{\mathrm{anal.\,cont.}}\, \varphi\big(U(t)\big) / \underset{t \longrightarrow i\beta}{\mathrm{anal.\,cont.}}\, \varphi_0\big(U(t)\big) \ .$$

Example, BMT sectors:

With $\varphi_{\beta,q}$ the β-state associated withe charge q, the chemical potential w.r.t. the charge q is given by

$$\mu_p(\varphi_{\beta,q}) = qp/\pi$$

By linearity μ_p is determined at $p = 1$, so we may put $\mu(\varphi_{\beta,q}) = q/\pi$.

2-dimensional CFT

$M = \mathbb{R}^2$ Minkowski plane.

$\begin{pmatrix} T_{00} & T_{10} \\ T_{01} & T_{11} \end{pmatrix}$ conserved and traceless stress-energy tensor.

As is well known, $T_+ = \frac{1}{2}(T_{00} + T_{01})$ and $T_- = \frac{1}{2}(T_{00} - T_{01})$ are chiral fields,

$$T_+ = T_+(t + x), \quad T_- = T_-(t - x).$$

Left and right movers.

Ψ_k family of conformal fields on M: T_{ij} + *relatively local fields*
$\mathcal{O} = I \times J$ double cone, I, J intervals of the chiral lines $t \pm x = 0$

$$\mathcal{A}(\mathcal{O}) = \{e^{i\Psi_k(f)}, \mathrm{supp} f \subset \mathcal{O}\}''$$

then by relative locality

$$\mathcal{A}(\mathcal{O}) \supset \mathcal{A}_L(I) \otimes \mathcal{A}_R(J)$$

$\mathcal{A}_L, \mathcal{A}_R$ chiral fields on $t \pm x = 0$ generated by T_L, T_R and other chiral fields

(completely) rational case: $\mathcal{A}_L(I) \otimes \mathcal{A}_R(J) \subset \mathcal{A}(\mathcal{O})$ *has finite Jones index.*

Phase boundaries (Bischoff, Kawahigashi, Rehren, L.)

$M_L \equiv \{(t,x) : x < 0\}$, $M_R \equiv \{(t,x) : x > 0\}$ left and right half Minkowski plane, with a CFT on each half.

Transmissive solution for left/right chiral stress energy tensors:

$$T_+^L(t) = T_+^R(t), \qquad T_-^L(t) = T_-^R(t).$$

‚A transpartent phase boundary is given by specifying two local conformal nets $\mathcal{B}^L$ and $\mathcal{B}^R$ on $M_{L/R}$ on the same Hilbert space $\mathcal{H}$;

$$M_L \supset O \mapsto \mathcal{B}^L(O) \ ; \qquad M_R \supset O \mapsto \mathcal{B}^R(O) \ ,$$

$\mathcal{B}^L$ and $\mathcal{B}^R$ both contain a common chiral subnet $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$.
so $\mathcal{B}^{L/R}$ extends on the entire M by covariance. By causality:

$$\left[\mathcal{B}^L(O_2), \mathcal{B}^R(O_1)\right] = 0, \quad O_1 \subset M_L,\ O_2 \subset M_R,\ O_1 \subset O_2'$$

By diffeomorphism covariance, $\mathcal{B}^R$ *is thus right local w.r.t.* $\mathcal{B}^L$

Given a phase boundary, we consider the von Neumann algebras

$$\mathcal{D}(O) \equiv \mathcal{B}^L(O) \vee \mathcal{B}^R(O) \ , \quad O \in \mathcal{K} \ .$$

$\mathcal{D}$ is in general non-local, but relatively local w.r.t. $\mathcal{A}$. $\mathcal{D}(O)$ may have non-trivial center. In the completely rational case, $\mathcal{A}(O) \subset \mathcal{D}(O)$ has finite Jones index, so the center of $\mathcal{D}(O)$ is finite dimensional; we may cut down by a minimal projection of the center *(a defect)* and assume $\mathcal{D}(O)$ to be a factor.

Universal construction (and classification) is done by considering the *braided product* of the Q-systems associated with $\mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{B}_L$ and $\mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{B}_R$.

Cf. Fröhlich, Fuchs, Runkel, Schweigert (Euclidean setting)

Non-equilubrium states in CFT (S. Hollands, R.L.)

Two local conformal nets $\mathcal{B}^L$ and $\mathcal{B}^R$ on the Minkowski plane M, both containing the same chiral net $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$. For the moment $\mathcal{B}^{L/R}$ is completely rational, so the KMS state is unique, later we deal wih chemical potentials.

Before contact. The two systems $\mathcal{B}^L$ and $\mathcal{B}^R$ are, separately, each in a thermal equilibrium state. KMS states $\varphi^{L/R}_{\beta_{L/R}}$ on $\mathfrak{B}^{L/R}$ at inverse temperature $\beta_{L/R}$ w.r.t. τ, possibly with $\beta_L \neq \beta_R$.

$\mathcal{B}^L$ and $\mathcal{B}^R$ live independently in their own half plane M_L and M_R and their own Hilbert space. The composite system on $M_L \cup M_R$ is given by

$$M_L \supset O \mapsto \mathcal{B}^L(O)\,, \qquad M_R \supset O \mapsto \mathcal{B}^R(O)$$

with C^*-algebra $\mathfrak{B}^L(M_L) \otimes \mathfrak{B}^R(M_R)$ and state

$$\varphi = \varphi^L_{\beta_L}|_{\mathfrak{B}^L(M_L)} \otimes \varphi^R_{\beta_R}|_{\mathfrak{B}^R(M_R)}\ ;$$

φ is a stationary but not KMS.

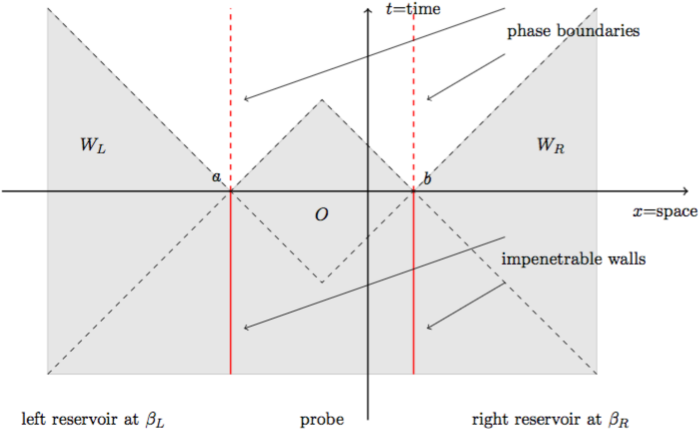


Figure 1: Spacetime diagram of our setup. The initial state ψ is set up in the shaded region before the system is in causal contact with the phase boundaries. In the shaded regions to the left/right of the probe, we have a thermal equilibrium state at inverse temperatures β_L/β_R. In the diamond shaped shaded region O, we have an essentially arbitrary probe state.

After contact.

At time $t = 0$ we put the two systems $\mathcal{B}^L$ on M_L and $\mathcal{B}^R$ on M_R in contact through a totally transmissible phase boundary. We are in the above phase boundary case, $\mathcal{B}^L$ and $\mathcal{B}^R$ are now nets on M acting on a common Hilbert space $\mathcal{H}$; the algebras $\mathfrak{B}^L(W_L)$ and $\mathfrak{B}^R(W_R)$ commute.

We want to describe the state ψ of the global system after contact at time $t = 0$. As above, we set

$$\mathcal{D}(O) \equiv \mathcal{B}^L(O) \vee \mathcal{B}^R(O)$$

ψ should be a natural state on the global algebra $\mathfrak{D}$ that satisfies

$$\psi|_{\mathfrak{B}^L(W_L)} = \varphi^L_{\beta_L}|_{\mathfrak{B}^L(W_L)}, \quad \psi|_{\mathfrak{B}^R(W_R)} = \varphi^R_{\beta_R}|_{\mathfrak{B}^R(W_R)} \ .$$

Since $\mathfrak{B}^L(M_L)$ and $\mathfrak{B}^R(M_R)$ are not independent, the existence of such state ψ is not obvious. Clearly the C^*-algebra on $\mathcal{H}$ generated by $\mathfrak{B}^L(W_L)$ and $\mathfrak{B}^R(W_R)$ is naturally isomorphic to $\mathfrak{B}^L(W_L) \otimes \mathfrak{B}^R(W_R)$ and the restriction of ψ to it is the product state $\varphi^L_{\beta_L}|_{\mathfrak{B}^L(W_L)} \otimes \varphi^R_{\beta_R}|_{\mathfrak{B}^R(W_R)}$.

$\exists$ a natural state $\psi \equiv \psi_{\beta_L,\beta_R}$ on $\mathfrak{D}$ s.t. $\psi|_{\mathfrak{B}(W_{L/R})}$ is $\varphi^{L/R}_{\beta_L/\beta_R}$ (natural on $\mathcal{D}(x \pm t \neq 0)$, our initial state.

The state ψ is given by $\psi \equiv \varphi \cdot \alpha_{\lambda_L,\lambda_R}$, where φ is the geometric state on $\mathfrak{D}$ (at inverse temperature 1) and $\alpha = \alpha_{\lambda_L,\lambda_R}$ is the above doubly scaling automorphism with $\lambda_L = \beta_L^{-1}$, $\lambda_R = \beta_R^{-1}$.

By inserting a probe ψ the state will be normal.

The large time limit. After a large time we expect the global system to reach a non equilibrium steady state ω.

Let $\varphi^+_{\beta_L}$, $\varphi^-_{\beta_R}$ be the geometric KMS states respectively on $\mathfrak{A}_+$ and $\mathfrak{A}_-$ with inverse temperature β_L and β_R; we define

$$\omega \equiv \varphi^+_{\beta_L} \otimes \varphi^-_{\beta_R} \cdot \varepsilon \ ,$$

so ω is the state on $\mathfrak{D}$ obtained by extending $\varphi^+_{\beta_L} \otimes \varphi^-_{\beta_R}$ from $\mathfrak{A}$ to $\mathfrak{D}$ by the conditional natural expectation $\varepsilon : \mathfrak{D} \to \mathfrak{A}$. Clearly ω is a stationary state, indeed:

ω is a NESS on $\mathfrak{D}$ with $\beta = \min\{\beta_L, \beta_R\}$.

	Initial state $\psi \equiv \psi_{\beta_L,\beta_R}$	Final state ω
Properties	β_L on W_L^a, β_R on W_R^b, unspecified on $W_L^b \cap W_R^a$	Determined on V_+
Definition on chiral subnet	doubly scaling the geometric KMS state by dilations	$\varphi^+_{\beta_L} \otimes \varphi^-_{\beta_R}$
Global definition	again by doubly scaling	extension by conditional expectation $\omega = \varphi^+_{\beta_L} \otimes \varphi^-_{\beta_R} \cdot \varepsilon$

We now want to show that the evolution $\psi \cdot \tau_t$ of the initial state ψ of the composite system approaches the non-equilibrium steady state ω as $t \to +\infty$.

Note that:

$$\psi|_{\mathcal{D}(O)} = \omega|_{\mathcal{D}(O)} \text{ if } O \in \mathcal{K}(V_+)$$

We have:

For every $Z \in \mathfrak{D}$ we have:

$$\lim_{t \to +\infty} \psi\big(\tau_t(Z)\big) = \omega(Z) \; .$$

Indeed, if $Z \in \mathcal{D}(O)$ with $O \in \mathcal{K}(M)$ and $t > t_O$, we have $\tau_t(Z) \in \mathfrak{D}(V_+)$ as said, so

$$\psi\big(\tau_t(Z)\big) = \omega\big(\tau_t(Z)\big) = \omega(Z) \; , \quad t > t_O \; ,$$

because of the stationarity property of ω.
See the picture.

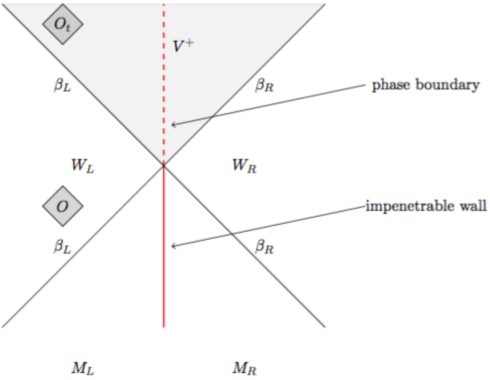


Figure 2: Spacetime diagram of simplified setup. There is just one phase boundary and no probe. Every time-translated diamond will eventually enter the future lightcone V^+.

Case with chemical potential

We suppose here that $\mathcal{A}_\pm$ in the net $\mathcal{C}$ contains is generated by the $U(1)$-current $J^\pm$ (thus $\mathcal{B}^{L/R}$ is non rational with central charge $c = 1$).

Given $q \in \mathbb{R}$, the β-KMS state $\varphi_{\beta,q}$ on $\mathfrak{D}$ with charge q is defined by

$$\varphi_{\beta,q} = \varphi^+_{\beta,q} \otimes \varphi^-_{\beta,q} \cdot \varepsilon \ ,$$

where $\varphi^\pm_{\beta,q}$ is the KMS state on $\mathcal{A}_\pm$ with charge q.

$\varphi_{\beta,q}$ satisfies the β-KMS condition on $\mathfrak{D}$ w.r.t. to τ.

Similarly as above we have:

Given $\beta_{L/R} > 0$, $q_{L/R} \in \mathbb{R}$, there exists a state ψ on $\mathfrak{D}$ such that

$$\psi|_{\mathfrak{B}^L(W_L)} = \varphi_{\beta_L, q_L}|_{\mathfrak{B}^L(W_L)} \ , \qquad \psi|_{\mathfrak{B}^R(W_R)} = \varphi_{\beta_R, q_R}|_{\mathfrak{B}^R(W_R)} \ .$$

and for every $Z \in \mathfrak{D}$ we have:

$$\lim_{t\to+\infty} \psi\big(\tau_t(Z)\big) = \omega(Z) \ .$$

We can explicitly compute the expected value of the asymptotic NESS state ω on the stress energy tensor and on the current (chemical potential enters):

Now $\omega = \varphi^{+}_{\beta_L,q_L} \otimes \varphi^{-}_{\beta_R,q_R} \cdot \varepsilon$ is a steady state is a NESS and ω is determined uniquely by $\beta_{L/R}$ and the charges $q_{L/R}$

$$\varphi^{+}_{\beta_L,q_L}\big(J^{+}(0)\big) = q_L \ , \qquad \varphi^{-}_{\beta_R,q_R}\big(J^{-}(0)\big) = q_R \ .$$

We also have

$$\varphi^{+}_{\beta_L,q_L}\big(T^{+}(0)\big) = \frac{\pi}{12\beta_L^2} + \frac{q_L^2}{2} \ , \qquad \varphi^{-}_{\beta_R,q_R}\big(T^{-}(0)\big) = \frac{\pi}{12\beta_R^2} + \frac{q_R^2}{2} \ .$$

In presence of chemical potentials $\mu_{L/R} = \frac{1}{\pi} q_{L/R}$, the large time limit of the two dimensional current density expectation value (x-component of the current operator J^{μ}) in the state ψ is, with $J^x(t,x) = J^{-}(t+x) - J^{+}(t-x)$

$$\lim_{t\to+\infty} \psi\big(J^x(t,x)\big) = \varphi^{-}_{\beta_L,q_L}\big(J^{-}(0)\big) - \varphi^{+}_{\beta_R,q_R}\big(J^{+}(0)\big) = -\pi(\mu_L - \mu_R) \ ,$$

whereas on the stress energy tensor

$$\lim_{t\to+\infty} \psi\big(T_{tx}(t,x)\big) = \varphi^{+}_{\beta_L,q_L}\big(T^{+}(0)\big) - \varphi^{-}_{\beta_R,q_R}\big(T^{-}(0)\big)$$
$$= \frac{\pi}{12}\big(\beta_L^{-2} - \beta_R^{-2}\big) + \frac{\pi^2}{2}\big(\mu_L^2 - \mu_R^2\big) \ ,$$

(cf. Bernard-Doyon)