

A Note on Uniform Power Connectivity in the SINR Model

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Abstract. Network connectivity is crucial for the operation of any communication network and in this paper we consider the connectivity of wireless networks. More precisely, when given a set of radio transmitters distributed in some area, we would like to build a directed connected communication graph and computing an edge coloring of this graph such that the transmitter-receiver pairs in each color class can communicate simultaneously.

Depending on the interference model, more or less colors, corresponding to the number of frequencies or time slots, are necessary. Graph-based interference model such as the Unit Disk Graph (UDG) model are not realistic enough, since they model interference as a binary property and ignore the accumulated interference of a large number of distant nodes. More realistic is the physical Signal to Interference plus Noise Ratio (SINR) model that compares the received power of a signal at a receiver to the sum of the strength of other signals. The strength of a signal is assumed to fade exponentially with the distance from the sender, depending on the so-called path-loss exponent α . This is the standard interference model used in the engineering community.

In this paper, we study the case where all transmitters use the same power. We show that the number of colors needed is constant in one-dimensional grids if $\alpha > 1$ as well as in two-dimensional grids if $\alpha > 2$. For a smaller path-loss exponents on the two-dimensional grid we prove upper and lower bounds in the order of $\mathcal{O}(\log n)$ and $\Omega(\log n / \log \log n)$ for $\alpha = 2$ and $\Theta(n^{1-\alpha/2})$ for $\alpha < 2$ respectively. If nodes are distributed on the interval $[0, 1]$ uniformly at random, a regular coloring of $\Theta(\log n)$ colors guarantees connectivity.

1 Introduction

The performance of wireless networks depends on the coordination of the timing and frequency bands of broadcasting nodes. This is due to the fact that if two nodes close to each other transmit concurrently, the chances are that neither of their signals can be received correctly because of interference. Thus, choosing an appropriate interference model is critical. The most popular models can be divided into two classes: graph-based models (protocol models) and fading channel models. Graph-based models, such as the unit disk graph (UDG) model [2], describe interference as a binary property by a set of interference edges. The existence of an edge between two communication pairs, usually based on the distance between nodes, implies that the two pairs cannot transmit successfully at the same time (or on the same frequency). Such models, which serve as a simple abstraction of wireless networks, have been very useful for the design of efficient distributed algorithms. Nevertheless, graph-based models bear the limitation of representing interference as a local property. In reality, the interference of several concurrent senders accumulates and can interrupt the reception at a far-away receiver. Therefore, the focus of the algorithmic networking community has recently shifted from graph-based models to the more realistic fading channel models, such as the physical Signal to Noise plus Interference (SINR) model [7] that we use in this paper. In this model, a message is received successfully if the ratio between the strength of the sender signal at the receiving location and the sum of interferences created by all other simultaneous senders plus ambient noise is larger than some hardware-defined threshold. Interference is modeled as continuous property, decreasing exponentially with the distance from the sender, according to the value of the so-called path-loss exponent α . More formally, a receiver r_i receives a sender s_i 's transmission if and only if

$$\frac{\frac{P(s_i)}{d(s_i, r_i)^\alpha}}{N + \sum_{j \neq i} \frac{P(s_j)}{d(s_j, r_i)^\alpha}} \geq \beta,$$

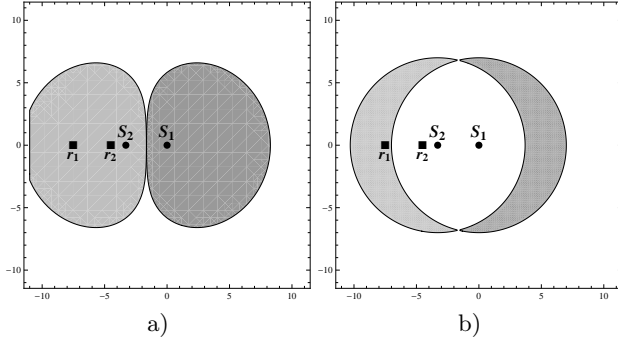


Fig. 1. Reception diagrams for scenario with two links, $l_1 = (s_1, r_1)$ and $l_2 = (s_2, r_2)$. The shaded areas denote where the signal of a sender can be decoded (the area in the lighter gray belongs to sender s_2), white indicates that the received signal power is too weak for reception. *a)* SINR model: only node r_2 receives a message from its sender, the interference is too high at r_1 . *b)* Unit Disk Graph model: neither r_1 nor r_2 receive a message from their corresponding senders.

where $P(s_k)$ denotes the transmission power of sender s_k , $d(s_k, r_i)$ is the distance between sender s_k and receiver r_i , N denotes the ambient noise power level and β is the minimum SINR required for a successful message reception.

In this paper we focus on the uniform power assignment, where every node transmits with the same power. This strategy has several important advantages due to its simplicity. While the benefits of power control are obvious, wireless devices that always transmit at the same power are less expensive and less complicated to build. Therefore, the uniform power assignment has been widely adopted in practical systems [18]. From the algorithmic perspective, the lack of freedom in choosing power levels makes reaching a decision much simpler. Moreover, recently a study of *SINR diagrams*¹ [1] showed that the reception zones of all senders are *convex* for a uniform scheme but not necessarily for non-uniform power assignments. This finding suggests that designing algorithms may be much simpler for uniform networks than for non-uniform networks. Figure 1 illustrates a setting with uniform power levels in the SINR and in the UDG model.

In any network, it is typically required that any pair of nodes can exchange message via relay nodes. In other words, the nodes have to be connected by a communication backbone, e.g., a spanning tree or a connected dominating set. In this paper, we investigate how many colors (time slots /frequencies) are necessary to guarantee that the resulting links (node pairs that can communicate) form a connected graph. [13] was the first to explore this question in the physical interference model. The authors suggest an algorithm that constructs a spanning tree, and assigns power levels and time slots to each link of the tree. This algorithm guarantees that at most $\mathcal{O}(\log^4 n)$ colors suffice for all transmissions to be received correctly, i.e., even in worse-case networks, the scheduling complexity of a strongly-connected topology is polylogarithmic in n and such topologies can thus be scheduled efficiently. The algorithm assigns many different power levels to the links and does not lend itself to a distributed implementation. As we discussed earlier, the study of the uniform case is still worthwhile, due to its simplicity and the way cheap commercial hardware is built. Therefore we aim at shedding light on the connectivity problem for uniform power assignments in this paper. More precisely, given a coloring we can construct a *SINR graph*, that represents which nodes can communicate concurrently. We examine the number of colors are necessary such that a strongly connected SINR graph can be built. We show that the number of colors needed is constant in one-dimensional grids if $\alpha > 1$ as well as in two-dimensional grids if $\alpha > 2$. For smaller path-loss exponents, more colors are necessary. If $\alpha = 2$ (i.e., the signal propagation in the vacuum), the upper and lower bounds for the number of colors

¹ The SINR diagram of a set of transmitters divides the plane into $n + 1$ *regions* or reception zones, one region for each transmitter that indicates the set of locations in which it can be heard successfully, and one more region that indicates the set of locations in which no sender can be heard. This concept is perhaps analogous to the role played by Voronoi diagrams in computational geometry.

are in the order of $\mathcal{O}(\log n)$ and $\Omega(\log n / \log \log n)$ respectively. Even smaller values of α have been measured for indoor propagation [16]. For $\alpha < 2$ we provide a tight bound of $\Theta(n^{1-\alpha/2})$. For the special case of $\alpha = 2$ we examined the connectivity of nodes distributed uniformly at random on the interval $[0,1]$. In this setting, a regular coloring of $\Theta(\log n)$ colors guarantees connectivity.

2 Related Work

The seminal work of Gupta and Kumar [7] initiated the study of the capacity of wireless networks. The authors bounded the throughput capacity in the best-case (i.e., optimal configurations) for the protocol and the physical models for $\alpha > 2$.

For both model classes, many scheduling algorithms have been suggested. E.g., [8, 10, 17] analyze algorithms in graph-based models. Typically, these algorithms employ a coloring strategy, which neglects the aggregated interference of nodes located further away. The resulting inefficiency of graph-based scheduling protocols in practice is well documented, both theoretically and by simulation [6, 14] as well as experimentally [15]. Recently, Lebar et al. [11] consider the case of $\alpha > 2$ and senders that are deployed uniformly at random in the area. They showed how a UDG protocol can be emulated when the network operates under the SINR model. Their emulation cost factor is $\mathcal{O}(\log^3 n)$. The fact that interference is continuous and accumulative as well as the geometric constraints lead to an increased difficulty of the scheduling task in the SINR model, even if the transmission power of the nodes is fixed. Two scheduling problems are shown to be NP-complete in the physical SINR model in [5]. Goussevskaja et al. propose in [4] a scheduling algorithm with an approximation guarantee independent of the network's topology. Their algorithm gives a constant approximation for the problem of maximizing the number of simultaneously feasible links and leads to a $\mathcal{O}(\log n)$ approximation for the problem of minimizing the number of time slots to schedule a given set of requests. Furthermore, in [9], the problem is shown to be in APX, thus precluding a PTAS. In the very recent work of Fanghänel et al. [3], a first non-trivial lower bound for the power-controlled oblivious scheduling problem is obtained. Moreover, for the bidirectional version of the problem, a polylogarithmic approximation algorithm is presented. Yet another line of research investigates static properties under the SINR model, e.g., the maximum achievable signal-to-interference-plus-noise ratio [19] or the shape of reception zones of nodes in a network [1].

Non-uniform power assignment can clearly outperform a uniform assignment [15, 14] and increase the capacity of the network, therefore the majority of the work on capacity and scheduling addressed non-uniform power. As mentioned in the introduction, Moscibroda et al. [13] were the first to raise the question of the complexity of connectivity in the SINR model. While their work applies for networks with devices that can adjust their transmission power, we address networks composed of devices that transmit with the same power.

3 Model

Let (M, d) be a metric space and $V \subseteq M$ a finite set of $n = |V|$ nodes. A node v_j successfully receives a message from node v_i depending on the set of concurrently transmitting nodes and the applied interference model. A standard interference model that captures some of the key characteristics of wireless communication and is sufficiently concise for rigorous reasoning is the physical SINR model [7]. In this model, the successful reception of a transmission depends on the strength of the received signal, the interference caused by nodes transmitting simultaneously, and the ambient noise level. The received power $P_r(s_i)$ of a signal transmitted by a sender s_i at an intended receiver r_i is $P_r(s_i) = P(s_i) \cdot g(s_i, r_i)$, where $P(s_i)$ is the transmission power of s_i and $g(s_i, r_i)$ is the propagation attenuation (link gain) modeled as $g(s_i, r_i) = d(s_i, r_i)^{-\alpha}$. The *path-loss exponent* $\alpha \geq 1$ is a constant typically between 1.6 and 6. The exact value of α depends on external conditions of the medium (humidity, obstacles, etc.) and on the exact sender-receiver distance. Measurements for indoor and outdoor path-loss exponents can be found in [16].

Given a sender and a receiver pair $l_i = (s_i, r_i)$, we use the notation $I_{r_i}(s_j) = P_r(s_j)$ for any other sender s_j concurrent to s_i in order to emphasize that the signal power transmitted by s_j is perceived at

r_i as interference. The *total interference* $I_{r_i}(L)$ experienced by a receiver r_i is the sum of the interference power values created by the set L of nodes transmitting simultaneously (except the intending sender s_i), i.e., $I_{r_i}(L) := \sum_{l_j \in L \setminus \{l_i\}} I_{r_i}(s_j)$. Finally, let N denote the ambient noise power level. Then, r_i receives s_i 's transmission if and only if

$$\text{SINR}(l_i) = \frac{P_{r_i}(s_i)}{N + I_{r_i}(L)} = \frac{P(s_i)g(s_i, r_i)}{N + \sum_{j \neq i} P(s_j)g(s_j, r_i)} = \frac{\frac{P(s_i)}{d(s_i, r_i)^\alpha}}{N + \sum_{j \neq i} \frac{P(s_j)}{d(s_j, r_i)^\alpha}} \geq \beta,$$

where $\beta \geq 1$ is the minimum SINR required for a successful message reception. For the sake of simplicity, we set $N = 0$ and ignore the influence of noise in the calculation of the SINR. However, this has no significant effect on the results: by scaling the power of all senders, the influence of ambient noise can be made arbitrarily small. Observe that for real scenarios with upper bounds on the maximum transmission power this is not possible, for our asymptotic calculations we can neglect this term.

The *scheduling complexity*, introduced in [13], describes the number of time slots or frequencies necessary to successfully transmit messages over a given set of communication links. More formally, we are given a network with a set of directed links representing communication requests. For each such link we assign a color (time slot/frequency) and a power level such that all simultaneous transmissions are successful, i.e., not violating the signal-to-interference plus noise ratio at any receiver.

The *connectivity problem* of a given set V of nodes located in the Euclidean plane is the scheduling complexity of a connected communication graph of V , i.e., an assignment of power levels and colors to each link of the directed strongly connected graph such all transmissions are received correctly.

In this paper, we investigate the uniform connectivity problem, i.e., the connectivity problem for a set V when only uniform power assignments are allowed. We give a formal definition of the graph we examine the connectivity of:

Definition 1 ((Uniform) SINR graph). Let (M, d) be a metric space, $V \subseteq M$ be a finite set of nodes, $c : V \rightarrow [k]$ be a coloring of the nodes, and $E \subseteq V^2$ be the set defined as follows

$$E = \left\{ (u, v) \in V^2 : \frac{1/d(u, v)^\alpha}{\sum_{w \in V \setminus \{u\} : c(w) = c(u)} 1/d(w, v)^\alpha} \geq \beta \right\} \quad (1)$$

We will refer to the directed graph $G = (V, E)$ as the (uniform) SINR graph.

In words, the definition of the graph says that a node v can *decode* a message coming from node u (i.e. there is an edge from u to v) if and only if the ratio between the *power* (i.e. $1/d(u, v)^\alpha$) at which v receives the message from u and the sum of the powers from the other *interfering* nodes (nodes w that use the same *frequency* of node u , i.e. $c(w) = c(u)$) is at least some fixed constant β .

The question we want to answer is the following: Given the metric space (M, d) and the set of nodes $V \subseteq M$, how many colors k do we need in order to be sure that a coloring $c : V \rightarrow [k]$ exists such that the resulting graph G is strongly connected?

In this paper, the set of nodes V will be located in \mathbb{R} or in \mathbb{R}^2 and d will always denote the Euclidean distance.

4 Connectivity in Grids

4.1 One-dimensional grid

Let $V = \{p_1, \dots, p_n\} \subseteq \mathbb{R}$ be a set of n nodes with $p_1 < p_2 < \dots < p_n$. We say that V is a *one-dimensional grid* if the nodes are equally spaced, i.e. $d(p_i, p_{i+1})$ is the same for every $i = 1, \dots, n$ (without loss of generality, we will assume $p_i = i$ for every i).

We say that a coloring $c : V \rightarrow [k]$ is a *regular k -coloring* if the points are colored in a *Round Robin* way, i.e. if $c(p_i) = (i \bmod k) + 1$ for $i = 1, \dots, n$.

Theorem 1. Let $V = \{p_1, \dots, p_n\}$ be a one-dimensional grid with $p_i = i$ for every $i = 1, \dots, n$. For any $\alpha > 1$ a constant k and a coloring $c : V \rightarrow [k]$ exist such that the corresponding SINR graph is strongly connected.

Proof. Consider a regular k -coloring, where k is a sufficiently large constant that we will choose later. Now we show that, for every $i = 1, \dots, n - 1$, in the SINR graph there is a directed edge from node p_i to node p_{i+1} . According to the definition of the SINR graph, we must show that

$$\frac{1/d(p_i, p_{i+1})^\alpha}{\sum_{j \in [n] \setminus \{i\} : c(p_j) = c(p_i)} 1/d(p_j, p_{i+1})^\alpha} > \beta \quad (2)$$

For the numerator, we have $1/d(p_i, p_{i+1})^\alpha = 1$ for any α . For the denominator, observe that the nodes with the same color of p_i are $\{\dots, p_{i-2k}, p_{i-k}, p_{i+k}, p_{i+2k}, \dots\}$. Thus, for any $j = 1, \dots, n$, we have at most 2 nodes at distance at least $j(k-1)$ from node p_i , hence

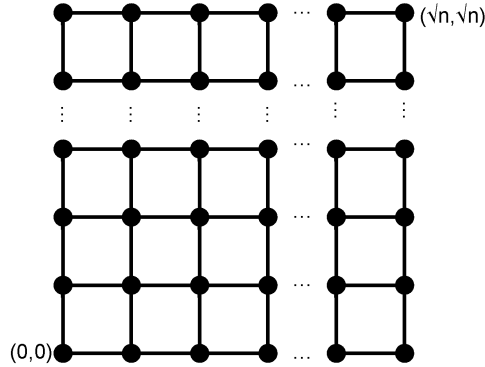
$$\sum_{j \in [n] \setminus \{i\} : c(p_j) = c(p_i)} \frac{1}{d(p_j, p_{i+1})^\alpha} \leq \sum_{j=1}^n \frac{2}{(j(k-1))^\alpha} = \frac{2}{(k-1)^\alpha} \sum_{j=1}^n \frac{1}{j^\alpha} < \frac{2}{(k-1)^\alpha} g(\alpha)$$

where we named $g(\alpha) = \sum_{j=1}^{\infty} j^{-\alpha}$. Observe that $g(\alpha) = \mathcal{O}(1)$ for any constant $\alpha > 1$. In order to satisfy (2) it is sufficient to choose $k \geq 1 + (2\beta g(\alpha))^{1/\alpha}$.

In exactly the same way we can show that for every $i = 2, \dots, n$, there is a directed edge from node p_i to node p_{i-1} , hence the the SINR graph is strongly connected. \square

4.2 Two-dimensional grid

Consider the following two dimensional grid topology of n nodes. An array of \sqrt{n} arrays containing \sqrt{n} nodes each, where the left most corner node is denoted by $(0,0)$.



A regular k^2 -coloring partitions the nodes into k^2 sets such that the closest distance between any two nodes of the same color is k . In other words, each set forms another grid with distance k . If α exceeds two, the number of colors required for connectivity is constant.

Theorem 2 (Bound 2D grids, $\alpha > 2$). Let $V = \{p_1, \dots, p_n\} \subseteq [0, \sqrt{n}]^2$ be a two-dimensional grid. For any $\alpha > 2$ a constant k and a coloring $c : V \rightarrow [k]$ exist such that the corresponding SINR graph is strongly connected.

Proof. Consider a regular k^2 -coloring for a grid consisting of n nodes. Let the node v at $(0,0)$ belong to color j . Without loss of generality we can assume that v is connected to the node at $(0,1)$ in the corresponding

interference graph. We now explore the interference accrued at node $(0,1)$ if all nodes of color j transmit simultaneously. In this case the total interference at $(0,1)$ is

$$\sum_{i=1}^{\sqrt{n}} \frac{2i+1}{(ki-1)^\alpha} < \frac{3}{(k/2)^\alpha} \sum_{i=1}^{\sqrt{n}} \frac{1}{i^{\alpha-1}} < \frac{3 \cdot 2^\alpha (\alpha-1)}{2k^\alpha (\alpha-2)},$$

for $\alpha > 2$ due to a standard bound for Riemann's zeta-function. Note that the node in the center of the grid faces less than four times the amount of interference that the node at $(0,1)$ is exposed to. This level of interference needs to be below $1/\alpha$, hence the distance k has to satisfy the following inequality:

$$k > (3 \cdot 2^\alpha \beta (\alpha-1) / (\alpha-2))^{1/\alpha}.$$

This procedure can be repeated to bound the interference at any node in the grid. In other words, a regular $(3 \cdot 2^\alpha \beta (\alpha-1) / (\alpha-2))^{2/\alpha}$ -coloring ensures connectivity in a constant number of rounds. \square

Observe that this result holds for infinite grids as well. In addition, it coincides with the UDG interference model, where a constant number of colors suffices as well. The situation changes dramatically if α is less than or equal to two. If $\alpha = 2$, the number of necessary colors increases logarithmically in the number of nodes.

Theorem 3 (Upper bound 2D grids, $\alpha = 2$). *Let $V = \{p_1, \dots, p_n\} \subseteq [0, \sqrt{n}]^2$ be a two-dimensional grid. For $\alpha = 2$ a regular $\mathcal{O}(\log n)$ -coloring ensures that the corresponding SINR graph is strongly connected.*

Proof. We start similarly to the proof for $\alpha > 2$ and sum up the interference accumulated at node $(0,1)$ under a regular k^2 -coloring. In this case the total interference at $(0,1)$ is less than

$$\sum_{i=1}^{\sqrt{n}} \frac{2i+1}{(ki-1)^\alpha} < \frac{3}{(k/2)^\alpha} \sum_{i=1}^{\sqrt{n}} \frac{1}{i^{\alpha-1}} = \frac{6 \log n}{k^2}.$$

Moreover, the total interference at $(0,1)$ exceeds

$$\sum_{i=1}^{\sqrt{n}} \frac{2i+1}{(\sqrt{2}ki)^\alpha} > \frac{\sqrt{2}^\alpha}{d^\alpha} \sum_{i=1}^{\sqrt{n}} \frac{1}{i^{\alpha-1}} = \frac{2 \log n}{k^2}.$$

Note that the node in the center of the grid faces less than four times the amount of interference that the node at $(0,1)$ is exposed to.

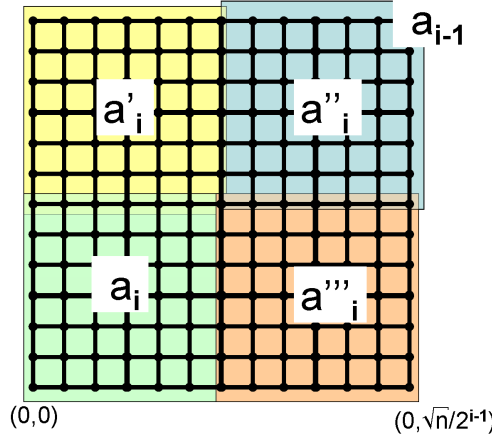
β being a constant entails that k^2 has to be in the order of $\Omega(\log n)$ if we want that a message from the node at $(0,0)$ can be decoded at $(0,1)$. There are $\mathcal{O}(k^2)$ nodes at a radius of k around $(0,1)$, consequently, we need $\Omega(\log n)$ frequencies if $\alpha = 2$ and we want all nodes to be able to send concurrently and form a connected structure. We achieve this goal by partitioning the existing grid into $\log n$ grids that send with distinct frequencies. \square

Theorem 4 (Lower bound 2D grids, $\alpha = 2$). *Let $V = \{p_1, \dots, p_n\} \subseteq [0, \sqrt{n}]^2$ be a two-dimensional grid and $\alpha = 2$. Let $c : V \rightarrow [k]$ be a coloring. If the corresponding SINR graph is strongly connected, then the number of colors is $k = \Omega\left(\frac{\log n}{\log \log n}\right)$.*

Proof. For the lower bound, we show that no matter how we distribute the colors on the grid, we need $\Omega\left(\frac{\log n}{\log \log n}\right)$ frequencies to ensure connectivity. More precisely, we show that in whatever way we position the nodes, we can always find a node where the interference experienced is at least as high as at $(0,0)$ in the grid situation.

Let us start by demonstrating the minimum interference accumulated at any node if we use three colors. Without loss of generality, there is at least one color j that is assigned to at least $\frac{n}{3}$ nodes. In the following,

we will only consider this color j . Let us divide the grid into 4 parts $(a_1, a'_1, a''_1, a'''_1)$ of equal size. Among these, there is at least one square with at least $n/12$ nodes with color j , because there would not be $\frac{n}{3}$ nodes of color j together with the other squares otherwise. Without loss of generality we can assume that this is the square a_1 anchored in $(0,0)$ and we denote the number of nodes in a_1 by $|a_1|$. We now want to compute the minimal interference that one of the nodes in a_1 experiences. To this end we assume that there are exactly $\frac{n}{3}$ nodes with color j and exactly $\frac{n}{12}$ nodes in a_1 (otherwise the interference for nodes in a_1 increases. By positioning all $\frac{n}{3} - |a_1| = \frac{n}{4}$ nodes that are not in a_1 into the corner (\sqrt{n}, \sqrt{n}) , i.e. the corner with the largest distance from $(0,0)$, the minimal interference any node in a_1 experiences exceeds $\frac{n}{4} \cdot \frac{1}{2n} = \frac{1}{8}$ because the largest distance between a point in a_1 and (\sqrt{n}, \sqrt{n}) is $\sqrt{2n}$. Let us now consider the interference the nodes in a_1 cause among themselves. We proceed as before by dividing the square a_1 into four squares $(a_2, a'_2, a''_2, a'''_2)$ of side length $\frac{\sqrt{n}}{4}$ each. Using the same arguments we know that one of them, let us say, a_2 contains at least $\frac{n}{48}$ nodes of color j and to minimize the interference within a_2 we look at the case where $|a_2|$ is $\frac{n}{48}$ and anchored at $(0,0)$. We can now compute the minimal amount of interference caused by the $\frac{n}{12} - \frac{n}{48} = \frac{n}{16}$ nodes in a_1 at $(0,0)$ to be at least $\frac{n}{16} \cdot \frac{2}{n} = 1/8$ because the maximal distance within a_1 is $\frac{\sqrt{n}}{\sqrt{2}}$. If we repeat these steps, it holds that in step i we have $\frac{n}{4^i}$ nodes in distance $\frac{\sqrt{2n}}{2^{i-1}}$ responsible for a sum of interference of $\frac{n}{4^i} \cdot \frac{4^{i-1}}{2n} = 1/8$. After $\lceil \log_4 n \rceil$ steps there is only one node left in a_i and we stop. The total interference is thus in $\Omega(\log n)$.



We can generalize this approach to more than three colors. If we use k colors and partition the square with most nodes into $k+1$ squares and proceed recursively, the number of nodes in a_{i-1} outside a_i is in the order of $\frac{n}{(k+1)^i}$, where a_i is the square with most nodes in the i^{th} step. These nodes are at most in distance $\frac{\sqrt{2n}}{(k+1)^{(i-1)/2}}$ from the nodes in a_i and thus cause interference of $\frac{n}{(k+1)^i} \cdot \frac{(k+1)^{i-1}}{2n} = \frac{1}{2(k+1)}$. The maximal number of recursions is $\lfloor \frac{\log n}{\log k} \rfloor$. Consequently, all the nodes are responsible for $\Omega(\frac{\log n}{k \log k})$ interference at $(0,0)$.

Typically the SINR threshold β that guarantees the reception of a message is a small constant. In order to allow a neighbor on the grid (at distance 1) to receive our message, we thus have to ensure that the total interference is at most a constant, i.e., for any number of colors k where

$$\frac{\log n}{k \log k} \in \omega(1) \quad (3)$$

a simultaneous transmission is not possible. Consider $k' := \frac{c \log n}{\log \log n}$ for some constant c . Depending on the value of c , this k' satisfies (3), thus the smallest suitable k is in $\Omega(\frac{\log n}{\log \log n})$. \square

Corollary 1 (Upper and lower bound 2D grids, $\alpha < 2$). Let $V = \{p_1, \dots, p_n\} \subseteq [0, \sqrt{n}]^2$ be a two-dimensional grid and $\alpha < 2$. Let $c : V \rightarrow [k]$ be a coloring. If the corresponding SINR graph is strongly connected, then the number of colors is $k = \Theta(n^{1-\alpha/2})$.

Proof. Given a regular k^2 -coloring, the total interference at any point in $[0, \sqrt{n}]$ is less than

$$\sum_{i=1}^{\sqrt{n}} \frac{2i+1}{(ki-1)^\alpha} < \frac{3}{(k/2)^\alpha} \sum_{i=1}^{\sqrt{n}} \frac{1}{i^{\alpha-1}} < \frac{6n^{1-\alpha/2}}{(2-\alpha)k^2}.$$

Due to the same arguments as in the proof of Theorem 3, this implies that a regular coloring using $\mathcal{O}(n^{1-\alpha/2})$ colors suffices for connectivity.

For the matching lower bound we adopt the same recursive strategy as in the proof of Theorem 4. In step i we have $\frac{n}{(k+1)^i}$ nodes in distance $\frac{\sqrt{2n}}{(k+1)^{(i-1)/2}}$ responsible for a sum of interference of

$$\frac{n}{(k+1)^i} \cdot \frac{(k+1)^{\alpha(i-1)/2}}{(2n)^{\alpha/2}} = \frac{n^{1-\alpha/2}}{2^{\alpha/2}(k+1)^{\alpha/2}} (k+1)^{i(\alpha/2-1)}.$$

After $\lceil \log_{k+1} n \rceil$ steps there is only one node left in a_i and we stop. The total interference is thus

$$\begin{aligned} I &= \frac{n^{1-\alpha/2}}{(2(k+1))^{\alpha/2}} \sum_{i=1}^{\log_{k+1} n} (k+1)^{i(\alpha/2-1)} \\ &= \frac{n^{1-\alpha/2}}{(2(k+1))^{\alpha/2}} \cdot c, \end{aligned}$$

for some constant c , since $(k+1)^{\alpha/2-1} < 1$. Hence, in order to make sure that a message in distance one from a sender can be received, i.e. $1/I > \beta$, the smallest possible k has to be in the order of $\Omega(n^{1-\alpha/2})$. \square

5 Connectivity for random instances: the one-dimensional case

In this section, we consider a set V of n nodes thrown independently and uniformly at random in $[0, 1]$, the unit interval.² We assume the path-loss exponent to be $\alpha = 2$.

Our first result shows that $\mathcal{O}(\log n)$ colors are enough to guarantee the strong connectivity of the corresponding SINR graph.

Theorem 5 (Upper bound). Let $V = \{p_1, \dots, p_n\} \subseteq [0, 1]$ where p_1, \dots, p_n are independent random variables uniformly distributed in $[0, 1]$. Then a coloring $c : V \rightarrow [k]$ exists, with number of colors k in the order of $\mathcal{O}(\log n)$, such that the corresponding SINR graph is strongly connected.

Idea of the proof. Consider a regular coloring of $c \log n$ colors, with a constant c sufficiently large so that we can partition the interval $[0, 1]$ in subintervals of length $\Theta(\log n/n)$, each one of them containing (i) $\Theta(\log n)$ nodes w.h.p. and (ii) no more than one node for each color w.h.p.

For any node p , we can take an interval of length $\Theta(\log n/n)$ containing $\Theta(\log n)$ nodes and such that every node in that interval is an out-neighbor of node p . Indeed, for any node q in that interval, the power at which q receives the signal from p is $\Omega(n^2/\log^2 n)$. For the nodes *interfering* with p , we have that for any h there are $\mathcal{O}(1)$ interfering nodes at distance $\Omega(h \log n/n)$ from q , hence the total *interfering power* at node q is $\mathcal{O}(n^2/\log^2 n)$. By choosing the constant c sufficiently large, we can make the ratio between the power at which q receives the signal from p and the interfering power an arbitrary large constant. \square

² In contrast to the grid, where we set the minimal distance between two nodes to be one, we consider the unit interval for the random case because of its direct correspondence to probability.

Proof. Let $I \subseteq [0, 1]$ be an interval of length $c \log n/n$, where c is a sufficiently large constant that we will choose later, and let X be the r.v. counting the number of nodes in I . The expectation of X is $\mathbf{E}[X] = c \log n$ and, since X can be written as sum of independent Bernoulli r.v., from Chernoff bound (6) with $\delta = 1/2$ and $\mu = c \log n$ we have that

$$\mathbf{P}\left(X > \frac{3c}{2} \log n\right) < n^{-c/12} \quad (4)$$

Set the number of colors to be $k = \frac{3c}{2} \log n$, and consider a *regular* coloring, i.e. the color of node p_i is $(i \bmod k) + 1$ for $i = 1, \dots, n$.

For a node p_i let I_i be an interval of length $32 \log n/n$ centered in p_i , possibly shorter if p_i is close to the boundary

$$I_i = \left[p_i - 16 \frac{\log n}{n}, p_i + 16 \frac{\log n}{n} \right] \cap [0, 1]$$

Let Y_i be the number of nodes in I_i and observe that $\mathbf{E}[Y_i] \geq 16 \log n$, and by using Chernoff bound (5) with $\delta = 1/2$ and $\mu = 16 \log n$, we have that

$$\mathbf{P}(Y_i < 8 \log n) < n^{-2}$$

Now we show that every node in interval I_i is w.h.p. an *out-neighbor* of node p_i in the SINR graph. This will prove that the SINR graph is strongly connected w.h.p. Consider the interval

$$J_i = \left[p_i - \frac{c \log n}{2n}, p_i + \frac{c \log n}{2n} \right] \cap [0, 1]$$

and partition all the rest of $[0, 1]$ with intervals of length $c \log n/n$ (possibly shorter for the two intervals on the boundary). Now observe that, from (4) and from the fact that we are using a k -regular coloring, it follows that the probability that a single interval contains more than one node with the same color is less than $n^{-c/12}$. By using union bound over all the intervals, the probability that one interval exists that contains more than one color is less than $n^{-c/12+1}$. Now, conditioning on the event that all such intervals contain no more than one node with the same color, we can show that every node in I_i is an out-neighbor of node p_i .

Let $q \in I_i$ be a node, as for the numerator in (2) we have that

$$\frac{1}{d(q, p_i)^2} \geq \frac{n^2}{16^2 \log^2 n}.$$

For the denominator, observe that, for any $h = 1, \dots, n$, we have at most 2 nodes with the same color of p_i , at distance at least $h(c/2 - 16) \log n/n \geq h(c/4) \log n/n$, where in the inequality we are assuming $c \geq 64$. Hence,

$$\sum_{j \in [n] \setminus \{i\} : c(p_j) = c(p_i)} \frac{1}{d(p_j, q)^2} \leq \sum_{h=1}^n \frac{2}{\left(h \frac{c \log n}{4n}\right)^2} = \frac{32}{c^2} \frac{n^2}{\log^2 n} \sum_{h=1}^n \frac{1}{h^2} \leq \frac{16\pi^2}{3c^2} \frac{n^2}{\log^2 n}.$$

In order to satisfy (2), it is sufficient to choose $c > 64\pi\sqrt{\beta/3}$, and it follows that

$$\frac{1/d(p_i, q)^2}{\sum_{j \in [n] \setminus \{i\} : c(p_j) = c(p_i)} 1/d(p_j, q)^2} \geq \frac{\frac{n^2}{16^2 \log^2 n}}{\frac{16\pi^2}{3c^2} \frac{n^2}{\log^2 n}} = \frac{3}{16^3 \pi^2} c^2 \geq \beta.$$

Hence, the probability that node p_i is not connected to all nodes in I_i , or I_i does not contain $c/8 \log n$ nodes is less than $n^{-2} + n^{-c/12}$. And by using union bound on all nodes we have that the probability that a node exists that is not connected to at least $\Omega(\log n)$ nodes in an interval of length $\mathcal{O}(\log n/n)$ is $\mathcal{O}(1/n)$. The SINR graph is thus connected w.h.p. \square

The previous theorem shows that, with a regular $\mathcal{O}(\log n)$ -coloring, the resulting SINR graph is strongly connected w.h.p. The rest of this section is devoted to prove that this is the best we can achieve with regular colorings.

The next lemma provides a condition implying that the SINR graph is not strongly connected. We will use it in the proof of Theorem 6.

Lemma 1. *Let $V = \{p_1, \dots, p_n\} \subseteq [0, 1]$ be an arbitrary set of nodes and let $c : V \rightarrow [k]$ be a regular coloring. Suppose that a length $0 < \ell < 1/3$ and a point $x \in [0, 1 - 3\ell]$ exist such that the following conditions hold:*

1. *In the subinterval $[x, x + \ell]$, there are at least $(4/\beta)k$ nodes;*
2. *In the subinterval $[x + \ell, x + 2\ell]$, there are no nodes;*
3. *In the subinterval $[x + 2\ell, x + 3\ell]$ there is at least one node.*

Then, the SINR graph is not strongly connected.

Proof. Let p be a node in $[0, x + \ell]$ and let q be a node in $[x + 2\ell, 1]$. From hypothesis (1) and the fact that the coloring is regular, there are at least $(4/\beta)$ nodes in the interval $[x, x + \ell]$ interfering with the transmission from p to q , that are at distance less than $\ell + d(p, q)$ from q . Hence, the interference ratio at node q is less than

$$\frac{\frac{1}{d(p,q)^2}}{\frac{4/\beta}{(d(p,q)+\ell)^2}} = \frac{\frac{1}{d(p,q)^2}}{\frac{4/\beta}{d(p,q)^2(1+\frac{\ell}{d(p,q)})^2}} = \frac{\beta \left(1 + \frac{\ell}{d(p,q)}\right)^2}{4} < \beta$$

In the last inequality we used $d(p, q) > \ell$. Thus, there are no edges from nodes in $[0, x + \ell]$ to nodes in $[x + 2\ell, 1]$ and by hypothesis (2) the graph is not strongly connected. \square

Theorem 6 (Lower Bound for regular colorings). *Let $V = \{p_1, \dots, p_n\} \subseteq [0, 1]$ where p_1, \dots, p_n are independent random variables uniformly distributed in $[0, 1]$, and let $c : V \rightarrow [k]$ be a regular coloring. If the corresponding SINR graph is strongly connected w.h.p., then the number of colors is $k = \Omega(\log n)$.*

Proof. Let $\ell = (4/\beta)(k/n)$ and let $I \subseteq [0, 1]$ be an interval of length 3ℓ . Consider the the event

$$\mathcal{E}_I = \text{“Interval } I \text{ satisfies conditions (1), (2), and (3) of Lemma 1”}$$

In what follows we prove that, if $k < (\beta/2) \log n$, then an interval I of length 3ℓ exists such that \mathcal{E}_I holds w.h.p. We will use the Poisson approximation (for a detailed description of such tool see, for example, Chapter 5.4 in [12]).

For $i = 1, \dots, n$ let X_i be the random variable counting the number of nodes in the interval $[(i-1)/n, i/n]$ and observe that $\mathbf{E}[X_i] = 1$. Consider the set $\{I_0, \dots, I_h\}$ of disjoint intervals of length 3ℓ , where $I_j = [3\ell j, 3\ell(j+1)]$ for $j = 0, 1, \dots, h$ and observe that, since $\ell = \mathcal{O}(\log n/n)$, then the number of such intervals is $h = \Omega(n/\log n)$. For interval I_j we can write the event \mathcal{E}_{I_j} as

$$\mathcal{E}_{I_j} = \left(\sum_{i=3n\ell j+1}^{3n\ell j+n\ell} X_i \geq n\ell \right) \cap \left(\sum_{i=3n\ell j+n\ell+1}^{3n\ell j+2n\ell} X_i = 0 \right) \cap \left(\sum_{i=3n\ell j+2n\ell+1}^{3n\ell j+3n\ell} X_i \geq 1 \right)$$

Now let Y_1, \dots, Y_n be i.i.d. Poisson random variables with $\mathbf{E}[Y_i] = 1$ and let \mathcal{F}_{I_j} , for $j = 0, 1, \dots, h$, be the events in the Poisson setting corresponding to the events \mathcal{E}_{I_j} , i.e.

$$\mathcal{F}_{I_j} = \left(\sum_{i=3n\ell j+1}^{3\ell j+n\ell} Y_i \geq n\ell \right) \cap \left(\sum_{i=3n\ell j+n\ell+1}^{3n\ell j+2n\ell} Y_i = 0 \right) \cap \left(\sum_{i=3n\ell j+2n\ell+1}^{3n\ell j+3n\ell} Y_i \geq 1 \right)$$

Since the Y_i s are independent, it holds that

$$\begin{aligned} \mathbf{P}(\mathcal{F}_{I_j}) &= \mathbf{P}\left(\left(\sum_{i=3n\ell j+1}^{3n\ell j+n\ell} Y_i \geq n\ell\right) \cap \left(\sum_{i=3n\ell j+n\ell+1}^{3n\ell j+2n\ell} Y_i = 0\right) \cap \left(\sum_{i=3n\ell j+2n\ell+1}^{3n\ell j+3n\ell} Y_i \geq 1\right)\right) \\ &= \mathbf{P}\left(\sum_{i=3n\ell j+1}^{3n\ell j+n\ell} Y_i \geq n\ell\right) \cdot \mathbf{P}\left(\sum_{i=3n\ell j+n\ell+1}^{3n\ell j+2n\ell} Y_i = 0\right) \cdot \mathbf{P}\left(\sum_{i=3n\ell j+2n\ell+1}^{3n\ell j+3n\ell} Y_i \geq 1\right) \\ &\geq \frac{1}{e} \cdot e^{-n\ell} \cdot (1 - e^{-n\ell}) \geq \frac{1}{2e} e^{-n\ell} = \frac{1}{2e} e^{-(4/\beta)k} \end{aligned}$$

Thus, if $k < (\beta/2) \log n$ then $\mathbf{P}(\mathcal{F}_{I_j}) \geq \frac{1}{2e\sqrt{n}}$ and, since the intervals I_j are disjoint, the probability that no one of the events \mathcal{F}_{I_j} holds is

$$\mathbf{P}\left(\bigcap_{j=0}^h \overline{\mathcal{F}_{I_j}}\right) = \prod_{j=0}^h \mathbf{P}(\overline{\mathcal{F}_{I_j}}) \leq \left(1 - \frac{1}{2e\sqrt{n}}\right)^h \leq e^{-\frac{h}{2e\sqrt{n}}}$$

By using the Poisson approximation, the probability that no one of the events \mathcal{E}_{I_j} holds is

$$\mathbf{P}\left(\bigcap_{j=0}^h \overline{\mathcal{E}_{I_j}}\right) \leq e\sqrt{n} \cdot \mathbf{P}\left(\bigcap_{j=0}^h \overline{\mathcal{F}_{I_j}}\right) \leq e\sqrt{n} e^{-\frac{h}{2e\sqrt{n}}}$$

Since $h = \Omega(n/\log n)$ this probability is exponentially small. Hence, at least one of the intervals I_j satisfies conditions (1), (2), and (3) of Lemma 1 w.h.p. \square

6 Conclusions and open problems

In this paper we initiate the study of connectivity in the uniform power SINR model. Clearly we can not achieve connectivity in the SINR model if we use only one frequency, since the SINR diagram is a partition of the plane. To overcome this problem we can either use a sophisticated scheduling algorithm or we can increase the number of frequencies. However those two actions are equivalent i.e., any schedule can be translated into a choice of frequencies and any frequency assignment can be translated into a schedule. Therefore we can defined the connectivity problem in the SINR model as the minimal number of frequency the network needs to use to maintain connectivity (the scheduling complexity of connectivity).

We provided upper and lower bounds for the number of time slots or frequencies to build a strongly connected graph of communication edges. We focused on nodes arranged in a regular grid or uniformly at random on the unit interval. We proved that if the nodes are located on a regular grid the number of frequencies needed to maintain connectivity is a function of the dimension of the grid and the path-loss exponent α . Apart from the special case $\alpha = 2$ these bounds are asymptotically tight. In contrast, there is a big gap between the lower and upper bounds, when transmitters are located uniformly at random on the interval $[0, 1]$. A natural open question is to close this gap. Other intriguing problems include determining upper and lower bounds for general colorings in the random two-dimensional case, or algorithms computing the uniform power complexity of connectivity of arbitrarily positioned nodes.

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A Appendix

Lemma 2 (Chernoff bounds). *Let X_1, \dots, X_n be independent Bernoulli random variables, and let $X = \sum_{i=1}^n X_i$. Then for any $0 < \delta < 1$ it holds that*

1. For any $\mu \leq \mathbf{E}[X]$,

$$\mathbf{P}(X > (1 - \delta)\mu) < e^{-\frac{\delta^2}{2}\mu}. \quad (5)$$

2. For any $\mu \geq \mathbf{E}[X]$,

$$\mathbf{P}(X > (1 + \delta)\mu) < e^{-\frac{\delta^2}{3}\mu}. \quad (6)$$

3. For $\mu = \mathbf{E}[X]$,

$$\mathbf{P}(X \notin [(1 - \delta)\mu, (1 + \delta)\mu]) < 2e^{-\frac{\delta^2}{3}\mu}. \quad (7)$$