Real Hilbert spaces, $SL(2,\mathbb{R})$ modular theory and CFT

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Toronto, July, 2007

Standard real Hilbert subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a real linear subspace.

Symplectic complement:

$$H' \equiv \{\xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \quad \forall \eta \in H\}.$$

$H' = (iH)^{\perp}$ (real orthogonal complement)

$$H_1 \subset H_2 \Rightarrow H_1' \supset H_2' \ .$$

A *standard* subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic ($\overline{H + iH} = \mathcal{H}$) and separating ($H \cap iH = \{0\}$). H is standard iff H' is standard.

H standard subspace $\rightarrow$ anti-linear operator $S \equiv S_H : D(S) \subset \mathcal{H} \rightarrow \mathcal{H}$, where $D(S) \equiv H + iH$,

$$S : \xi + i\eta \mapsto \xi - i\eta \ , \quad \xi, \eta \in H \ .$$

$S^2 = 1{\restriction}_{D(S)}$. S is closed and densely defined.

Conversely, S densely defined, closed, anti-linear involution on $\mathcal{H}$ gives

$$H = \{\xi : S\xi = \xi\} \quad \text{is a standard subspace}$$

$$H \longleftrightarrow S \quad \text{bijection}$$

Modular theory. Set

$$S_H = J_H \Delta_H^{1/2}$$

polar decomposition of $S = S_H$. Then J_H is an anti-unitary involution $\Delta \equiv S^* S > 0$

$$\Delta_H^{-it} H = H, \quad J_H H = H'$$

Borchers theorem (real subspace version)
H standard subspace, U a one-parameter group with positive generator

$$U(s)H \subset H \quad s \geqslant 0.$$

Then:

$$\begin{cases} \Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s), \\ J_H U(s) J_H = U(-s), \qquad t, s \in \mathbb{R}. \end{cases}$$

Note: Setting $K \equiv U(1)H$ we have

$$\Delta_H^{-it}K = \Delta_H^{-it}U(1)H = U(e^{2\pi t})\Delta_H^{-it}H$$
$$= U(e^{2\pi t})H \subset K, \quad t \geq 0\,.$$

$K \subset H$ is a half-sided modular inclusion.

About the proof (adapted from Florig). With $\xi \in H, \xi' \in H'$

$$f_U(z) = (\Delta^{i\bar{z}}\xi', U(e^{2\pi z}s)\Delta^{-iz}\xi).$$

is analytic in $\mathbb{S}_{1/2} = \{z \in \mathbb{C} :\ 0 < \Im\ z < \frac{1}{2}\}$ (the generator of $U(t)$ is positive and $\Im e^{2\pi z}s \geqslant 0$ for $z \in \mathbb{S}_{1/2}$).

$V(t) = JU(-t)J$ satisfies the same assumptions then U because of $JH = H'$

$$\begin{aligned} f_U\left(t + \frac{i}{2}\right) &= (\Delta^{-1/2}\Delta^{-it}\xi', U(e^{2\pi t + i\pi}s)\Delta^{-it}\Delta^{1/2}\xi) \\ &= (\Delta^{-1/2}\Delta^{-it}\xi', JV(e^{2\pi t}s)\Delta^{-it}\xi) \\ &= (\Delta^{-it}\xi', (J\Delta^{1/2})V(e^{2\pi t}s)\Delta^{-it}\xi) \\ &= (\Delta^{-it}\xi', V(e^{2\pi t}s)\Delta^{-it}\xi) = f_V(t) \end{aligned}$$

(KMS and positivity of energy) analogously $V(t) = JU(-t)J$ satisfies the same assumptions then U because of $JH = H'$

$$f_V\left(t + \frac{i}{2}\right) = f_U(t)$$

f_U and f_V glue to an entire bounded function, thus constant.

Converse: Wiesbrock, Borchers, Araki-Zsido theorem (real subspace version)

Let H, K be standard subspaces. Assume half-sided modular inclusion:

$$\Delta_H^{-it} K \subset K, \qquad t \geq 0$$

Then $\{\Delta_K^{it}, \Delta_H^{is}\}$ generates a unitary represen-taion of the "$ax+b$" group with positive energy

$$\text{dilation group} = \Delta_H^{-is/2\pi}$$

$$\text{gen. of translations } P = \frac{1}{2\pi}(\log \Delta_K - \log \Delta_H)$$

Conclusion:

$$\text{positive energy rep. of "}ax+b\text{" group}$$
$$\Updownarrow$$
$$\text{Borchers pair } (U,H)$$
$$\Updownarrow$$
$$\text{half-sided modular inclusion of standard subspace}$$

therefore, if U has no non-zero fixed vector, (U,H) is *unique* up to multiplicity.

von Neumann algebras and real Hilbert subspaces

M von Neumann algebra on $\mathcal{H}$, $\Omega \in \mathcal{H}$ a cyclic separating vector,

$$H_M \equiv \overline{M_{sa}\Omega}$$

is a standard subspace of H

$$\Delta_M = \Delta_{H_M}, \quad J_M = J_{H_M}$$

In particular

$$H'_M = H_{M'}$$

Borchers theorem (original for vN algebras)
M von Neumann algebra, Ω cyclic serating vector, U a one-parameter group with positive generator with $U(s)\Omega = \Omega$ and

$$U(s)MU(-s) \subset M \quad s \geqslant 0.$$

Then:

$$\begin{cases} \Delta_M^{it} U(s) \Delta_M^{-it} = U(e^{-2\pi t}s), \\ J_M U(s) J_M = U(-s), \qquad t, s \in \mathbb{R}. \end{cases}$$

Note: If Ω is the unique U-fixed vector then M is a type III_1 factor.

Wiesbrock, Borchers, Araki-Zsido theorem (original for vN algebras)

Let M, N be vN algebras, Ω jointly cyclic and separating vector. Assume half-sided modular inclusion:

$$\Delta_M^{-it} N \Delta_M^{it} \subset N\,, \qquad t \geq 0\,.$$

Then $\{\Delta_N^{it}, \Delta_M^{is}\}$ generates a unitary representaion of the "$ax+b$" group with positive energy

$$\text{dilations} = \Delta_M^{-is/2\pi}$$

gen. of translations $P = \dfrac{1}{2\,\pi}(\log \Delta_N - \log \Delta_M)$

Therefore Borchers triple $\Leftrightarrow$ Wiesbrock triple.

How many Borchers triples there are?

Is is possible that $U(s)MU(-s)' \cap M = \mathbb{C}$ for $s > 0$?

Möbius covariant nets of real Hilbert subspaces

A *local Möbius covariant net* of standard subspaces $\mathcal{A}$ of real Hilbert subspaces on the intervals of S^1 is a map

$$I \to H(I)$$

with

1. Isotony : *If* I_1, I_2 *are intervals and* $I_1 \subset I_2$, *then*

$$H(I_1) \subset H(I_2) \ .$$

2. Möbius invariance**:** *There is a unitary representation* U *of* $\mathbf{Mob}$ *on* $\mathcal{H}$ *such that*

$$U(g)H(I) = H(gI) \ , \qquad g \in \mathbf{Mob}, \ I \in \mathcal{I}.$$

Here $\mathbf{Mob} \simeq PSL(2, \mathbb{R})$ acts on S^1 as usual.

3. Positivity of the energy : $L_0 \geq 0$

4. Cyclicity : *the* complex *linear span of all spaces* $H(I)$ *is dense in* $\mathcal{H}$.

5. Locality : *If* I_1 *and* I_2 *are disjoint intervals then*

$$H(I_1) \subset H(I_2)'$$

First consequences

Irreducibility: $\overline{\text{real lin.span}}_{I \in \mathcal{I}} \mathcal{H}(I) = H$.

Reeh-Schlieder theorem: $H(I)$ is a standard subspace for every I.

Bisognano-Wichmann property: Tomita-Takesaki modular operator Δ_I and conjugation J_I of

$H(I)$, are

$$U(\Lambda_I(2\pi t)) = \Delta_I^{-it},\ t \in \mathbb{R}, \qquad \text{dilations}$$
$$U(r_I) = J_I \qquad \text{reflection}$$

($\Lambda_{I_1}(t)x = e^t x, x \in \mathbb{R}$, $I_1 \simeq \mathbb{R}^+$ upper semi-circle)

Haag duality: $H(I)' = H(I') \quad (I' \equiv S^1 \smallsetminus I)$.

Factoriality: $H(I) \cap H(I)' = 0$

Additivity: $I \subset \cup_i I_i \implies H(I) \subset \overline{\text{real lin.span}_i H(I_i)}$.

Modular theory and representations of $SL(2,\mathbb{R})$ (Brunetti, Guido, L.)

U a unitary, positive energy representation of $\mathbf{Mob}$ on $\mathcal{H}$ and J anti-unitary involution on $\mathcal{H}$ s.t.

$$JU(g)J = U(rgr), \quad g \in \mathbf{Mob}$$

where $r : z \mapsto \bar{z}$ reflection on S^1 w.r.t. the upper semicircle I_1. Then define

$$J_I \equiv U(g) J U(g)^*$$

where $g \in \mathbf{Mob}$ maps I_1 onto I.

$$\Delta_I^{it} \equiv U(\Lambda_I(-2\pi t)), \quad t \in \mathbb{R}$$

namely $-\frac{1}{2\pi} \log \Delta_I$ generator of dilations of I,

$$S_I \equiv J_I \Delta_I^{1/2}$$

is a densely defined, antilinear, closed involution on $\mathcal{H}$.

$$H(I) \text{ standard subspace associated with } S_I$$
$$\downarrow$$
Möbius covariant local net of real Hilbert spaces

A $\pm$*hsm factorization of real subspaces* is a triple K_0, K_1, K_2, where $\{K_i, i \in \mathbb{Z}_3\}$ is a set of standard subspaces s.t. $K_i \subset K'_{i+1}$ is a $\pm$hsm inclusion.

$$\text{Factorization}$$
$$\Updownarrow$$
$$\text{Local Möbius covariant net of real Hilbert spaces}$$
$$\Updownarrow$$
$$\text{Positive energy representation of } SL(2,\mathbb{R})/\{1,-1\}$$

Note: Irr. positive energy rep. of $SL(2,\mathbb{R})/\{1,-1\}$ are parametrized by $\mathbb{N}$

Möbius covariant nets of vN algebras. A (local) *Möbius covariant net* $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

A. *Isotony.* $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$

B. *Locality.* $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$

C. *Möbius covariance.* $\exists$ unitary rep. U of the Möbius group $\mathbf{Mob}$ on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \mathbf{Mob},\ I \in \mathcal{I}.$$

D. *Positivity of the energy.* Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.

E. *Existence of the vacuum.* $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}}\mathcal{A}(I)$ and unique U-invariant.

First consequences

Irreducibility: $\bigvee_{I\in\mathcal{I}}\mathcal{A}(I) = B(\mathcal{H})$.

Reeh-Schlieder theorem: Ω is cyclic and separating for each $\mathcal{A}(I)$.

Bisognano-Wichmann property: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I), \Omega)$, are

$$\begin{aligned} U(\Lambda_I(2\pi t)) &= \Delta_I^{it},\ t \in \mathbb{R}, & \text{dilations} \\ U(r_I) &= J_I & \text{reflection} \end{aligned}$$

(Guido-L., Frölich-Gabbiani)

Haag duality: $\mathcal{A}(I)' = \mathcal{A}(I')$

Factoriality: $\mathcal{A}(I)$ is III_1-factor (or $\mathcal{A}(I) = \mathbb{C}$).

Additivity: $I \subset \cup_i I_i \implies \mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$ (Fredenhagen, Jorss).

- Net of factors on $\mathcal{H}$ $\rightarrow$ Net of standard subspaces (not one-to-one) on $\mathcal{H}$

- Net of standard subspaces on $\mathcal{H}$ $\rightarrow$ Net of factors on on $e^{\mathcal{H}}$ (second quantization)

$$\mathcal{A}(I) \equiv \{W(h): \ h \in H(I)\}''$$

Further selection properties.

• *Split property.* $\mathcal{A}$ is *split* if the von Neumann algebra

$$\boxed{\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)}$$

(natural isomorphism) if $\bar{I}_1 \cap \bar{I}_2 = \varnothing$.

- Split is a property of the net (not of U).

- Split is crucial, e.g. for local charges, complete rationality, hypefinetness, classification...

- *Trace class condition*.

$$\boxed{\mathrm{Tr}(e^{-tL_0}) < \infty,\ \forall t > 0}$$

- Trace class condition is standard in CFT

- Trace class condition $\Longrightarrow$ split

- Trace class condition can be refined to *log-ellipticity*

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t^\alpha}(a_0 + a_1 t + \cdots) \quad \text{as } t \to 0^+$$

$\alpha = 1$ (Kawahigashi,L.)

- Trace class is a property of U (not of the net).

- *Buchholz-Wichmann nuclearity*:

$$\boxed{\Phi_I^{\mathrm{BW}}(\beta) : x \in \mathcal{A}(I) \to e^{-\beta P} x\Omega \in \mathcal{H}}$$

is nuclear, I interval of $\mathbb{R}$, $\beta > 0$. P translation generator (Hamiltonian).

Recall: $A : X \to Y$ is nuclear if $\exists$ sequences $f_k \in X^*$ and $y_k \in Y$ s.t. $\sum_k ||f_k|| \, ||y_k|| < \infty$ and

$$Ax = \sum_k f_k(x) y_k \ .$$

$||A||_1 \equiv \inf \sum_k ||f_k|| \, ||y_k||$.

- BW-nuclearity is a physical property (Haag-Swieca): essentially finately many localized states in a finite volume.

- BW-nuclearity is a property of the full Möbius covariant net.

- Can be refined with $||\Phi_I^{\mathrm{BW}}(\beta)||_1 \leq e^{cr^m/\beta^n}$ as $\beta \to 0^+$ and $\to$ *KMS states for translations* (Buchholz-Junglas).

Derive BW-nuclearity from the trace class condition (Buchholz, D'Antoni, L.)

- *Modular nuclearity*

M von Neumann algebra, Ω cyclic separating unit vector. Set

$$L^\infty(M) = M, \qquad L^2(M) = \mathcal{H}, \qquad L^1(M) = M_* \,.$$

Then we have the embeddings

$$\begin{tikzcd}[column sep=small] L^\infty(M) \arrow[rr, "x\to(x\Omega, J\cdot\Omega)", "\Phi^M_{\infty,1}"'] \arrow[dr, "x\to\Delta^{1/4}x\Omega"', "\Phi^M_{\infty,2}"] & & L^1(M) \\ & L^2(M) \arrow[ur, "\xi\to(\xi, J\cdot\Omega)"', "\Phi^M_{2,1}"] & \end{tikzcd}$$

Now let $N \subset M$ be an inclusion of vN algebras with cyclic and separating unit vector Ω.

$L^{p,q}$-*nuclearity* if $\Phi^M_{p,q}|_N$ is a nuclear operator.

$L^{\infty,2}$-nuclearity was called *modular nuclearity*, i.e.

$$\boxed{\Phi^M_{\infty,2}|_N : x \in N \to \Delta_M^{1/4} x\Omega}$$

is nuclear.

As $\Phi^M_{\infty,1} = \Phi^M_{2,1}\Phi^M_{\infty,2}$, we have

$$||\Phi^M_{\infty,1}|_N||_1 \leq ||\Phi^M_{2,1}||\cdot||\Phi^M_{\infty,2}|_N||_1 \leq ||\Phi^M_{\infty,2}|_N||_1\ ,$$

Thus

Modular nuclearity $\Rightarrow L^{\infty,1} -$ nuclearity.

indeed $\Phi^M_{\infty,1}|_N = \Phi^N_{2,1} \cdot \Phi^M_{\infty,2}|_N$ and $||\Phi^N_{2,1}|| \leq 1$ so $||\Phi^M_{\infty,1}|_N||_1 \leq ||\Phi^M_{\infty,2}|_N||_1$. (A certain converse holds) .

- If N or M is a factor and $\Phi^M_{\infty,1}|_N$ is nuclear then $N \subset M$ is a split inclusion ($N \vee M' \simeq N \otimes M'$).

Short proof. By definition $\Phi^M_{\infty,1}|_N$ nuclear means: $\exists$ sequences of elements $\varphi_k \in N^*$ and $\psi_k \in M'_*(\simeq L^1(M))$ such that $\sum_k ||\varphi_k||\ ||\psi_k|| < \infty$ and

$$\omega(nm') = \sum_k \varphi_k(n)\psi_k(m') \ , \quad n \in N,\, m' \in M' \ .$$

where $\omega \equiv (\,\cdot\,\Omega, \Omega)$. As $\Phi^M_{\infty,1}|_N$ is normal the φ_k can be chosen normal (take the normal part). Thus the state ω on $N \odot M'$ extends to $N \otimes M'$ and this gives the split property.

Consider now the commutative diagram

$$\begin{tikzcd}
L^\infty(N) \arrow[rr, "\Phi^M_{\infty,1}|_N"] \arrow[d, "\Phi^N_{\infty,2}"'] & & L^1(M) \\
L^2(N) \arrow[rr, "T_{M,N}\equiv\Delta_M^{1/4}\Delta_N^{-1/4}"] & & L^2(M) \arrow[u, "\Phi^M_{2,1}"']
\end{tikzcd}$$

$T_{M,N} \equiv \Phi^M_{2,2}|_N$. L^2-*nuclearity* condition (or $L^{2,2}$-nuclearity) means that

$$\boxed{||T_{M,N}||_1 < \infty}$$

- L^2-nuclearity $\Rightarrow$ modular nuclearity,

indeed $||\Phi^M_{\infty,2}|_N||_1 \leq ||T_{M,N}||_1$ because $\Phi^M_{\infty,2}|_N = T_{M,N} \cdot \Phi^N_{\infty,2}$ and $||\Phi^N_{\infty,2}|| \leq 1$.

L^2-**Nuclearity.** Let $H \subset \tilde{H}$ be an inclusion of standard subspaces. Set

$$T_{\tilde{H},H} \equiv \Delta_{\tilde{H}}^{1/4}\Delta_H^{-1/4}$$

then $||T_{\tilde{H},H}|| \leq 1$. The inclusion is *nuclear* if $T_{\tilde{H},H}$ is a nuclear (i.e. trace class) operator.

U unitary, positive energy representation of $\mathbf{Mob}$, $H(I)$ the associated net of standard subspaces. U satisfies L^2 *nuclearity* if $H(I) \subset H(\tilde{I})$ is nuclear if $I \subset\subset \tilde{I}$.

$SL(2,\mathbb{R})$ **identities.**

Formula 0 (Schroer-Wiesbrock)

U positive energy unitary $\mathbf{Mob}$ rep., $\forall s \geq 0$:

$$\boxed{\Delta_1^{1/4} \Delta_2^{-is} \Delta_1^{-1/4} = e^{-2\pi s L_0}}$$

$\Delta_1 = \Delta_{I_1}$, $\Delta_2 = \Delta_{I_2}$, with I_1, I_2 upper and right semicircles.

About the proof. Use of double interpretation of Δ_1, Δ_2: modular (analyticity) and $SL(2,\mathbb{R})$ (Lie algebra relations)

Formula 1 U positive energy unitary representation:

$$\boxed{T_{\tilde{I},I} = e^{-sL_0}\Delta_2^{is/2\pi}}$$

$s = \ell(\tilde{I}, I)$ is the inner distance (if $I = (-1,1)$ and $\tilde{I} = (-e^s, e^s)$ on the real line, then $\ell(\tilde{I}, I) = s$) thus

$$||T_{\tilde{I},I}||_1 = ||e^{-sL_0}||_1$$

About the proof.

$$e^{-2\pi sL_0} = \Delta_1^{1/4}\Delta_2^{-is}\Delta_1^{-1/4} =$$
$$\Delta_1^{1/4}\Delta_2^{-is}\Big(\Delta_1^{-1/4}\Delta_2^{is}\Big)\Delta_2^{-is} = T_{I_1,I_{1,s}}\Delta_2^{-is}$$

Formula 2

$$T_{I,I_{a',a}} = e^{-a'P'_I}e^{-aP_I}e^{-iaP_I}e^{ia'P'_I}\ .$$

$I_{a',a} \equiv \tau'_{-a'}\tau_a I$ with $a, a' > 0$.

$$\boxed{e^{-2sL_0} = e^{-\tanh(\frac{s}{2})P}e^{-\sinh(s)P'}e^{-\tanh(\frac{s}{2})P}}$$

therefore

$$\boxed{e^{-2sL_0} \leq e^{-2\tanh(\frac{s}{2})P}}$$

in particular $e^{-i\pi L_0} = e^{iP}e^{iP'}e^{iP}$.

About the proof. Consider $\tilde{I} = (0, \infty)$, $I = (t, \infty)$, then

$$\begin{aligned} T_{\tilde{I},I} &= \Delta_{\tilde{I}}^{1/4}\Delta_I^{-1/4} \\ &= \left(\Delta_{\tilde{I}}^{1/4}U(t)\Delta_{\tilde{I}}^{-1/4}\right)U(-t) \\ &= e^{-tP}e^{itP} \end{aligned}$$

where we have used the Borchers commutation relation $\Delta_{\tilde{I}}^{is}e^{itP}\Delta_{\tilde{I}}^{-is} = e^{i(e^{-2\pi s})tP}$. Any $I \subset\subset \tilde{I}$ is obtain by iteration the above, get a formula and compare with formula 1.

Formula 3

$$||e^{-\tan(2\pi\lambda)d_I P}\Delta_I^{-\lambda}|| \leq 1 \ , \qquad 0 < \lambda < 1/4 \ .$$

with d_I the usual lenght. Thus

$$\boxed{e^{-2\tan(2\pi\lambda)d_I P} \leq \Delta_I^{2\lambda} \ .}$$

so we have

$$\boxed{e^{-2d_I P} \leq \Delta_I^{1/4} \leq e^{\frac{2}{d_I}P'}\ .}$$

Modular nuclearity and L^2-nuclearity

L^2-nuclearity implies modular nuclearity and $||\Delta_{\tilde{H}}^{1/4} E_H||_1 \leq ||T_{\tilde{H},H}||_1$.

Comparison of nuclearity conditions

Let H be a Möbius covariant net of real Hilbert subspaces of a Hilbert space $\mathcal{H}$. Consider the following nuclearity conditions for H.

Trace class condition: $\mathrm{Tr}(e^{-sL_0}) < \infty$, $s > 0$;

L^2-nuclearity: $||T_{\tilde{I},I}||_1 < \infty$, $\forall I \subset\subset \tilde{I}$;

Modular nuclearity: $\Xi_{\tilde{I},I} : \xi \in H(I) \to \Delta_{\tilde{I}}^{1/4}\xi \in \mathcal{H}$ is nuclear $\forall I \subset\subset \tilde{I}$;

Buchholz-Wichmann nuclearity: $\Phi_I^{\mathrm{BW}}(s) : \xi \in H(I) \to e^{-sP}\xi \in \mathcal{H}$ is nuclear, I interval of $\mathbb{R}$, $s > 0$ (P the generator of translations);

Conformal nuclearity: $\Psi_I(s) : \xi \in H(I) \to e^{-sL_0}\xi \in \mathcal{H}$ is nuclear, I interval of S^1, $s > 0$.

We shall show the following chain of implications:

$$\begin{gathered}
\text{Trace class condition} \\
\Updownarrow \\
L^2-\text{nuclearity} \\
\Downarrow \\
\text{Modular nuclearity} \\
\Downarrow \\
\text{Buchholz-Wichmann nuclearity} \\
\Downarrow \\
\text{Conformal nuclearity}
\end{gathered}$$

Where all the conditions can be understood for a specific value of the parameter, that will be determined, or for all values in the parameter range.

We have already discussed the implications "Trace class condition $\Leftrightarrow$ L^2-nuclearity $\Rightarrow$ Modular nuclearity".

Modular nuclearity $\Rightarrow$ BW-nuclearity

We have

$$||\Phi_{I_0}^{\mathrm{BW}}\Big(d_I\Big)||_1 \leq ||\Xi_{I,I_0}||_1$$

where d_I is the length of I on $\mathbb{R}$.

BW-nuclearity $\Rightarrow$ Conformal nuclearity

By formula 2 there exists a bounded operator B with norm $||B|| \leq 1$ such that $e^{-sL_0} = Be^{-\tanh(\frac{s}{2})H}$, therefore

$$\Psi_I(s) = B\Phi_I^{\mathrm{BW}}(\tanh(s/2))$$

$$||\Psi_I(s)||_1 \leq ||\Phi_I^{\mathrm{BW}}(\tanh(s/2))||_1.$$

Consequences

• *Distal split property.* If $\text{Tr}(e^{-sL_0}) < \infty$ for a fixes $s > 0$, then $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ is split if $I \subset \tilde{I}$ and $\ell(\tilde{I}, I) > s$ e.g free probability nets (D'Antoni, Radulescu, L.).

• *Constructing KMS states.* $\mathcal{A}|_{\mathbb{R}}$ restriction of $\mathcal{A}$ to $\mathbb{R} \simeq S^1 \smallsetminus \{-1\}$, $\mathcal{A}_0$ the quasi-local C*-algebra. i.e. the norm closure of $\cup_I \mathcal{A}(I)$ as I varies in the bounded intervals of $\mathbb{R}$. Let $\mathfrak{A} \subset \mathcal{A}_0$ the C*-algebras of elements with norm continuous orbit, namely

$$\mathfrak{A} = \{X \in \mathcal{A}_0 : \lim_{t \to 0} ||\tau_t(X) - X|| = 0\}$$

τ translation automorphism group.

Thm. If the trace class condition holds for $\mathcal{A}$ with the asymptotic bound

$$\text{Tr}(e^{-sL_0}) \leq e^{\text{const.} \frac{1}{s^\alpha}}, \quad s \to 0^+$$

for some $\alpha > 0$, then the BW-nuclearity holds with $m = n = \alpha$.

If the trace class condition holds with log-ellipticity (above asymptotics) then for every $\beta > 0$ there exists a translation β-KMS state on $\mathfrak{A}$.

- *L^2-Nuclearity and KMS states in higher dimensions.*

$\mathcal{O}$ a double cone in the Minkowski spacetime $\mathbb{R}^{d+1}$, $\mathcal{A}(\mathcal{O})$ the local von Neumann algebra associated with $\mathcal{O}$ by the $d+1$-dimensional scalar, massless, free field.

With I an interval of the time-axis $\{x = \langle x_0, \mathbf{x}\rangle : \mathbf{x} = 0\}$ we set

$$\mathcal{A}_0(I) \equiv \mathcal{A}(\mathcal{O}_I)$$

where $\mathcal{O}_I$ is the double cone $I'' \subset \mathbb{R}^{d+1}$, the causal envelope of I. Then $\mathcal{A}_0$ is a translation-dilation covariant net on $\mathbb{R}$. $\mathcal{A}_0$ is local if d is odd and twisted local if d is even. Moreover $\mathcal{A}_0$ extends to a Möbius covariant net on S^1 (d odd) as one has a natural factorization.

We have:

$$\mathcal{A}_0 = \bigotimes_{k=0}^{\infty} N_d(k)\mathcal{A}^{(k)}$$

where $\mathcal{A}^{(k)}$ is the Möbius covariant net on S^1 associated with the k^{th}-derivative of the $U(1)$-current algebra and $N_d(k)$ is a multiplicity factor (see below).

This follows because the one-particle $\mathbf{Mob}$ representation U_0 decomposes

$$U_0 = \bigoplus_{k=1}^{\infty} N_d(k)U^{(k)}$$

where $U^{(k)}$ is the positive energy irreducible representation of $PSL(2,\mathbb{R})$ with lowest weight k.

A spherical harmonics computations determines the multiplicity factor $N_d(k)$. As $k \to \infty$:

$$N_d(k+1) = \mathrm{dim}(\mathcal{P}_k \ominus \mathcal{P}_{k-2})$$
$$= m_{d-1}(k-1) + m_{d-1}(k) \sim \frac{2}{(d-2)!}k^{d-2},$$

with $\mathcal{P}_k \ominus \mathcal{P}_{k-2}$ the k-spherical harmonics and $m_d(k) \sim \frac{1}{(d-1)!}k^{d-1}$. Thus

$$\log \mathrm{Tr}(e^{-sL_0}) \sim \frac{2}{s^d} \qquad s \to 0^+ \ ,$$

where L_0 is the conformal Hamiltonian of $\mathcal{A}_0$.

Problems.

- $\mathrm{Tr}(e^{-sL_0}) < \infty \Leftrightarrow$ split property?

- e^{-sL_0} compact $\Leftrightarrow$ split property?

- $\mathrm{Tr}(e^{-sL_0}) < \infty \Rightarrow \mathrm{Tr}(e^{-sL_{0,\rho}}) < \infty$ in every irreducible representation ρ of $\mathcal{A}$?