Mixing Time and Stationary Expected Social Welfare of Logit Dynamics

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Abstract

We study *logit dynamics* [3] for strategic games. At every stage of the game a player is selected uniformly at random and she plays according to a *noisy* best-response dynamics where the noise level is tuned by a parameter β. Such a dynamics defines a family of ergodic Markov chains, indexed by β, over the set of strategy profiles. Our aim is twofold: On the one hand, we are interested in the expected social welfare when the strategy profiles are random according to the stationary distribution of the Markov chain, because we believe it gives a meaningful description of the long-term behavior of the system. On the other hand, we want to estimate how long it takes, for a system starting at an arbitrary profile and running the logit dynamics, to get close to the stationary distribution; i.e., the *mixing time* of the chain.

In this paper we study the stationary expected social welfare for the 3-player CK game [5], for 2-player coordination games (the same class of games studied in [3]), 2-player anti-coordination games, and for a simple n-player game. For all these games, we give almost-tight upper and lower bounds on the mixing time of logit dynamics.

1 Introduction

The evolution of a system is determined by its dynamics and complex systems are often described by looking at the equilibrium states induced by their dynamics. Once the system enters an equilibrium state, it stays there and thus it can be rightly said that an equilibrium state describes the long-term behavior of the system. In this paper we are mainly interested in *selfish* systems whose individual components are selfish agents. The state of a selfish system is fully described by a vector of *strategies*, each controlled by one agent, and each state assigns a payoff to each agent. The agents are selfish in the sense that they pick their strategy so to maximize their payoff, given the strategies of the other agents. The notion of a Nash equilibrium is the classical notion of equilibrium for selfish systems and it corresponds to the equilibrium induced by the *best-response* dynamics. The observation that selfish systems are described by their equilibrium states (that is, by the Nash equilibria) has motivated the notion of Price of Anarchy [14] (and Price of Stability [1]) and the efficiency analysis of selfish systems based on such notions.

The analysis based on Nash equilibria inherits some of the shortcomings of the concept of a Nash equilibrium. First of all, the best-response dynamics assumes that the selfish agents have complete knowledge of the current state of the system; that is, of the payoff associated with each possible choice and of the strategies chosen by the other agents. Instead, in most cases, agents have only an approximate knowledge of the system state. Moreover, in presence of multiple equilibria, it is not clear which equilibrium will be reached by the system as it may depend on the initial state of the system. The notion of Price of Anarchy solves this problem by considering the worst case equilibrium whereas Price of Stability focuses on the best case equilibrium. Finally, Nash equilibria are hard to compute [6, 4] and thus for some system it might take very long to enter a Nash equilibrium. In this case using equilibrium states to describe the system performance is not well justified. Rather, one would like to analyze the performance of a system by using a dynamics (and its related equilibrium notion) that has the following three properties: the dynamics takes into account the fact that the system components might have a perturbed or noisy knowledge of the system; the equilibrium state exists and is unique for every system; independently from the starting state, the system enters the equilibrium very quickly.

In this paper, we consider *noisy best-response dynamics* in which the behavior of the agents is described by a parameter $\beta \geqslant 0$ (β is sometimes called the *inverse temperature*). The case $\beta = 0$ corresponds to agents picking their strategies completely at random (that is, the agents have no knowledge of the system) and the case $\beta = \infty$ corresponds to the best-response dynamics (in which the agents have full and complete knowledge of the system). The intermediate values of β correspond to agents that are roughly guided by the best-response dynamics but can make a sub-optimal response with some probability that depends on β (and on the associated payoff). We will study a specific noisy best-response dynamics for which the system evolves according to an ergodic Markov chain for all $\beta \geqslant 0$. For these systems, it is natural to look at the stationary distribution (which is the equilibrium state of the Markov chain) and to analyze the performance of the system at the stationary distribution. We stress that the noisy best-response dynamics well models agents that only have approximate or noisy knowledge of the system and that for ergodic Markov chains (such as the ones arising in our study) the stationary distribution is known to exist and to be unique. Moreover, to justify the use of the stationary distribution for analyzing the performance of the system, we will study how fast the Markov chain converges to the stationary distribution.

Related Works and Our Results. Several dynamics, besides the best-response dynamics, and several notions of equilibrium, besides Nash equilibria, have been considered to describe the evolution of a selfish system and to analyze its performance. See, for example, [10, 19, 18].

Equilibrium concepts based on the best-response. In case the game does not possess a Pure Nash equilibrium, the best-response dynamics will eventually cycle over a set of states (in a Nash equilibrium the set is a singleton). These states are called *sink equilibria* [11]. Sink equilibria exist for all games and, in some context, they seem a better approximation of the real setting than *mixed* Nash equilibria. Unfortunately, sink equilibria share two undesirable properties with Nash equilibria: a game can have more that one sink equilibrium and sink equilibria seem hard to compute [8]. Other notions of equilibrium state associated with best-response dynamics are the *unit-recall equilibria* and *component-wise unit-recall equilibria* (see [8]). We point out though that the former does not always exist and that the latter imposes too strict limitations on the players.

No-Regret Dynamics. Another broadly explored set of dynamics are the no-regret dynamics (see, for example, [10]). The regret of a user is the difference between the long term average cost and average cost of the best strategy in hindsight. In the no-regret dynamics the regret of every player after t step is $o(t)$ (sublinear with time). In [9, 13] it is showed that the no-regret dynamics converges to the set of Correlated Equilibria. Note that the convergence is to the *set* of Correlated Equilibria and not to a specific correlated equilibrium.

Our work. In this paper we consider a specific noisy best-response dynamics called the *logit* dynamics (see [3]) and we study its mixing time (that is, the time it takes to converge to the stationary distribution) for various games. Specifically,

- We start by analyzing the logit dynamics for a simple 3-player linear congestion game (the CK game [5]) which exhibits the worst Price of Anarchy among linear congestion games. We show that the convergence time to stationarity of the logit dynamics is upper bounded by a constant independent of β. Moreover, we show that the expected social cost at stationarity is smaller than the cost of the worst Nash equilibrium for all β.

- We then study the 2×2 coordination games studied by [3]. Here we show that, under some conditions, the expected social welfare at stationarity is better than the social welfare of the worst Nash equilibrium. We give exponential in β upper and lower bounds on the convergence time to stationarity for all values of β. We also study anti-coordination games and we show that, under some conditions, the logit dynamics guarantees each player a utility greater than the one of the worst Nash equilibrium.

- Finally, we apply our analysis to a simple n-player game, the OR-game, and give upper and lower bound on the convergence time to stationarity. In particular, we prove that for $\beta = \mathcal{O}(\log n)$ the convergence time is polynomial in n.

The *logit* dynamics has been first studied by Blume [3] who showed that, for 2×2 games, the long-term behaviour of the Markov chain is concentrated in the risk dominant equilibrium (see [12]) for sufficiently large β. Ellison [7] studied different noisy best-response dynamics for 2×2 games and assumed that interaction among players were described by a graph; that is, the utility of a player is determined only by the strategies of the adjacent players. Specifically, Ellison [7] studied interaction modeled by rings and showed that some large fraction of the players will eventually choose the risk dominant strategy. Similar results were obtained by Peyton Young [20] for the logit dynamics and for more general families of graphs. Montanari and Saberi [16] gave bounds on the hitting time of the risk dominant equilibrium states for the logit dynamics in terms of some graph theoretic properties of the underlying interaction network. Asadpour and Saberi [2] studied the hitting time for a class of congestion games. We notice that none of [3, 7, 20] gave any bound on the convergence time to the risk dominant equilibrium. Montanari and Saberi [16] were the first to do so but their study focuses on the hitting time of a specific configuration.

From a technical point of view, our work follows the lead of [3, 7, 20] and extends their technical findings by giving bounds on the mixing time of the Markov chain of the logit dynamics. We stress that previous results only proved that, for sufficiently large β, *eventually* the system concentrates around certain states without further quantifying the rate of convergence nor the asymptotic behaviour of the system for small values of β. Instead, we identify the stationary distribution of the logit dynamics as the *global* equilibrium and we evaluate the social welfare at stationarity and the time it takes the system to reach it (the mixing time) as explicit functions of the inverse temperature β of the system.

We choose to start our study from the class of coordination games considered in [3] for which we give tight upper and lower bound on the mixing time and then look also at other 2-player games and a simple n-player game (the OR-game). Despite its game-theoretic simplicity, the analytical study of the mixing time of the Markov chain associated with the OR-game as a function of β is far from trivial. Also we notice that the results of [16] cannot be used to derive upper bounds on the mixing time as in [16] the authors give a tight estimation of the hitting time only for a specific state of the Markov chain. The mixing time instead is upper bounded by the *maximum* hitting time.

From a more conceptual point of view, our work tries (similarly to [11, 8, 17]) to introduce a solution concept that well models the behaviour of selfish agents, is uniquely defined for any game and is quickly reached by the game. We propose the stationary distribution induced by the logit dynamics as a possible solution concept and exemplify its use in the analysis of the performance of some 2×2 games (as the ones considered in [3, 7, 20]), in games used to obtain tight bounds on the Price of Anarchy and on a simple multiplayer game.

Organization of the paper. In Section 2 we formally describe the logit dynamics Markov chain for a strategic game. We also describe the coupling we will repeatedly use in the proofs of the upper bounds on mixing times. In Sections 3, 4, and 5 we study the stationary expected social welfare and the mixing time of the logit dynamics for CK game, coordination and anti-coordination games, and the OR-game respectively. Finally, in Section 6 we present conclusions and some open problems.

Notation. We write $\overline{S}$ for the complementary set of a set S, we write $|S|$ for its size. We use bold symbols for vectors, when $\mathbf{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ we write $|\mathbf{x}|$ for the number of 1s in $\mathbf{x}$; i.e., $|\mathbf{x}| = |\{i \in [n] : x_i = 1\}|$. We use the standard game theoretic notation $(\mathbf{x}_{-i}, y)$ to mean the vector obtained from $\mathbf{x}$ by replacing the i-th entry with y, i.e. $(\mathbf{x}_{-i}, y) = (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$. We use standard Markov chain terminology (see [15]) and review the needed technical facts in Appendix A.

2 The Model and the problem

A *strategic game* is a triple $([n], \mathcal{S}, \mathcal{U})$, where $[n] = \{1, \ldots, n\}$ is a finite set of *players*, $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a family of non-empty finite sets (S_i is the set of *strategies* for player i), and $\mathcal{U} = \{u_1, \ldots, u_n\}$ is a family of *utility functions* (or *payoffs*), where $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ is the utility function of player i.

Consider the following *noisy* best-response dynamics, introduced in [3] and known as *logit dynamics*: At every time step

1. Select one player $i \in [n]$ uniformly at random;

2. Update the strategy of player i according to the following probability distribution over the set S_i of her strategies. For every $y \in S_i$

$$\sigma_i(y \mid \mathbf{x}) = \frac{1}{T_i(\mathbf{x})}\, e^{\beta u_i(\mathbf{x}_{-i}, y)} \tag{1}$$

where $\mathbf{x} \in S_1 \times \cdots \times S_n$ is the strategy profile played at the current time step, $T_i(\mathbf{x}) = \sum_{z \in S_i} e^{\beta u_i(\mathbf{x}_{-i}, z)}$ is the normalizing factor, and $\beta \geqslant 0$ is the *inverse noise*.

From (1) it is easy to see that, for $\beta = 0$ player i selects her strategy uniformly at random, for $\beta > 0$ the probability is biased toward strategies promising higher payoffs, and for $\beta \to \infty$ player i chooses her best response strategy (if more than one best response is available, she chooses uniformly at random one of them). Moreover observe that probability $\sigma_i(y \mid \mathbf{x})$ does not depend on the strategy x_i currently adopted by player i.
The above dynamics defines an ergodic finite Markov chain with the set of strategy profiles as state space, and where the transition probability from profile $\mathbf{x} = (x_1, \ldots, x_n)$ to profile $\mathbf{y} = (y_1, \ldots, y_n)$ is zero if the two profiles differ at more than one player and it is $\frac{1}{n}\sigma_i(y_i \mid \mathbf{x})$ if the two profiles differ exactly at player i. More formally, we have the following definition.

Definition 1 (Logit dynamics [3]) *Let $\mathcal{G} = ([n], \mathcal{S}, \mathcal{U})$ be a strategic game and let $\beta \geqslant 0$ be the inverse noise. The* logit dynamics *for $\mathcal{G}$ is the Markov chain $\mathcal{M}_\beta = \{X_t : t \in \mathbb{N}\}$ with state space $\Omega = S_1 \times \cdots \times S_n$ and transition matrix*

$$P(\mathbf{x}, \mathbf{y}) = \frac{1}{n} \sum_{i=1}^{n} \frac{e^{\beta u_i(\mathbf{x}_{-i}, y_i)}}{T_i(\mathbf{x})} \mathbb{I}_{\{y_j = x_j \text{ for every } j \neq i\}} \tag{2}$$

It is easy to see that, if $([n], \mathcal{S}, \mathcal{U})$ is a potential game with exact potential Φ, then the Markov chain given by (2) is reversible and its stationary distribution is the Gibbs measure

$$\pi(\mathbf{x}) = \frac{1}{Z} e^{\beta \Phi(\mathbf{x})} \tag{3}$$

where $Z = \sum_{\mathbf{y} \in S_1 \times \cdots \times S_n} e^{\beta \Phi(\mathbf{y})}$ is the normalizing constant (the *partition function* in physicists' language). Except for the Matching Pennies example in Subsection 2.1, all the games we analyse in this paper are potential games.
Let $W : S_1 \times \cdots \times S_n \longrightarrow \mathbb{R}$ be a *social welfare function* (in this paper we assume that W is simply the sum of all the utility functions $W(\mathbf{x}) = \sum_{i=1}^{n} u_i(\mathbf{x})$, but clearly any other function of interest can be analysed). We study the *stationary expected social welfare*, i.e. the expectation of W when the strategy profiles are random according to the stationary distribution π of the Markov chain,

$$\mathbf{E}_\pi[W] = \sum_{\mathbf{x} \in S_1 \times \cdots \times S_n} W(\mathbf{x}) \pi(\mathbf{x})$$

Since the Markov chain defined in (2) is irreducible and aperiodic, from every initial profile $\mathbf{x}$ the distribution $P^t(\mathbf{x}, \cdot)$ of chain X_t starting at $\mathbf{x}$ will eventually converge to π as t tends to infinity. We will be interested in the *mixing time* t_{mix} of the chain, i.e. the time needed to have that $P^t(\mathbf{x}, \cdot)$ is *close* to π for every initial configuration $\mathbf{x}$. More formally, we define

$$t_{\mathrm{mix}}(\varepsilon) = \min_{t \in \mathbb{N}} \max_{\mathbf{x} \in \Omega} \left\{ \|P^t(\mathbf{x}, \cdot) - \pi\|_{\mathrm{TV}} \leqslant \varepsilon \right\}$$

where $\|P^t(\mathbf{x}, \cdot) - \pi\|_{\mathrm{TV}} = \frac{1}{2} \sum_{\mathbf{y} \in \Omega} |P^t(\mathbf{x}, \mathbf{y}) - \pi(\mathbf{y})|$ is the *total variation distance*, and we set $t_{\mathrm{mix}} = t_{\mathrm{mix}}(1/4)$.

2.1 An example: Matching Pennies

As an example consider the classical *Matching Pennies* game.

	H	T
H	$+1, -1$	$-1, +1$
T	$-1, +1$	$+1, -1$

(4)

The update probabilities (1) for the logit dynamics are, for every $x \in \{H, T\}$

$$\sigma_1(H \mid (x,H)) = \sigma_1(T \mid (x,T)) = \quad \tfrac{1}{1+e^{-2\beta}} \quad = \sigma_2(T \mid (H,x)) = \sigma_2(H \mid (T,x))$$

$$\sigma_1(T \mid (x,H)) = \sigma_1(H \mid (x,T)) = \quad \tfrac{1}{1+e^{2\beta}} \quad = \sigma_2(H \mid (H,x)) = \sigma_2(T \mid (T,x))$$

So the transition matrix (2) is

$$P = \left(\begin{array}{c|cccc} & HH & HT & TH & TT \\ \hline HH & 1/2 & b/2 & (1-b)/2 & 0 \\ HT & (1-b)/2 & 1/2 & 0 & b/2 \\ TH & b/2 & 0 & 1/2 & (1-b)/2 \\ TT & 0 & (1-b)/2 & b/2 & 1/2 \end{array} \right)$$

Where we named $b = \frac{1}{1+e^{-2\beta}}$ for readability sake.

Since every column of the matrix adds up to 1, the uniform distribution π over the set of strategy profiles is the stationary distribution for the logit dynamics. The expected stationary social welfare is thus 0 for every inverse noise β.

As for the mixing time, it is easy to see that it is upper bounded by a constant independent of β. Indeed, a direct calculation shows that, for every $\mathbf{x} \in \{HH, HT, TH, TT\}$ and for every $\beta \geqslant 0$ it holds that

$$\|P^3(\mathbf{x}, \cdot) - \pi\|_{\text{TV}} \leqslant \frac{7}{16} < \frac{1}{2}$$

2.2 Coupling

Throughout the paper we will use the path-coupling technique (see Theorem 20 in the Appendix) to give upper bounds on mixing times. Since we will use the same coupling idea in several proofs, we describe it here and we will refer to this description when we will need it.

For a game $([n], \mathcal{S}, \mathcal{U})$ let us define the *Hamming graph* $G = (\Omega, E)$ of the game where $\Omega = S_1 \times \cdots \times S_n$ is the set of strategy profiles, and two profiles $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \Omega$ are adjacent if they differ only for the strategy of one player, i.e.

$$\{\mathbf{x}, \mathbf{y}\} \in E \iff |\{j \in [n] \,:\, x_j \neq y_j\}| = 1 \tag{5}$$

For every edge $\{\mathbf{x}, \mathbf{y}\} \in E$ we define a coupling (X, Y) of the two probability distributions $P(\mathbf{x}, \cdot)$ and $P(\mathbf{y}, \cdot)$ where P is the transition matrix defined in (2). The coupling proceeds as follows: Pick $i \in [n]$ uniformly at random. If $i = j$ then, since $\mathbf{x}_{-i} = \mathbf{y}_{-i}$, it holds that $\sigma_i(\cdot \mid \mathbf{x}) = \sigma_i(\cdot \mid \mathbf{y})$, hence we can set the same value for X and Y according to distribution $\sigma_i(\cdot \mid \mathbf{x})$. More formally, for every $z_1, z_2 \in S_i$ we set

$$\mathbf{P}_{\mathbf{x},\mathbf{y}}\left(X = (\mathbf{x}_{-i}, z_1) \text{ and } Y = (\mathbf{y}_{-i}, z_2)\right) = \begin{cases} 0 & \text{if } z_1 \neq z_2 \\ \sigma_i(z \mid \mathbf{x}) = \sigma_i(z \mid \mathbf{y}) & \text{if } z_1 = z_2 =: z \end{cases}$$

If $i \neq j$ then we cannot always use the same strategy for player i because $\sigma_i(z \mid \mathbf{x})$ and $\sigma_i(z \mid \mathbf{y})$ may be different, so we use a standard way to couple them: consider two copies I_X, I_Y of the interval $[0,1]$.

Intuition box. We want to partition the two intervals I_X and I_Y in little sub-intervals labeled with the strategies for player i, in a way that, for every $z \in S_i$, the sum of the lengths of sub-intervals of I_X labeled z is $\sigma_i(z \mid \mathbf{x})$, and the sum of the lengths of sub-intervals of I_Y labeled z is $\sigma_i(z \mid \mathbf{y})$. Indeed, in this way if we name $J_X(z)$ the union of the sub-intervals of I_X with label z and $J_Y(z)$ the union of the the sub-intervals of I_Y with label z, and if U is a random variable uniform over $[0,1]$, we have that $\mathbf{P}(U \in J_X(z)) = \sigma_i(z \mid \mathbf{x})$ and $\mathbf{P}(U \in J_Y(z)) = \sigma_i(z \mid \mathbf{y})$. Moreover, we want $J_X(\cdot)$ and $J_Y(\cdot)$ to "overlap" as much as possible, so that both X and Y have the same strategy for player i as many times as possible.

Order the strategies for player i arbitrarily, say $S_i = \{z_1, \ldots, z_{|S_i|}\}$. For $k = 1, \ldots, |S_i|$, label with z_k an interval of length $\min\{\sigma_i(z_k \mid \mathbf{x}), \sigma_i(z_k \mid \mathbf{y})\}$ on the leftmost side (not yet labeled) of both I_X and I_Y. If $\sigma_i(z_k \mid \mathbf{x}) > \sigma_i(z_k \mid \mathbf{y})$ then label with z_k also an interval of length $\sigma_i(z_k \mid \mathbf{x}) - \sigma_i(z_k \mid \mathbf{y})$ on the rightmost side (not yet labeled) of I_X, otherwise (if $\sigma_i(z_k \mid \mathbf{x}) < \sigma_i(z_k \mid \mathbf{y})$) label with z_k also an interval of length

$\sigma_i(z_k \mid \mathbf{y}) - \sigma_i(z_k \mid \mathbf{x})$ on the rightmost side (not yet labeled) of I_Y. At the end of this procedure, both the intervals I_X and I_Y are partitioned into sub-intervals labeled with the strategies of S_i. Let us name $h_X : I_X \to S_i$ and $h_Y : I_Y \to S_i$ the functions that for a point $s \in [0,1]$ return the labels $h_X(s)$ and $h_Y(s)$ of the sub-intervals at which s belongs to in I_X and I_Y respectively. Moreover, observe that there is a point $r \in (0,1)$ such that $h_X(s) = h_Y(s)$ for every $s \leqslant r$ and $h_X(s) \neq h_Y(s)$ for every $s > r$.

Now let U be a uniform random variable over the interval $[0,1]$, and define $(X,Y) = (h_X(U), h_Y(U))$. By construction we have that (X,Y) is a coupling of $P(\mathbf{x},\cdot)$ and $P(\mathbf{y},\cdot)$.

3 Warm up: A 3-player congestion game

In this section we study the CK game, a simple 3-player linear congestion game introduced in [5] that exhibits the worst Price of Anarchy of the average social welfare among linear congestion games with 3 or more players. This game has two equilibria: one with social welfare -6 (which is also optimal) and one with social welfare -15. As we shall see briefly, the stationary expected social welfare of the logit dynamics is always larger than the social welfare of the worst Nash equilibrium and, for large enough β, players spend most of the time in the best Nash equilibrium. Moreover, we will show that the mixing time of the logit dynamics is bounded by a constant independent of β; that is, the stationary distribution guarantees a good social welfare and it is quickly reached by the system.

Let us now describe the CK game. We have 3 players and 6 facilities divided into two sets: $G = \{g_1, g_2, g_3\}$ and $H = \{h_1, h_2, h_3\}$. Player $i \in \{0,1,2\}$ has two strategies: Strategy "0" consists in selecting facilities (g_i, h_i); Strategy "1" consists in selecting facilities $(g_{i+1}, h_{i-1}, h_{i+1})$ (index arithmetic is modulo 3). The cost of a facility is the number of players choosing such facility, and the welfare of a player is minus the sum of the costs of the facilities she selected. It easy to see that this game has two pure Nash equilibria: when every player plays strategy 0 (each player pays 2, which is optimal), and when every player plays strategy 1 (each player pays 5). The game is a congestion game, and thus a potential game with following potential function:

$$\Phi(\mathbf{x}) = \sum_{j \in G \cup H} \sum_{i=1}^{L_{\mathbf{x}}(j)} i$$

where $L_{\mathbf{x}}(j)$ is the number of players using facility j in configuration $\mathbf{x}$.

Stationary expected social welfare. The logit dynamics for this game gives the following update probabilities (see Equation 1)

$$\begin{array}{rclrcl}
\sigma_i(0 \mid |\mathbf{x}_{-i}| = 0) & = & \frac{1}{1+e^{-4\beta}} & \sigma_i(1 \mid |\mathbf{x}_{-i}| = 0) & = & \frac{1}{1+e^{4\beta}} \\
\sigma_i(0 \mid |\mathbf{x}_{-i}| = 1) & = & \frac{1}{1+e^{-2\beta}} & \sigma_i(1 \mid |\mathbf{x}_{-i}| = 1) & = & \frac{1}{1+e^{2\beta}} \\
\sigma_i(0 \mid |\mathbf{x}_{-i}| = 2) & = & \frac{1}{2} & \sigma_i(1 \mid |\mathbf{x}_{-i}| = 2) & = & \frac{1}{2}
\end{array}$$

It is easy to check that the following distribution is stationary for the logit dynamics:

$$\begin{array}{rcl}
\pi[(0,0,0)] & = & \frac{e^{-6\beta}}{Z(\beta)} \\
\pi[(0,0,1)] = \pi[(0,1,0)] = \pi[(1,0,0)] & = & \frac{e^{-10\beta}}{Z(\beta)} \\
\pi[(0,1,1)] = \pi[(1,1,0)] = \pi[(1,0,1)] = \pi[(1,1,1)] & = & \frac{e^{-12\beta}}{Z(\beta)}
\end{array}$$

where $Z(\beta) = e^{-6\beta} + 3e^{-10\beta} + 4e^{-12\beta}$. Let k be the number of players playing strategy 1; the social welfare is -6 when $k = 0$, it is -13 if $k = 1$, it is -16 if $k = 2$, and -15 when $k = 3$. Thus the stationary expected social welfare is

$$\mathbf{E}_\pi[W] = -\frac{6e^{-6\beta} + 39e^{-10\beta} + (48+15)e^{-12\beta}}{e^{-6\beta} + 3e^{-10\beta} + 4e^{-12\beta}} = -\frac{3[2 + 13e^{-4\beta} + 21e^{-6\beta}]}{1 + 3e^{-4\beta} + 4e^{-6\beta}}.$$

For $\beta = 0$, we have $\mathbf{E}_\pi[W] = -27/2$ which is better than the social welfare of the worst Nash equilibrium. As β tends to ∞, $\mathbf{E}_\pi[W]$ approaches the optimal social welfare. Furthermore, we observe that $\mathbf{E}_\pi[W]$ increases with β and thus we can conclude that the long-term behavior of the logit dynamics gives a better social welfare than the worst Nash equilibrium for any $\beta \geqslant 0$.

Mixing time. Now we study the mixing time of the logit dynamics and we show that it is bounded by a constant for any $\beta \geqslant 0$.

Theorem 2 (Mixing time) *There exists a constant τ such that the mixing time t_{mix} of the logit dynamics of the* CK *game is upper bounded by τ for every $\beta \geqslant 0$.*

Proof. Observe that $\sigma_i(0 \mid \mathbf{x}) \geqslant 1/2$ for every player i, every profile $\mathbf{x}$, and every inverse temperature β. Hence, if X_t and Y_t are two copies of the Logit dynamics Markov chain starting in states $\mathbf{x}$ and $\mathbf{y}$ respectively, we can define a coupling of the two chains with the following properties: At every step we choose the same player in both chains, and such player chooses strategy 0 in both chains with probability at least 1/2.

Now consider the probability that after three steps the two chains have coupled. It is at least as large as the probability that we choose three different players and all of them play strategy 0 at their turn, i.e.

$$\mathbf{P}_{\mathbf{x},\mathbf{y}}\left(X_3 = Y_3\right) \geqslant \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{36}$$

Since this bound holds for every starting pair $(\mathbf{x}, \mathbf{y})$, we have that the probability the two chains have not yet coupled after $3t$ steps is

$$\mathbf{P}_{\mathbf{x},\mathbf{y}}\left(X_{3t} \neq Y_{3t}\right) \leqslant \left(1 - \frac{1}{36}\right)^t \leqslant e^{-t/36}$$

The thesis follows from the well-known upper bound on the total variation distance by coupling time (e.g., see Theorem 5.2 in [15]). $\square$

4 Two player games

In this section we analyse the performance of the logit dynamics for 2×2 coordination games (the same class studied in [3]) and 2×2 anti-coordination games .

4.1 Coordination games

Coordination Games are two-player games in which the players have an advantage in selecting the same strategy. They are often used to model the spread of a new technology [20]: two players have to decide whether to adopt or not a new technology. We assume that the players would prefer choosing the same technology and that choosing the new technology is risk dominant.

We analyse the mixing time of the logit dynamics for 2×2 coordination games and compute the stationary expected social welfare of the game as a function of β. We show that, for large enough β, players will spend most of the time in the risk dominant equilibrium and the expected utility is better than the one associated with the worst Nash equilibrium.

We denote by 0 the `NEW` strategy and by 1 the `OLD` strategy. The game is formally described by the following payoff matrix

	0	1
0	(a, a)	(c, d)
1	(d, c)	(b, b)

(6)

We assume that $a > d$ and $b > c$ (meaning that they prefer to coordinate) and that $a - d > b - c$ (meaning that strategy 0 is the risk dominant strategy for each player). Notice that we do not make any assumption on the relation between a and b. It is easy to see that this game is a potential game and the following function is an exact potential for it:

$$\Phi(0,0) = a - d \qquad \Phi(0,1) = \Phi(1,0) = 0 \qquad \Phi(1,1) = b - c.$$

This game has two pure Nash equilibria: $(0, 0)$, where each player has utility a, and $(1, 1)$, where each player has utility b. As $d + c < a + b$, the social welfare is maximized in correspondence of one the two equilibria and the Price of Anarchy is equal to $\max\{b/a, a/b\}$.

Stationary expected social welfare. The logit dynamics for the game defined by the payoffs in Table 6 gives the following update probabilities for any strategy $x \in \{0,1\}$ (see Equation (1))

$$\begin{aligned} \sigma_1(0 \mid (x,0)) = \sigma_2(0 \mid (0,x)) &= \tfrac{1}{1+e^{-(a-d)\beta}} & \sigma_1(1 \mid (x,0)) = \sigma_2(1 \mid (0,x)) &= \tfrac{1}{1+e^{(a-d)\beta}} \\ \sigma_1(0 \mid (x,1)) = \sigma_2(0 \mid (1,x)) &= \tfrac{1}{1+e^{(b-c)\beta}} & \sigma_1(1 \mid (x,1)) = \sigma_2(1 \mid (1,x)) &= \tfrac{1}{1+e^{-(b-c)\beta}}. \end{aligned}$$

Theorem 3 (Expected social welfare) *The stationary expected social welfare $\mathbf{E}_\pi[W]$ of the logit dynamics for the coordination game is*

$$\mathbf{E}_\pi[W] = 2 \cdot \frac{a + be^{-((a-d)-(b-c))\beta} + (c+d)e^{-(a-d)\beta}}{1 + e^{-((a-d)-(b-c))\beta} + 2e^{-(a-d)\beta}}.$$

Proof. The stationary distribution π of the logit dynamics is

$$\pi(0,0) = \frac{e^{(a-d)\beta}}{Z(\beta)} \qquad \pi(1,1) = \frac{e^{(b-c)\beta}}{Z(\beta)} \qquad \pi(0,1) = \pi(1,0) = \frac{1}{Z(\beta)}$$

where $Z(\beta) = e^{(a-d)\beta} + e^{(b-c)\beta} + 2$. Observe that the probability that at the stationary distribution the players are in one of the two states that are not Nash equilibria, $(0,1)$ or $(1,0)$, is $\frac{1}{2}$ for $\beta = 0$, (that is when players randomly select their strategies) and it goes down to 0 as β increases.

We compute the expected utility $\mathbf{E}_\pi[u_i]$ of player i at the stationary distribution from which we obtain the expected social welfare $\mathbf{E}_\pi[W]$ as $\mathbf{E}_\pi[W] = 2 \cdot \mathbf{E}_\pi[u_i]$.

$$\begin{aligned} \mathbf{E}_\pi[u_i] &= \textstyle\sum_{\mathbf{x}\in\{0,1\}^2} u_i(\mathbf{x})\pi(\mathbf{x}) \\ &= \tfrac{ae^{(a-d)\beta}+be^{(b-c)\beta}+c+d}{e^{(a-d)\beta}+e^{(b-c)\beta}+2} \\ &= \tfrac{a+be^{-(a-d-b+c)\beta}+(c+d)e^{-(a-d)\beta}}{1+e^{-(a-d-b+c)\beta}+2e^{-(a-d)\beta}} \end{aligned} \tag{7}$$

□

The following observation 4 gives conditions on β and the players' utility for which the expected social welfare $\mathbf{E}_\pi[W]$ obtained by the logit dynamics is better than the social welfare SW_N of the worst Nash Equilibrium.

Observation 4 *For the coordination game described in Table 6, we have*

- *if $a > b$ and $b \leqslant \max\{\frac{a+c+d}{3}, \frac{c+d}{2}\}$ then $\mathbf{E}_\pi[W] > \mathsf{SW}_N$ for all β;*
- *if $a > b$ and $b > \max\{\frac{a+c+d}{3}, \frac{c+d}{2}\}$ then $\mathbf{E}_\pi[W] > \mathsf{SW}_N$ for all sufficiently large β;*
- *if $a < b$ and $a \leqslant \max\{\frac{b+c+d}{3}, \frac{c+d}{2}\}$ then $\mathbf{E}_\pi[W] > \mathsf{SW}_N$ for all β;*
- *if $a < b$ and $a > \max\{\frac{b+c+d}{3}, \frac{c+d}{2}\}$ then $\mathbf{E}_\pi[W] > \mathsf{SW}_N$ for all sufficiently large β;*
- *if $a = b$ then $\mathbf{E}_\pi[W] < \mathsf{SW}_N$ for any β, a, c and d.*

Proof. We consider first the case $a > b$ in which case the worst Nash Equilibrium is $(1,1)$ with social welfare $2b$. It is easy to see that if $b \leqslant \frac{a+c+d}{3}$ or $b \leqslant \frac{c+d}{2}$, then $\mathbf{E}_\pi[u_i] > b$ and thus $\mathbf{E}_\pi[W] > \mathsf{SW}_N$. Otherwise, we observe that

$$\frac{(a-b) - (2b-c-d)e^{-(a-d)\beta}}{1 + e^{-(a-d-b+c)\beta} + 2e^{-(a-d)\beta}} > 0$$

for

$$\beta > \frac{1}{a-d} \log \frac{2b-c-d}{a-b}.$$

Consider now the case $a < b$; that is when the social welfare of the risk dominant equilibrium $(0,0)$ is lower than at equilibrium $(1,1)$. In this case, the expected utility of the players goes down to the utility of the worst Nash equilibrium, as β grows. However, as before, we can see that when $3a \leqslant b+c+d$ or $2a \leqslant c+d$ then $\mathbf{E}_\pi[u_i]$ is always greater than a. Moreover, the condition holds for $\beta > \frac{1}{b-c} \log \frac{2a-c-d}{b-a}$.

Finally, consider the case $a = b$. In this case, it is easy to see that for any β we have that $\mathbf{E}_\pi[u_i] < a$ (since we are assuming $a > d$ and $b > c$). Then the expected social welfare obtained by our dynamics is always worse than the social welfare of the Nash Equilibrium. □

Mixing time. Now we study the mixing time of the logit dynamics for coordination games and we show that it is exponential in β.

Theorem 5 (Mixing Time) *The mixing time of the logit dynamics with parameter β for the coordination game of Table 6 is $\Theta\left(e^{(b-c)\beta}\right)$.*

Proof. Upper bound: We apply the Path Coupling technique (see Theorem 20 in Appendix A) with the Hamming graph defined in (5) and all the edge-weights set to 1. Let $\mathbf{x}$ and $\mathbf{y}$ be two profiles differing only for the player j and consider the coupling defined in Section 2.2 for this pair of profiles. Now we bound the expected distance of the two coupled chains after one step.

We denote by $b_i(\mathbf{x}, \mathbf{y})$ the probability that both chains perform the same update given that player i has been selected for strategy update. Clearly, $b_j(\mathbf{x}, \mathbf{y}) = 1$. For $i \neq j$, we have

$$\begin{aligned} b_i(\mathbf{x}, \mathbf{y}) &= \min\{\sigma_i(0 \mid \mathbf{x}), \sigma_i(0 \mid \mathbf{y})\} + \min\{\sigma_i(1 \mid \mathbf{x}), \sigma_i(1 \mid \mathbf{y})\} \\ &= \frac{1}{1 + e^{(a-d)\beta}} + \frac{1}{1 + e^{(b-c)\beta}} \end{aligned} \tag{8}$$

For sake of compact notation we set

$$p = \frac{1}{1 + e^{(a-d)\beta}} \qquad \text{and} \qquad q = \frac{1}{1 + e^{(b-c)\beta}}$$

and thus $b_i(\mathbf{x}, \mathbf{y}) = p + q$. To compute $\mathbf{E}_{\mathbf{x},\mathbf{y}}\left[\rho(X_1, Y_1)\right]$, we observe that the logit dynamics chooses player j with probability 1/2. In this case, we have $b_j(\mathbf{x}, \mathbf{y}) = 1$ and thus the coupling updates both chains in the same way, resulting in $X_1 = Y_1$. Similarly, player $i \neq j$ is chosen for strategy update with probability 1/2. In this case, with probability $b_i(\mathbf{x}, \mathbf{y})$ the coupling performs the same update in both chains resulting in $\rho(X_1, Y_1) = 1$. Instead with probability $1 - b_i(\mathbf{x}, \mathbf{y})$, the coupling performs different updates on the chains resulting in $\rho(X_1, Y_1) = 2$. Therefore we have,

$$\begin{aligned} \mathbf{E}_{\mathbf{x},\mathbf{y}}\left[\rho(X_1, Y_1)\right] &= \frac{1}{2} b_i(\mathbf{x}, \mathbf{y}) + 2 \cdot \frac{1}{2}(1 - b_i(\mathbf{x}, \mathbf{y})) \\ &= 1 - \frac{1}{2} b_i(\mathbf{x}, \mathbf{y}) = 1 - \frac{1}{2}(p + q) \leqslant e^{-\frac{1}{2}(p+q)}. \end{aligned}$$

From Theorem 20, with $\alpha = \frac{1}{2}(p + q)$ and $\operatorname{diam}(\Omega) = 2$, it follows that

$$t_{\text{mix}}(\varepsilon) \leqslant \frac{2\left(\log 2 + \log(1/\varepsilon)\right)}{p + q} = \frac{1}{p + q} \log \frac{4}{\varepsilon^2} \tag{9}$$

Lower bound: We use the *relaxation time* bound (see Theorem 22 in Appendix A). The transition matrix of the logit dynamics is

$$P = \left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1-p & p/2 & p/2 & 0 \\ 01 & \frac{1-p}{2} & \frac{p+q}{2} & 0 & \frac{1-q}{2} \\ 10 & \frac{1-p}{2} & 0 & \frac{p+q}{2} & \frac{1-q}{2} \\ 11 & 0 & q/2 & q/2 & 1-q \end{array}\right)$$

It is easy to see that the second largest eigenvalue of P is $\lambda_\star = \frac{(1-p)+(1-q)}{2}$, hence the relaxation time is $t_{\text{rel}} = 1/(1 - \lambda_\star) = \frac{2}{p+q}$, and for the mixing time we have

$$\begin{aligned} t_{\text{mix}}(\varepsilon) &\geqslant (t_{\text{rel}} - 1) \log \frac{1}{2\varepsilon} = \frac{2 - (p+q)}{p + q} \log \frac{1}{2\varepsilon} \\ &\geqslant \frac{1}{p + q} \log \frac{1}{2\varepsilon} \end{aligned} \tag{10}$$

In the last inequality we used that p and q are both smaller than 1/2. The theorem follows by observing that

$$\frac{1}{p+q} = \frac{1}{\frac{1}{1+e^{(a-d)\beta}} + \frac{1}{1+e^{(b-c)\beta}}} = \Theta\left(e^{(b-c)\beta}\right)$$

□

We observe that the upper bound on the mixing time obtained using the relaxation time (see Theorem 22 in Appendix A) is not tight, hence in the above proof we had to resort to the path coupling for the upper bound.

4.2 Anti-coordination games

Anti-coordination games are two-player games in which the players have an advantage in selecting different strategies. They model many settings where there is a common and exclusive resource: two players have to decide whether to use the resource or to drop it. If they both try to use it, then a deadlock occurs and this is bad for both players. Moreover, they prefer to use the resource.

We analyze the mixing time of the logit dynamics for 2×2 anti-coordination games and we compute the expected social welfare of the game as a function of β. It turns out that the stationary expected social welfare is *fair* (every player has the same expected utility) whereas Nash equilibria are *unfair*. Moreover, the stationary expected social welfare tends to the social welfare of the Nash equilibria.

We denote by 0 the `USE` strategy and by 1 the `DROP` strategy. The game is formally described in the following payoff matrix

	0	1
0	(a,a)	(d,c)
1	(c,d)	(b,b)

(11)

We assume that $c > a$ and $d > b$ (meaning that players prefer not to choose the same strategy), and that $c - a > d - b$ (meaning that strategy 0 is the risk dominant action for each player). Notice that we do not make any assumption of the relation between c and d. It is easy to see that the following function is an exact potential for this game:

$$\Phi(0,0) = a - c \qquad \Phi(0,1) = \Phi(1,0) = 0 \qquad \Phi(1,1) = b - d.$$

This game has two pure Nash equilibria: $(0,1)$ and $(1,0)$. Note that Nash Equilibria are *unfair*, as one player has utility $\max\{c,d\}$ and the other $\min\{c,d\}$.

Stationary expected social welfare. The logit dynamics for the game defined by the payoffs in Table 11 gives the following update probabilities for any strategy $x \in \{0,1\}$ (see Equation (1))

$$\begin{array}{rclrcl} \sigma_1(0 \mid (x,0)) = \sigma_2(0 \mid (0,x)) & = & \frac{1}{1+e^{(c-a)\beta}} & \sigma_1(1 \mid (x,0)) = \sigma_2(1 \mid (0,x)) & = & \frac{1}{1+e^{-(c-a)\beta}} \\ \sigma_1(0 \mid (x,1)) = \sigma_2(0 \mid (1,x)) & = & \frac{1}{1+e^{-(d-b)\beta}} & \sigma_1(1 \mid (x,1)) = \sigma_2(1 \mid (1,x)) & = & \frac{1}{1+e^{(d-b)\beta}}. \end{array}$$

Theorem 6 (Expected social welfare) *The stationary expected social welfare $\mathbf{E}_\pi[W]$ of the logit dynamics for the anti-coordination game is*

$$\mathbf{E}_\pi[W] = 2 \cdot \mathbf{E}_\pi[u_i] = 2 \cdot \frac{c + d + ae^{(a-c)\beta} + be^{(b-d)\beta}}{2 + e^{(a-c)\beta} + e^{(b-d)\beta}}$$

Proof. The stationary distribution of this Markov chain is

$$\begin{array}{rcl} \pi(0,0) & = & \frac{e^{(a-c)\beta}}{Z(\beta)} \\ \pi(1,1) & = & \frac{e^{(b-d)\beta}}{Z(\beta)} \\ \pi(0,1) = \pi(1,0) & = & \frac{2}{Z(\beta)} \end{array}$$

where $Z(\beta) = e^{(a-c)\beta} + e^{(b-d)\beta} + 2$. Observe that the probability that at the stationary distribution the players are in one of the two states that are not Nash equilibria, $(0,0)$ or $(1,1)$ is $\frac{1}{2}$ for $\beta = 0$, that is when players select randomly the strategy to play and it goes down to 0 as β increases. The expected utility is the same for every player $i \in \{0,1\}$

$$\mathbf{E}_\pi[u_i] = \sum_{\mathbf{x} \in \{0,1\}^2} u_i(\mathbf{x})\pi(\mathbf{x}) = \frac{c + d + ae^{(a-c)\beta} + be^{(b-d)\beta}}{2 + e^{(a-c)\beta} + e^{(b-d)\beta}} \tag{12}$$

□

Observe that $\mathbf{E}_\pi[u_i] \leqslant \frac{c+d}{2}$ and thus $\mathbf{E}_\pi[W] \leqslant c+d$. In other words, for all β, the stationary expected social welfare is worse than the one guaranteed by a Nash equilibrium. On the other hand, for sufficiently large β we have that $\mathbf{E}_\pi[u_i] \geqslant \min\{c,d\}$. That is, in the logit dynamics each player expects to gain more than in the worst Nash equilibrium.

Now we study the mixing time of the logit dynamics and we show that it is exponential in β. The proof is similar to the one of Theorem 5 and it is omitted.

Theorem 7 (Mixing time) *The mixing time of the logit dynamics with parameter β for the anti-coordination game in Table 11 is $\Theta(e^{(d-b)\beta})$.*

5 A simple n-players game: OR-game

In this section we consider the following simple n-player potential game that we here call *OR-game*. Despite its simplicity, the analysis of the mixing time is far from trivial. For the upper bound we use the path coupling techniques on the Hamming graph with carefully chosen edge weights. Every player has two strategies, say $\{0,1\}$, and each player pays the OR of the strategies of all players (including herself). More formally, the utility function of player $i \in [n]$ is

$$u_i(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{0}; \\ -1, & \text{otherwise.} \end{cases}$$

Notice that the OR-game has $2^n - n$ Nash equilibria. The only profiles that are not Nash equilibria are the n profiles with exactly one player playing 1. Nash equilibrium $\mathbf{0}$ has social welfare 0, while all the others have social welfare $-n$.

In Theorem 8 we show that the stationary expected social welfare is always better than the social welfare of the worst Nash equilibrium, and it is *significantly* better for large β. Unfortunately, in Theorem 9 we show that, if β is large enough to guarantee a *good* stationary expected social welfare, then the time needed to get close to the stationary distribution is exponential in n. Finally, in Theorem 10 we give upper bounds on the mixing time showing that, if β is relatively small then the mixing time is polynomial in n, while for large β the upper bound is exponential in n and it is almost-tight with the lower bound.

Theorem 8 (Expected social welfare) *The stationary expected social welfare of the logit dynamics for the OR-game is $\mathbf{E}_\pi[W] = -\alpha n$ where $\alpha = \alpha(n,\beta) = \frac{(2^n-1)e^{-\beta}}{1+(2^n-1)e^{-\beta}}$.*

Proof. Observe that the OR-game is a potential game with exact potential Φ where $\Phi(\mathbf{0}) = 0$ and $\Phi(\mathbf{x}) = -1$ for every $\mathbf{x} \neq \mathbf{0}$. Hence, the stationary distribution is

$$\pi(\mathbf{x}) = \begin{cases} 1/Z, & \text{if } \mathbf{x} = \mathbf{0}; \\ e^{-\beta}/Z, & \text{if } \mathbf{x} \neq \mathbf{0}; \end{cases}$$

where the normalizing factor is $Z = 1 + (2^n-1)e^{-\beta}$. The expected social welfare is thus

$$\mathbf{E}_\pi[W] = \sum_{\mathbf{x} \in \{0,1\}^n} W(\mathbf{x})\pi(\mathbf{x}) = -n \cdot \frac{(2^n-1)e^{-\beta}}{1+(2^n-1)e^{-\beta}}.$$

□

In the next theorem we show that the mixing time can be polynomial in n only if $\beta \leqslant c\log n$ for some constant c.

Theorem 9 (Lower bound on mixing time) *The mixing time of the logit dynamics for the OR-game is*

1. *$\Omega(e^\beta)$ if $\beta < \log(2^n-1)$;*
2. *$\Omega(2^n)$ if $\beta > \log(2^n-1)$.*

Proof. Consider the set $S \subseteq \{0,1\}^n$ containing only the state $\mathbf{0} = (0,\dots,0)$ and observe that $\pi(\mathbf{0}) \leqslant 1/2$ for $\beta \leqslant \log(2^n - 1)$. The bottleneck ratio is

$$\Phi(\mathbf{0}) = \frac{1}{\pi(\mathbf{0})} \sum_{\mathbf{y} \in \{0,1\}^n} \pi(\mathbf{0}) P(\mathbf{0},\mathbf{y}) = \sum_{\mathbf{y} \in \{0,1\}^n \,:\, |\mathbf{y}|=1} P(\mathbf{0},\mathbf{y}) = n \cdot \frac{1}{n} \frac{1}{1+e^{\beta}}$$

Hence the mixing time is

$$t_{\text{mix}} \geqslant \frac{1}{\Phi(\mathbf{0})} = 1 + e^{\beta}$$

When $\beta > \log(2^n - 1)$ consider the set $R \subseteq \{0,1\}^n$ containing all states except state $\mathbf{0}$, and observe that

$$\pi(R) = \frac{1}{Z}(2^n - 1)e^{-\beta} = \frac{(2^n-1)e^{-\beta}}{1 + (2^n - 1)e^{-\beta}}$$

and it is less than $1/2$ for $\beta > \log(2^n - 1)$. It holds that

$$Q(R, \overline{R}) = \sum_{\mathbf{x} \in R} \pi(\mathbf{x}) P(\mathbf{x}, \mathbf{0}) = \sum_{\mathbf{x} \in \{0,1\}^n \,:\, |\mathbf{x}|=1} \pi(\mathbf{x}) P(\mathbf{x},\mathbf{0}) = n \frac{e^{-\beta}}{Z} \frac{1}{n} \frac{1}{1+e^{-\beta}}$$

The bottleneck ratio is

$$\Phi(R) = \frac{Q(R,\overline{R})}{\pi(R)} = \frac{Z}{(2^n-1)e^{-\beta}} \frac{e^{-\beta}}{Z} \frac{1}{1+e^{-\beta}} = \frac{1}{(2^n-1)(1+e^{-\beta})} < \frac{1}{2^n - 1}$$

Hence the mixing time is

$$t_{\text{mix}} \geqslant \frac{1}{\Phi(R)} > 2^n - 1$$

$\square$

In the next theorem we give upper bounds on the mixing time depending on the value of β. The theorem shows that, if $\beta \leqslant c \log n$ for some constant c, the mixing time is effectively polynomial in n with degree depending on c. The use of the path coupling technique in the proof of the theorem requires a careful choice of the edge-weights.

Theorem 10 (Upper bound on mixing time) *The mixing time of the logit dynamics for the OR-game is*

1. $\mathcal{O}(n \log n)$ *if* $\beta < (1-\varepsilon)\log n$, *for an arbitrary small constant* $\varepsilon > 0$;
2. $\mathcal{O}(n^{c+3} \log n)$ *if* $\beta \leqslant c \log n$, *where* $c \geqslant 1$ *is an arbitrary constant.*

Moreover the mixing time is $\mathcal{O}(n^{5/2} 2^n)$ *for every* β.

Proof. We apply the path coupling technique (see Theorem 20 in Appendix A) with the Hamming graph defined in (5). Let $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$ be two profiles differing at only one player $j \in [n]$, without loss of generality let us assume $|\mathbf{x}| = k-1$ and $|\mathbf{y}| = k$ for some $k = 1, \dots, n$. We set the weight of edge $\{\mathbf{x}, \mathbf{y}\}$ depending only on k, i.e. $\ell(\{\mathbf{x},\mathbf{y}\}) = \delta_k$ where $\delta_k \geqslant 1$ will be chosen later. Consider the coupling defined in Subsection 2.2.

Now we evaluate the expected distance after one step $\mathbf{E}_{\mathbf{x},\mathbf{y}}[\rho(X_1, Y_1)]$ of the two coupled chains (X_t, Y_t) starting at $(\mathbf{x}, \mathbf{y})$. Let i be the player chosen for the update. Observe that if $i = j$, i.e. if we update the player where $\mathbf{x}$ and $\mathbf{y}$ are different (this holds with probability $1/n$), then the distance after one step is zero, otherwise we distinguish four cases depending on the value of k.

Case $k = 1$: In this case profile $\mathbf{x}$ is all zeros and profile $\mathbf{y}$ has only one 1 and the length of edge $\{\mathbf{x}, \mathbf{y}\}$ is $\ell(\mathbf{x}, \mathbf{y}) = \delta_1$. When choosing a player $i \neq j$ (this happens with probability $(n-1)/n$), at the next step the two chains will be at distance δ_1 (if in both chains player i chooses strategy 0, and this holds with probability $\min\{\sigma_i(0 \mid \mathbf{x}),\, \sigma_i(0 \mid \mathbf{y})\}$), or at distance δ_2 (if in both chains player i chooses strategy 1, and this holds with probability $\min\{\sigma_i(1 \mid \mathbf{x}),\, \sigma_i(1 \mid \mathbf{y})\}$), or at distance $\delta_1 + \delta_2$ (if player i chooses strategy 0 in chain X_1 and strategy 1 in chain Y_1, and this holds with the remaining probability). Notice that, from the definition of the coupling, it will never happen that player i chooses strategy 1 in chain X_1 and strategy 0 in chain Y_1, indeed we have that

$$\min\{\sigma_i(0\,|\,\mathbf{x}),\, \sigma_i(0\,|\,\mathbf{y})\} = \sigma_i(0\,|\,\mathbf{y}) = \frac{1}{2} \qquad \text{and} \qquad \min\{\sigma_i(1\,|\,\mathbf{x}),\, \sigma_i(1\,|\,\mathbf{y})\} = \sigma_i(1\,|\,\mathbf{x}) = \frac{1}{1+e^{\beta}} \tag{13}$$

Hence the expected distance after one step is

$$\begin{aligned} \mathbf{E}_{\mathbf{x},\mathbf{y}}\left[\rho(X_1,Y_1)\right] &= \frac{n-1}{n}\left(\frac{1}{2}\delta_1 + \frac{1}{1+e^{\beta}}\delta_2 + \left(1-\frac{1}{2}-\frac{1}{1+e^{\beta}}\right)(\delta_1+\delta_2)\right) \\ &= \frac{n-1}{n}\left(\frac{\delta_1}{1+e^{-\beta}} + \frac{\delta_2}{2}\right) \end{aligned} \tag{14}$$

Case $k=2$: In this case we have $x_j = 0$ and $y_j = 1$, there is another player $h \in [n] \setminus \{j\}$ where $x_h = y_h = 1$, and for all the other players $i \in [n] \setminus \{j,h\}$ it holds $x_i = y_i = 0$. Hence the length of edge $\{\mathbf{x},\mathbf{y}\}$ is $\ell(\mathbf{x},\mathbf{y}) = \delta_2$.

When player h is chosen (this holds with probability $1/n$) we have that $\sigma_h(s\,|\,\mathbf{x})$ and $\sigma_h(s\,|\,\mathbf{y})$ for $s = 0,1$ are the same as in (13). At the next step the two chains will be at distance δ_2 (if player h stays at strategy 1 in both chains), or at distance δ_1 (if player h chooses strategy 0 in both chains), or at distance $\delta_1+\delta_2$ (if player h stays at strategy 0 in chain X_1 and chooses strategy 1 in chain Y_1).

When a player $i \notin \{h,j\}$ is chosen (this holds with probability $(n-2)/n$) we have that $\sigma_i(0,\mathbf{x}) = \sigma_i(1,\mathbf{x}) = \sigma_i(0,\mathbf{y}) = \sigma_i(1,\mathbf{y}) = 1/2$. Thus in this case the two coupled chains always perform the same choice at player i, and at the next step they will be at distance δ_2 (if player i stays at strategy 0 in both chains) or at distance δ_3 (if player i chooses strategy 1 in both chains).
Hence the expected distance after one step is

$$\begin{aligned} \mathbf{E}_{\mathbf{x},\mathbf{y}}\left[\rho(X_1,Y_1)\right] &= \frac{1}{n}\left(\frac{1}{2}\delta_1 + \frac{1}{1+e^{\beta}}\delta_2 + \left(1-\frac{1}{2}-\frac{1}{1+e^{\beta}}\right)(\delta_1+\delta_2)\right) + \frac{n-2}{n}\left(\frac{1}{2}\delta_2+\frac{1}{2}\delta_3\right) \\ &= \frac{1}{2n}\left(\frac{2}{1+e^{-\beta}}\delta_1 + (n-1)\delta_2 + (n-2)\delta_3\right) \end{aligned} \tag{15}$$

Case $3 \leqslant k \leqslant n-1$: When a player $i \neq j$ is chosen such that $x_i = y_i = 1$ (this holds with probability $(k-1)/n$) then at the next step the two chains will be at distance δ_k (if i stays at strategy 1) or at distance δ_{k-1} (if i moves to strategy 0). When a player $i \neq j$ is chosen such that $x_i = y_i = 0$ (this holds with probability $(n-k)/n$) then at the next step the two chains will be at distance δ_k (if i chooses to stay at strategy 0) or at distance δ_{k+1} (if i chooses to move to strategy 0). Hence the expected distance after one step is

$$\begin{aligned} \mathbf{E}_{\mathbf{x},\mathbf{y}}\left[\rho(X_1,Y_1)\right] &= \frac{k-1}{n}\left(\frac{1}{2}\delta_k+\frac{1}{2}\delta_{k-1}\right) + \frac{n-k}{n}\left(\frac{1}{2}\delta_k+\frac{1}{2}\delta_{k+1}\right) \\ &= \frac{1}{2n}\left((n-1)\delta_k + (k-1)\delta_{k-1} + (n-k)\delta_{k+1}\right) \end{aligned} \tag{16}$$

Case $k=n$: When a player $i \neq j$ is chosen, then at the next step the two chains will be at distance δ_n or at distance δ_{n-1}. Hence the expected distance after one step is

$$\mathbf{E}_{\mathbf{x},\mathbf{y}}\left[\rho(X_1,Y_1)\right] = \frac{n-1}{n}\left(\frac{1}{2}\delta_n+\frac{1}{2}\delta_{n-1}\right) = \frac{n-1}{2n}(\delta_n+\delta_{n-1}) \tag{17}$$

In order to apply Theorem 20 we now have to show that it is possible to choose the edge weights $\delta_1,\dots,\delta_n$ and a parameter $\alpha > 0$ such that

$$\begin{array}{rcll} \frac{n-1}{n}\left(\frac{\delta_1}{1+e^{-\beta}}+\frac{\delta_2}{2}\right) & \leqslant & \delta_1 e^{-\alpha} & \\ \frac{1}{2n}\left(\frac{2}{1+e^{-\beta}}\delta_1 + (n-1)\delta_2 + (n-2)\delta_3\right) & \leqslant & \delta_2 e^{-\alpha} & \\ \frac{1}{2n}\left((n-1)\delta_k + (k-1)\delta_{k-1} + (n-k)\delta_{k+1}\right) & \leqslant & \delta_k e^{-\alpha} & k = 3,\dots,n-1 \\ \frac{n-1}{2n}(\delta_n+\delta_{n-1}) & \leqslant & \delta_n e^{-\alpha} & \end{array} \tag{18}$$

For different values of β, we make different choices for α and for the weights δ_k. For clarity's sake we split the proof in three different lemmas. We denote by $\delta^{\max}$ the largest δ_k.

In Lemma 11 we show that Inequalities (18) are satisfied for every value of β by choosing the weights as follows

$$\delta_k = \begin{cases} \frac{1}{2}[(n-1)\delta_2 + 1], & \text{if } k = 1; \\ \frac{n-k}{k}\delta_{k+1} + 1, & \text{if } 2 \leqslant k \leqslant n-1; \\ 1, & \text{if } k = n; \end{cases}$$

and by setting $\alpha = 1/(2n\delta^{\max})$. Since the diameter of the Hamming graph is $\sum_{i=1}^{n} \delta_i \leqslant n\delta^{\max}$, from Theorem 20 and Corollary 16 in the appendix, we obtain $t_{\text{mix}} = \mathcal{O}(n^{5/2}2^n)$.

In Lemma 12 we show that, if $\beta < (1-\epsilon)\log n$ for an arbitrarily small constant $\varepsilon > 0$, Inequalities (18) are satisfied for every value of β by choosing weights $\delta_1 = n^{1-\varepsilon}$, $\delta_2 = 4/3, \delta_3 = \cdots = \delta_n = 1$ and $\alpha = 1/n$. This gives $t_{\text{mix}} = \mathcal{O}(n \log n)$.

In Lemma 13 we show that, Inequalities (18) are satisfied by choosing weights as follows

$$\delta_k = \begin{cases} \frac{1+e^{-\beta}}{2}\left[\frac{a_1}{b_1}\delta_2 + 1\right], & \text{if } k = 1; \\ \frac{a_k}{b_k}\delta_{k+1} + 1, & \text{if } 2 \leqslant k \leqslant n-1; \\ 1, & \text{if } k = n; \end{cases}$$

where $a_1 = n-1$ and $b_1 = ne^{-\beta} + 1$ and, for every $k = 2, \dots, n-1$

$$a_k = (n-k)b_{k-1} \qquad b_k = (n+1)b_{k-1} - (k-1)a_{k-1}$$

and by setting $\alpha = 1/(2n\delta^{\max})$. From Corollary 19 it follows that, if $\beta \leqslant c \log n$ for a constant $c \in \mathbb{N}$, we have that $\delta_{\max} = \mathcal{O}(n^{c+2})$. Since the diameter of the Hamming graph is $\sum_{i=1}^{n} \delta_i \leqslant n\delta^{\max}$, from Theorem 20 in Appendix A it follows that $t_{\text{mix}} = \mathcal{O}(n^{c+3} \log n)$. □

5.1 Technical lemmas

In this section we prove the technical lemmas needed for completing the proof of Theorem 10.

Lemma 11 *Let $\delta_1, \dots, \delta_n$ be as follows*

$$\delta_k = \begin{cases} \frac{1}{2}[(n-1)\delta_2 + 1], & \text{if } k = 1; \\ \frac{n-k}{k}\delta_{k+1} + 1, & \text{if } 2 \leqslant k \leqslant n-1; \\ 1, & \text{if } k = n; \end{cases} \tag{19}$$

and let $\alpha = 1/(2n\delta^{\max})$ where $\delta^{\max} = \max\{\delta_k \ : \ k = 1, \dots, n\}$. Then Inequalities (18) are satisfied for every $\beta \geqslant 0$.

Proof. Observe that, for every $k = 1, \dots, n$, the right-hand side of the k-th inequality in (18) is

$$\delta_k e^{-\alpha} = \delta_k e^{-1/(2n\delta^{\max})} \geqslant \delta_k \left(1 - \frac{1}{2n\delta^{\max}}\right) = \delta_k - \frac{\delta_k}{2n\delta^{\max}} \geqslant \delta_k - \frac{1}{2n} \tag{20}$$

Now we check that the left-hand side is at most $\delta_k - 1/(2n)$.

<u>First inequality ($k = 1$):</u> $\frac{n-1}{n}\left(\frac{\delta_1}{1+e^{-\beta}} + \frac{\delta_2}{2}\right) \leqslant \delta_1 e^{-\alpha}$

From the definition of δ_1 in (19) we have that

$$\delta_2 = \frac{2\delta_1 - 1}{n-1}$$

Hence the left-hand side is

$$\frac{n-1}{n}\left(\frac{\delta_1}{1+e^{-\beta}} + \frac{\delta_2}{2}\right) \leqslant \frac{n-1}{n}\left(\delta_1 + \frac{\delta_2}{2}\right) = \frac{n-1}{n}\left(\delta_1 + \frac{2\delta_1 - 1}{2(n-1)}\right) = \frac{1}{2n}(2n\delta_1 - 1) = \delta_1 - \frac{1}{2n}$$

Second inequality ($k=2$): $\frac{1}{2n}\left(\frac{2}{1+e^{-\beta}}\delta_1+(n-1)\delta_2+(n-2)\delta_3\right)\leqslant \delta_2 e^{-\alpha}$
From the definition of δ_2 in (19) we have that

$$\delta_3=\frac{2}{n-2}(\delta_2-1)$$

Hence the left-hand side of the second inequality is

$$\begin{aligned}\frac{1}{2n}\left(\frac{2}{1+e^{-\beta}}\delta_1+(n-1)\delta_2+(n-2)\delta_3\right) &\leqslant \frac{1}{2n}\left(2\delta_1+(n-1)\delta_2+(n-2)\delta_3\right)\\ &= \frac{1}{2n}\left((n-1)\delta_2+1+(n-1)\delta_2+2(\delta_2-1)\right)\\ &= \frac{1}{2n}(2n\delta_2-1)=\delta_2-\frac{1}{2n}\end{aligned}$$

Other inequalities ($k=3,\ldots,n-1$): $\frac{1}{2n}\left((n-1)\delta_k+(k-1)\delta_{k-1}+(n-k)\delta_{k+1}\right)\leqslant \delta_k e^{-\alpha}$
From the definition of δ_k in (19) we have that

$$\delta_{k+1}=\frac{k}{n-k}(\delta_k-1)$$

Hence the left-hand side is

$$\begin{aligned}\frac{1}{2n}\left((n-1)\delta_k+(k-1)\delta_{k-1}+(n-k)\delta_{k+1}\right) &= \frac{1}{2n}\left((n-1)\delta_k+(n-k+1)\delta_k+(k-1)+k\delta_k-k\right)\\ &= \frac{1}{2n}(2n\delta_k-1)=\delta_k-\frac{1}{2n}\end{aligned}$$

Last inequality ($k=n$): $\frac{n-1}{2n}(\delta_n+\delta_{n-1})\leqslant \delta_n e^{-\alpha}$
Since $\delta_n=1$ and $\delta_{n-1}=\frac{1}{n-1}\delta_n+1=\frac{n}{n-1}$, the left-hand side of the last inequality is

$$\frac{n-1}{2n}(\delta_n+\delta_{n-1})=\frac{n-1}{2n}(1+\frac{n}{n-1})=1-\frac{1}{2n}$$

□

Lemma 12 *Let $\delta_1,\ldots,\delta_n$ be as follows*

$$\delta_1=n^{1-\varepsilon},\ \delta_2=4/3,\ \delta_3=\cdots=\delta_n=1$$

where $\varepsilon>0$ is an arbitrary small constant and let $\alpha=1/n$. Then Inequalities (18) are satisfied for every $\beta\leqslant(1-\varepsilon)\log n$.

Proof. We check that all the inequalities in (18) are satisfied.
First inequality ($k=1$): $\frac{n-1}{n}\left(\frac{\delta_1}{1+e^{-\beta}}+\frac{\delta_2}{2}\right)\leqslant \delta_1 e^{-\alpha}$
For the left-hand side we have

$$\begin{aligned}\frac{n-1}{n}\left(\frac{\delta_1}{1+e^{-\beta}}+\frac{\delta_2}{2}\right) &= \left(1-\frac{1}{n}\right)\left(\frac{n^{1-\varepsilon}}{1+e^{-\beta}}+\frac{2}{3}\right)\\ &\leqslant \left(1-\frac{1}{n}\right)\left(\frac{n^{1-\varepsilon}}{1+\frac{1}{n^{1-\varepsilon}}}+\frac{2}{3}\right)=\left(1-\frac{1}{n}\right)\left(\frac{n^{2(1-\varepsilon)}}{n^{1-\varepsilon}+1}+\frac{2}{3}\right)\\ &= \left(1-\frac{1}{n}\right)\left(\frac{(n^{1-\varepsilon}+1)(n^{1-\varepsilon}-1)+1}{n^{1-\varepsilon}+1}+\frac{2}{3}\right)=\left(1-\frac{1}{n}\right)\left(n^{1-\varepsilon}+\frac{1}{n^{1-\varepsilon}+1}-\frac{1}{3}\right).\end{aligned}$$

For the right-hand side we have

$$\delta_1 e^{-\alpha}=n^{1-\varepsilon}e^{-1/n}\geqslant n^{1-\varepsilon}\left(1-\frac{1}{n}\right).$$

Hence the left-hand side is smaller than the right-hand one (for n sufficiently large).

Second inequality ($k = 2$): $\frac{1}{2n}\left(\frac{2}{1+e^{-\beta}}\delta_1 + (n-1)\delta_2 + (n-2)\delta_3\right) \leqslant \delta_2 e^{-\alpha}$

For the left-hand side we have

$$\begin{aligned}\frac{1}{2n}\left(\frac{2}{1+e^{-\beta}}\delta_1 + (n-1)\delta_2 + (n-2)\delta_3\right) &= \frac{1}{2n}\left(\frac{2}{1+e^{-\beta}}n^{1-\varepsilon} + (n-1)\frac{4}{3} + (n-2)\right)\\ &\leqslant \frac{1}{2n}\left(2n^{1-\varepsilon} + \frac{7}{3}n\right) = \frac{7}{6} + \frac{1}{n^{\varepsilon}}\end{aligned}$$

And for the right-hand side we have

$$\delta_2 e^{-\alpha} = \frac{4}{3}e^{-1/n} \geqslant \frac{4}{3}\left(1 - \frac{1}{n}\right) \geqslant \frac{4}{3} - \frac{1}{n}$$

Hence the left-hand side is smaller than the right-hand one (for n sufficiently large).

Third inequality ($k = 3$): $\frac{1}{2n}\left((n-1)\delta_3 + 2\delta_2 + (n-3)\delta_4\right) \leqslant \delta_3 e^{-\alpha}$

For the left-hand side we have

$$\begin{aligned}\frac{1}{2n}\left((n-1)\delta_3 + 2\delta_2 + (n-3)\delta_4\right) &= \frac{1}{2n}\left((n-1) + 2\frac{4}{3} + (n-3)\right)\\ &= \frac{1}{2n}(2n-3) \leqslant \frac{n-1}{n}\end{aligned}$$

And for the right-hand side we have

$$\delta_3 e^{-\alpha} = e^{-1/n} \geqslant \left(1 - \frac{1}{n}\right)$$

Hence the left-hand side is smaller than the right-hand one.

Other inequalities ($k \geqslant 4$): $\frac{1}{2n}\left((n-1)\delta_k + (k-1)\delta_{k-1} + (n-k)\delta_{k+1}\right) \leqslant \delta_k e^{-\alpha}$

Since $\delta_k = \delta_{k-1} = \delta_{k+1} = 1$ the left-hand side is equal to $\frac{n-1}{n}$ and the right-hand side is $e^{-1/n} \geqslant \frac{n-1}{n}$ □

Lemma 13 *Let $\delta_1, \ldots, \delta_n$ be as follows*

$$\delta_k = \begin{cases} \frac{1+e^{-\beta}}{2}\left[\frac{a_1}{b_1}\delta_2 + 1\right], & \text{if } k = 1;\\ \frac{a_k}{b_k}\delta_{k+1} + 1, & \text{if } 2 \leqslant k \leqslant n-1;\\ 1, & \text{if } k = n; \end{cases} \tag{21}$$

where $a_1 = n - 1$ and $b_1 = ne^{-\beta} + 1$ and for every $k = 2, \ldots, n-1$

$$a_k = (n-k)b_{k-1} \qquad b_k = (n+1)b_{k-1} - (k-1)a_{k-1}$$

and let $\alpha = 1/(2n\delta^{\max})$ where $\delta^{\max} = \max\{\delta_k \ : \ k = 1, \ldots, n\}$. Then Inequalities (18) are satisfied for every $\beta \geqslant 0$.

Proof. Observe that, as in Equation (20), for every $k = 1, \ldots, n$, the right-hand side of the k-th inequality in (18) is

$$\delta_k e^{-\alpha} \geqslant \delta_k - \frac{1}{2n}$$

Now we check that the left-hand side is at most $\delta_k - 1/(2n)$.

First inequality ($k = 1$): $\frac{n-1}{n}\left(\frac{\delta_1}{1+e^{-\beta}} + \frac{\delta_2}{2}\right) \leqslant \delta_1 e^{-\alpha}$

From the definition of δ_1 in (21) we have that

$$\delta_2 = \frac{ne^{-\beta}+1}{n-1}\left(\frac{2\delta_1}{1+e^{-\beta}} - 1\right)$$

Hence the left-hand side is

$$\begin{aligned}
\frac{n-1}{n}\left(\frac{\delta_1}{1+e^{-\beta}}+\frac{\delta_2}{2}\right) &= \frac{n-1}{n}\left[\frac{\delta_1}{1+e^{-\beta}}+\frac{ne^{-\beta}+1}{n-1}\left(\frac{\delta_1}{1+e^{-\beta}}-\frac{1}{2}\right)\right] \\
&= \frac{n-1}{n}\frac{\delta_1}{1+e^{-\beta}}\left(1+\frac{ne^{-\beta}+1}{n-1}\right)-\frac{ne^{-\beta}+1}{2n} \\
&\leqslant \delta_1-\frac{1}{2n}
\end{aligned}$$

Second inequality $(k=2)$: $\frac{1}{2n}\left(\frac{2}{1+e^{-\beta}}\delta_1+(n-1)\delta_2+(n-2)\delta_3\right)\leqslant \delta_2 e^{-\alpha}$
From the definition of δ_2 in (21) we have that

$$\delta_3=\frac{b_2}{a_2}(\delta_2-1)=\frac{(n+1)b_1-a_1}{(n-2)b_1}(\delta_2-1)$$

Hence the left-hand side is

$$\begin{aligned}
\frac{1}{2n}\left(\frac{2}{1+e^{-\beta}}\delta_1+(n-1)\delta_2+(n-2)\delta_3\right) &= \frac{1}{2n}\left[\left(\frac{a_1}{b_1}\delta_2+1\right)+(n-1)\delta_2+\frac{(n+1)b_1-a_1}{b_1}(\delta_2-1)\right] \\
&= \delta_2-\frac{1}{2n}\frac{nb_1-a_1}{b_1}=\delta_2-\frac{1}{2n}\left(n-\frac{n-1}{ne^{-\beta}+1}\right) \\
&\leqslant \delta_2-\frac{1}{2n}
\end{aligned}$$

For the next inequalities we need the following claim.

Claim 14 *For every $k\geqslant 2$, it holds that $b_k\geqslant kb_{k-1}$.*

Proof. We proceed by induction on k. The base case $k=2$ follows from

$$b_2=(n+1)(ne^{-\beta}+1)-(n-1)=(n+1)ne^{-\beta}+2>2(ne^{-\beta}+1)=2b_1.$$

Now suppose the claim holds for $k-1$, that is $b_{k-1}\geqslant (k-1)b_{k-2}$. Then

$$\begin{aligned}
b_k &= (n+1)b_{k-1}-(k-1)a_{k-1} \\
&= (n+1)b_{k-1}-(k-1)(n-k+1)b_{k-2} \\
&\geqslant [(n+1)-(n-k+1)]\,b_{k-1}=kb_{k-1}
\end{aligned}$$

□

Other inequalities $(k=3,\ldots,n-1)$: $\frac{1}{2n}\left((n-1)\delta_k+(k-1)\delta_{k-1}+(n-k)\delta_{k+1}\right)\leqslant \delta_k e^{-\alpha}$
From the definition of δ_k in (21) we have that

$$\delta_{k+1}=\frac{b_k}{a_k}(\delta_k-1)=\frac{(n+1)b_{k-1}-(k-1)a_{k-1}}{(n-k)b_{k-1}}(\delta_k-1)$$

Hence the left-hand side is

$$\begin{aligned}
\frac{1}{2n}\left((n-1)\delta_k+(k-1)\delta_{k-1}+(n-k)\delta_{k+1}\right) &= \frac{1}{2n}\left[(n-1)\delta_k+(k-1)\left(\frac{a_{k-1}}{b_{k-1}}\delta_k+1\right)\right. \\
&+ \left.\frac{(n+1)b_{k-1}-(k-1)a_{k-1}}{b_{k-1}}(\delta_k-1)\right] \\
&= \delta_k-\frac{1}{2n}\frac{(n-k+2)b_{k-1}-(k-1)a_{k-1}}{b_{k-1}} \\
&= \delta_k-\frac{1}{2n}\left((n-k+2)-(k-1)(n-k+1)\frac{b_{k-2}}{b_{k-1}}\right) \\
&\leqslant \delta_2-\frac{1}{2n}
\end{aligned}$$

where the inequality follows from the above claim.
Last inequality ($k = n$): $\frac{n-1}{2n}(\delta_n + \delta_{n-1}) \leqslant \delta_n e^{-\alpha}$
Since $\delta_n = 1$ and $\delta_{n-1} = \frac{a_{n-1}}{b_{n-1}}\delta_n + 1 = \frac{a_{n-1}}{b_{n-1}} + 1$, the left-hand side of the last inequality is

$$\begin{aligned}\frac{n-1}{2n}(\delta_n + \delta_{n-1}) &= \frac{n-1}{2n}\left(2 + \frac{a_{n-1}}{b_{n-1}}\right) = \frac{n-1}{2n}\left(2 + \frac{b_{n-2}}{b_{n-1}}\right) \\ &\leqslant \frac{n-1}{2n}\left(2 + \frac{1}{n-1}\right) = 1 - \frac{1}{2n}\end{aligned}$$

where the inequality follows from the above claim. □

Observation 15 *Let $\delta_1, \ldots, \delta_n$ be defined recursively as follows: $\delta_n = 1$ and*

$$\delta_k = \gamma_k \delta_{k+1} + 1$$

where $\gamma_k > 0$ for every $k = 1, \ldots, n-1$. Let $\delta^{\max} = \max\{\delta_k \ : \ k = 1, \ldots, n\}$. Then

$$\delta^{\max} \leqslant n \max\left\{\prod_{i=h}^{j} \gamma_i \ : \ 1 \leqslant h \leqslant j \leqslant n-1\right\}$$

Proof. The observation follows from the fact that, for $k = 1, \ldots, n-1$, we have

$$\delta_k = 1 + \sum_{j=k}^{n-1} \prod_{i=k}^{j} \gamma_i$$

□

Corollary 16 *Let $\delta_1, \ldots, \delta_n$ be defined as in Lemma 11. Then $\delta^{\max} \leqslant c\sqrt{n}2^n$ for a suitable constant c.*

Proof. By Observation 15 and the definition of $\delta_1, \ldots, \delta_n$, it holds that

$$\begin{aligned}\delta^{\max} &\leqslant n \max\left\{\prod_{i=h}^{j} \frac{n-i}{i} \ : \ 1 \leqslant h \leqslant j \leqslant n\right\} \\ &\leqslant n \prod_{i=1}^{\lfloor n/2 \rfloor} \frac{n-i}{i} \leqslant n\binom{n}{\lfloor n/2 \rfloor} \leqslant c\sqrt{n}2^n\end{aligned}$$

for a suitable constant c. □

Lemma 17 *Let $\delta_1, \ldots, \delta_n$ be defined as in Lemma 13 and $\beta \leqslant c \log n$ for a constant $c \in \mathbb{N}$ and define $\gamma_k = \frac{a_k}{b_k}$. Then*

$$\begin{cases} \gamma_k < n & \forall\, k; \\ \gamma_k < 1 & \text{if } k > c+2; \\ \gamma_{c+2} = \mathcal{O}(1); \end{cases}$$

Proof. Let us write γ_k as

$$\frac{p_k e^{-\beta} + l_k}{q_k e^{-\beta} + r_k}$$

where $p_1 = 0$, $q_1 = n$, $l_1 = n-1$, $r_1 = 1$ and

$$\begin{aligned} p_k &= (n-k)q_{k-1} & q_k &= (n+1)q_{k-1} - (k-1)p_{k-1} \\ l_k &= (n-k)r_{k-1} = (n-k)(k-1)! & r_k &= (n+1)r_{k-1} - (k-1)l_{k-1} = k! \end{aligned}$$

Observe that for every k, p_k is always less than q_k, hence $[nq_k - p_k]e^{-\beta} > 0$; indeed, $l_k - nr_k = (k-1)!(n-k-nk) < 0$. Hence we have for every k

$$\gamma_k = \frac{p_k e^{-\beta} + l_k}{q_k e^{-\beta} + r_k} < n.$$

Moreover, observe that if exists k^* such that $\gamma_{k^*} < 1$, then for every $k \geqslant k^*$ we have $\gamma_{k^*} < 1$. Indeed

$$b_k - a_k = (k+1)b_{k-1} - (k-1)a_{k-1}$$

is larger than 1 since $b_{k-1} > a_{k-1}$ by hypothesis. In order to complete the proof, we need to show that $\gamma_{c+3} < 1$ and $\gamma_{c+2} = \mathcal{O}(1)$. We need the following claim.

Claim 18 *For $1 \leqslant k \leqslant c+2$, we have $q_k \geqslant 2^{-k} n^k$.*

Proof. We proceed by induction on k, with the base $k = 1$ being obvious. Suppose the claim holds for $k-1$, that is $q_{k-1} \geqslant 2^{-(k-1)} n^{k-1}$, then

$$q_k = (n+1)q_{k-1} - (k-1)p_{k-1} \geqslant \frac{n}{2} q_{k-1} \geqslant 2^{-k} n^k.$$

□

From the above claim, since $e^{-\beta} \geqslant n^{-c}$, we have that

$$(q_{c+3} - p_{c+3})e^{-\beta} = [(n+1)q_{c+2} - (c+2)p_{c+2} - (n-c-3)q_{c+2}]e^{-\beta} \geqslant 2q_{c+2}e^{-\beta} \geqslant 2^{-(c+1)} n^2.$$

Indeed, $l_{c+3} - r_{c+3} = (c+2)!(n-2c-6) \leqslant (c+2)! \cdot n$. Hence, for n sufficiently large we have $\gamma_{c+3} < 1$. Similarly, we have $[q_{c+2} - p_{c+2}]e^{-\beta} \geqslant 2^{-c} n$ and $l_{c+2} - r_{c+2} \leqslant (c+1)! \cdot n$. Hence,

$$\gamma_{c+2} \leqslant (c+1)! \cdot 2^c = \mathcal{O}(1).$$

□

Corollary 19 *Let $\delta_1, \ldots, \delta_n$ and c be defined as in Lemma 13. Then $\delta^{\max} = \mathcal{O}(n^{c+2})$.*

Proof. By Observation 15, Lemma 17 and the definition of $\delta_1, \ldots, \delta_n$ it holds that

$$\begin{aligned} \delta^{\max} &\leqslant n \max \left\{ \prod_{i=h}^{j} \frac{a_i}{b_i} \ : \ 1 \leqslant h \leqslant j \leqslant n \right\} \\ &\leqslant n \prod_{i=1}^{c+2} \frac{a_i}{b_i} = \mathcal{O}(n^{c+2}). \end{aligned}$$

□

6 Conclusions and open problems

In this paper we studied strategic games where at every run a player is selected uniformly at random and she is assumed to choose her strategy for the next run according to a *noisy best-response*, where the noise level is tuned by a parameter β. Such dynamics defines a family of ergodic Markov chains, indexed by β, over the set of strategy profiles. We study the long-term behavior of the system by analysing the expected social welfare when the strategy profiles are random according to the stationary distribution of such chains and we compare it with the social welfare at Nash equilibria.

In order for such analysis to be meaningful we are also interested in the *mixing time* of the chains, i.e. how long it takes, for a chain starting at an arbitrary profile, to get close to its stationary distribution. The analysis of the mixing time is usually far from trivial even for very simple games.

We study several examples of applications of this approach to games with two and three players and to a simple n-players game. We started by showing that the social welfare at stationarity for the 3-player linear congestion game that attains the maximum Price of Anarchy is larger than the social welfare of the worst Nash equilibrium. This result is made significant by the fact that, for all β, the logit dynamics converges at the stationary distribution in constant time. For 2-player coordination and anti-coordination games the mixing time turns out to be exponential in β and we give conditions for the expected social welfare at stationarity to be smaller than the social welfare of the worst Nash equilibrium. In the n-player OR-game, the mixing time is $\mathcal{O}(n \log n)$ for β up to $\log n$; if $\beta < c \log n$ with $c > 1$ constant, the mixing time is polynomial in n with the degree depending on constant c; finally, for large β the mixing time is exponential in n.

We leave several open questions for further investigation. For example, we would like to close gaps between upper and lower bounds for the mixing time of the OR-game. Moreover, we would like to investigate logit dynamics for notable classes of n-player games.

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Appendix

A Markov Chains' Summary

We summarize the main tools we use to bound the mixing time of Markov chains (for a complete description of such tools see, for example, Chapters 7.2, 12.2 and 14.2 of [15]).

For an irreducible and aperiodic Markov chain $\mathcal{M}$ with finite state space Ω, transition matrix P, and stationary distribution π, the *mixing time* is defined as

$$t_{\mathrm{mix}}(\varepsilon) = \min\left\{t \in \mathbb{N} \,:\, \max\{\|P^t(x,\cdot) - \pi\|_{\mathrm{TV}} \,:\, x \in \Omega\} \leqslant \epsilon\right\}$$

where $P^t(x,\cdot)$ is the distribution at time t of the chain starting at x, $\|P^t(x,\cdot)-\pi\|_{\mathrm{TV}} = \frac{1}{2}\sum_{y\in\Omega}|P^t(x,y) - \pi(y)|$ is the *total variation distance*, and $\varepsilon > 0$ is a constant smaller than $1/2$. For convenience, it is usually set to $\varepsilon = 1/4$ or $\varepsilon = 1/2e$. If not explicitly specified, when we write t_{mix} we mean $t_{\mathrm{mix}}(1/4)$.

Theorem 20 (Path coupling) *Let $\mathcal{M} = \{X_t \,:\, t \in \mathbb{N}\}$ be an irreducible and aperiodic Markov chain with finite state space Ω and transition matrix P. Let $G = (\Omega, E)$ be a connected graph, let $\ell : E \to \mathbb{R}$ be a function assign* weights *to edges such that $\ell(e) \geqslant 1$ for every edge $e \in E$, and let $\rho : \Omega \times \Omega \to \mathbb{R}$ be the corresponding path distance, i.e. $\rho(x,y)$ is the length of the (weighted) shortest path in G between x and y.*
Suppose that for every edge $\{x,y\} \in E$ a coupling (X,Y) of distributions $P(x,\cdot)$ and $P(y,\cdot)$ exists such that $\mathbf{E}_{x,y}\left[\rho(X,Y)\right] \leqslant \ell(\{x,y\})e^{-\alpha}$ for some $\alpha > 0$. Then the mixing time $t_{mix}(\varepsilon)$ of $\mathcal{M}$ is

$$t_{mix}(\varepsilon) \leqslant \frac{\log(diam(G)) + \log(1/\varepsilon)}{\alpha}$$

where $diam(G)$ is the (weighted) diameter of G.

Theorem 21 (Bottleneck ratio) *Let $\mathcal{M} = \{X_t \,:\, t \in \mathbb{N}\}$ be an irreducible and aperiodic Markov chain with finite state space Ω, transition matrix P and stationary distribution π. Let $S \subseteq \Omega$ be any set with $\pi(S) \leqslant 1/2$. Then the mixing time is*

$$t_{mix}(\varepsilon) \geqslant \frac{1-2\epsilon}{2\Phi(S)}$$

where

$$\Phi(S) = \frac{Q(S,\overline{S})}{\pi(S)} \qquad \text{and} \qquad Q(S,\overline{S}) = \sum_{x\in S,\, y\in\overline{S}} \pi(x)P(x,y).$$

For a reversible transition matrix P of a Markov chain with finite state space Ω, the *relaxation time* t_{rel} is defined as

$$t_{\mathrm{rel}} = \frac{1}{1-\lambda^\star}$$

where $\lambda^\star$ is the largest absolute value of an eigenvalue other than 1,

$$\lambda^\star = \max\{|\lambda| \,:\, \lambda \text{ is an eigenvalue of } P,\, \lambda \neq 1\}\,.$$

Notice that all the eigenvalues have absolute value at most 1, $\lambda = 1$ is an eigenvalue, and for irreducible and aperiodic chains, -1 is not an eigenvalue. Hence t_{rel} is positive and finite.

We have the following theorem.

Theorem 22 (Relaxation time) *Let P be the transition matrix of a reversible, irreducible, and aperiodic Markov chain with state space Ω and stationary distribution π. Then*

$$(t_{rel} - 1)\log\left(\frac{1}{2\epsilon}\right) \leqslant t_{mix}(\epsilon) \leqslant \log\left(\frac{1}{\epsilon\pi_{min}}\right) t_{rel}$$

where $\pi_{min} = \min_{x\in\Omega}\pi(x)$.