József Wildt International Mathematical Competition

The Edition XXVIIth , 2017

The solution of the problems W.1 - W.62 must be mailed before 30. September 2017, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania, E-mail: 1benczemihaly@gmail.com; benczemihaly@yahoo.com

W1. If x, y, z are positive real numbers, and $x^2 + xy + \frac{y^2}{3} = 25$; $\frac{y^2}{3} + z^2 = 9$ and $z^2 + xz + x^2 = 16$ then compute $xy + 2yz + 3zx$.

Chang-Jian Zhao

W2. Prove that $x^x \leq x^2 - x + 1 - x(1-x)^4$ for all $0 \leq x \leq 1$.

Perfetti Paolo

W3. Let $a, b, c, d, e \geq 0$ and

$$\sum a \doteq a + b + c + d + e$$

$$\sum ab = ab + ac + ad + ae + bc + bd + be + cd + ce + de$$

$$\sum abc \doteq abc + abd + abe + bcd + bde + bce + acd + ace + ade + cde$$

Prove that

$$6\left(\sum a\right)^3 + 25\sum abc \geq 20\sum a\sum ab$$

Perfetti Paolo

W4. Let $p \in N, p \geq 2$. Determine $f : R \to R$ continuous functions, derivable in $x_0 = 0$, which ferifies the relationship: $f(px) = f^p(x)$, for any $x \in R$.

Nicolae Papacu

W5. 1. Prove that for any natural number $n \geq 2$ and for any natural numbers a_k, $k = \overline{1, n}$, $\{a_1, a_2, ..., a_n\} = \{1, 2, ..., n\}$, there are positive integers $x_1, x_2, ..., x_n \geq 1$ so that

$$x_1 x_2 ... x_n = a_1 x_1 + a_2 x_2 + ... + a_n x_n.$$

2. Determine the natural numbers $n \geq 2$, so that for any natural numbers a_k, $k = \overline{1, n}$, $\{a_1, a_2, ..., a_n\} = \{1, 2, ..., n\}$, there are positive integers $x_1, x_2, ..., x_n \geq 1$ such that $1 \leq x_1 < x_2 < ... < x_n$ and

$$x_1 x_2 ... x_n = a_1 x_1 + a_2 x_2 + ... + a_n x_n.$$

Nicolae Papacu

W6. a). Study the monotonity of the function $f : (0, +\infty) \to R$ where

$$f(x) = 4x arctg x + 2x arctg \frac{2x}{3} - 3\pi x + 2\pi$$

b). Solve in $(0, +\infty)$ the equation

$$4x arctg x + 2x arctg \frac{2x}{3} = (3x - 2)\pi.$$

Ionel Tudor

W7. Let $a > 0$ be a real number. Compute the value of the following integral

$$\int_{-a}^{a} \frac{\cos t}{\pi^{1/t} + 1}\, dt$$

José Luis Díaz-Barrero

W8. Let p, q be integer numbers and let

$$A = \left\{ x \in \mathbb{R} \,|\, x = \frac{3^p + 7^q}{3^p 7^q}, \quad p, q \geq 1 \right\}$$

Show that A is neither an open set nor a closed set in $\mathbb{R}$ with the usual topology.

José Luis Díaz-Barrero

W9. Let $p \geq 5$ be a prime number. Prove that p^3 divides $\binom{2p}{p} - 2$.

José Luis Díaz-Barrero

W10. Calculate

$$\sum_{n=2}^{\infty} H_n \left(\zeta(n) - \zeta(n+1) \right),$$

where ζ denotes the Riemann zeta function and H_n is the nth harmonic number.

Ovidiu Furdui

W11. Calculate

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1 + \sqrt{\sin(2x)})^2} \mathrm{d}x.$$

Ovidiu Furdui

W12. Let $p \geq 3$ be a prime number. Solve in $\mathcal{M}_2\left(\mathbb{Z}_p\right)$ the equation

$$X^p = \begin{pmatrix} \widehat{p-1} & \widehat{2} \\ \widehat{p-1} & \widehat{1} \end{pmatrix}.$$

Ovidiu Furdui

W13. Let $\Delta\left(x, y, z\right) := 2(xy + yz + xz) - (x^2 + y^2 + z^2)$ and let a, b, c be sidelengths of a triangle. Prove that

$$\Delta\left(a, b, c\right) \cdot \Delta\left(a^3, b^3, c^3\right) \geq 3\Delta\left(a^4, b^4, c^4\right)$$

Arkady Alt

W14. Let $f\left(u, v\right)$ be continuous in $(1, 0)$ homogeneous function of order m (that is for any $t > 0$ holds $f\left(tu, tv\right) = t^m f\left(u, v\right)$) and let D_1 be set of strictly decreasing sequences of positive real numbers with first term equal to 1. For any sequence $x_N := \left(x_1, x_2, ..., x_n, ...\right) \in D_1$ let

$$S_f\left(x_N\right) = \sum_{n=1}^{\infty} f\left(x_n, x_{n+1}\right)$$

if series $\sum\limits_{n=1}^{\infty} f\left(x_n, x_{n+1}\right)$ converges and $S_f\left(x_N\right) = \infty$ if it diverges. Prove that

$$\inf\left\{S_f\left(x_N\right) \mid x_N \in D_1\right\} = \min_{x \in [0,1)} \frac{f\left(1, x\right)}{1 - x^m}.$$

Arkady Alt

W15. For any given natural numbers $m \geq 1, k \geq 2$ prove that

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}...\sum_{i_k=1}^{n}\min{}^m\{i_1, i_2, ..., i_k\} =$$

$$= \sum_{i=1}^{m}(-1)^{m-i}\binom{m}{i}\left((n+1)^i - n^i\right)s_{k+m-i}(n),$$

or

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}...\sum_{i_k=1}^{n}\min{}^m\{i_1, i_2, ..., i_k\} = \sum_{i=1}^{m}\sum_{j=1}^{i}(-1)^{m-i}\binom{m}{i}\binom{i}{j}n^{i-j}s_{k+m-i}(n)$$

where $s_p(n) = \sum_{k=1}^{n} k^p, p \in N \cup \{0\}$.

Arkady Alt

W16. Find number of elements in image of function

$$k \mapsto \left[\frac{k^2}{n}\right] : \{1, 2, ..., n\} \longrightarrow N \cup \{0\}$$

Arkady Alt

W17. For any natural $n \geq 3$ solve the system.

$$\begin{cases} x_{k+2} - x_{k+1} - x_k = f_k, k = 1, 2, ..., n-2 \\ x_1 - x_n - x_{n-1} = f_{n-1} \\ x_2 - x_1 - x_n = f_n \end{cases},$$

where $f_n, n \in N$ Fibonacci number, that is $f_0 = 0, f_1 = 1$ and $f_{n+1} - f_n - f_{n-1} = 0, n \in Z$

Arkady Alt

W18. Let p be an integer and a positive real number. Prove that

$$\sum_{n=-\infty}^{\infty} \arctan\left(\frac{a^p - a^{-p}}{a^n + a^{-n}}\right) = \pi p.$$

Ángel Plaza

W19. Prove that if $a, b, c, d \in \mathbb{R}; (a^2+b^2)(c^2+d^2) \neq 0$ then:

$$\left|\frac{a(c+d) - b(c-d)}{\sqrt{(a^2+b^2)(c^2+d^2)}}\right| \leq \left|1 + \frac{(ad-bc)(ac+bd)}{(a^2+b^2)(c^2+d^2)}\right|$$

Daniel Sitaru

W20. Prove that in any ΔABC the following relationship holds:

$$4\sum \frac{(r_a^2 + r)^2}{r_a^2 + 16rr_a + 12r^2} \geq 9 + \sum \frac{r_a - 8r}{r_a + r}$$

Daniel Sitaru

W21. Prove that if $0 < a < b < \frac{\pi}{2}$ then:

$$\frac{3}{2}\int_a^b \frac{1}{\sqrt[3]{1-\cos 4x}}dx > \cot 2a + \cot 2b$$

Daniel Sitaru

W22. Prove that if $x, y \in \mathbb{R}; z \in [0, \infty)$ then:

$$z \sin(x-y)\cos(x+y) + \cos x + \cos y \leq \cos(x+z) + \cos(y+z) + 2\sqrt{2}z$$

Daniel Sitaru

W23. Prove that if $a, b, c \in (0, 1]$ then:

$$(e^{a^2}-1)(e^{b^2}-1)(e^{c^2}-1) \leq (e-1)^3 a^2 b^2 c^2$$

Daniel Sitaru

W24. $C_{2\pi}^{p0}(R, R)$ the set of pice wise continuous function $2\pi-$ periodic. Let $t > 2\pi$, find the optimal constant $m_1(t)$, $m_2(t)$ such that $\forall f \in C_{2\pi}^{po}(R, R)$,

$$m_1(t)\int\limits_0^{2\pi} |f|^2 \leq \int\limits_0^{t} |f|^2 \leq m_2(t)\int\limits_0^{2\pi} |f|^2.$$

Moubinool Omarjee

W25. $A \in M_n(C)$ such that $\exp(A) = -I_n$. Prove A is diagonalisable.

Moubinool Omarjee

W26. $A \in GL_n(k)$ when K community field with characteristic different than 2. $B \in M_{n,1}(k)$, $C \in M_{1,n}(k)$. Suppose $1_k + CA^{-1}B = O_k$. Compute $\det(A + BC)$.

Moubinool Omarjee

W27. Find $\lim\limits_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\sqrt[3]{3!}\ldots\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}}$.

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W28. Let $\gamma_n = -\ln n + \sum\limits_{k=1}^{n}\frac{1}{k}$, $\lim\limits_{n\to\infty}\gamma_n = \gamma$. Find $\lim\limits_{n\to\infty}(\gamma_n - \gamma)\sqrt[n]{(2n-1)!!}$.

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W29. Let $S_n = -2\sqrt{n} + \sum\limits_{k=1}^{n}\frac{1}{\sqrt{k}}$, $\lim\limits_{n\to\infty} s_n = s$ (Ioachimescu constant). Find

$$\lim_{n\to\infty}(s_n - s)\sqrt[2n]{(2n-1)!!}.$$

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W30. Find the probability , so that throwing two dice, their sum to be equal with the last digit of the number 2017^{2017}.

Laurenţiu Modan

W31. Let M be the set $M = \{f : X \to Y \backslash |X| = n, |Y| = m, f \text{ surjective}\}$. Study if there are sets X, so that $|M| = 10$ and $C_{m+2}^4 = m^2 - 1$.

Laurenţiu Modan

W32. Let G be an unoriented graph, without multiedges and loops, hawing n vertices. Let A be the adjacency matrix of G, with

i). $A^3 = \begin{pmatrix} 4 & 5 & 5 & 5 \\ 5 & 4 & 5 & 5 \\ 5 & 5 & 2 & 2 \\ 5 & 5 & 2 & 2 \end{pmatrix}$

ii). $P_G(\lambda) = \lambda^4 - (n+1)\lambda^2 + \alpha\lambda$, $n, \alpha \in N$ is the characteristic polynomial of G.

Find the spectrum, $Spec(G)$ draw the graph and establish its planarity. Find another graph G', cospectral and non-isomorphic with G.

Laurenţiu Modan

W33. Let $K \supset N$ be a field with the characteristic p, where p is a prime number. Prove:

i). $3^p, 4^p, 5^p$ determine an arithmetic progression,

ii). $3^{p+1} = 1 + 2^{2p+1}, 2^{2p} = 1 + 3^p, \left(\frac{3}{4}\right)^{p-1} = 1.$

Laurenţiu Modan

W34. Let $A, B \in M_2(R)$ two matrices, at least one noninverted so that $A^2 + 3AB + B^2 = BA$. Prove that

$$Tr(AB) = Tr(A)\,Tr(B).$$

Stănescu Florin

W35. Let $f, g : [0,1] \to R$ two convex functions and continuous on $[0,1]$ such that f it is differentiable, with f' concave, and g the positive on $[0,1]$. If $0 \le (f(1) - f(0)) \cdot g(0) = f'(0) \int_0^1 g(x)\,dx$, then

i). Give an example of such functions, where g is not increasin on $[0,1]$

ii). Show that $\int_0^1 f(x)\,g(x)\,dx \ge \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx$.

Stănescu Florin

W36. Is considered a, b, c three complex numbers, which have that following properties:

a). $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{b}{a} + \frac{c}{b} + \frac{a}{c} = \frac{10}{3}$

b). $\max(\arg a, \arg b, \arg c) \le \frac{\pi}{2}$

If $A = \sum_{cyclic}^{3} \left(\frac{a-b}{a+b}\right)^2$, show that the inequality $|\pi - \arg A| < \arccos\left(\frac{1}{2}|A|\right)$

Stănescu Florin

W37. Prove that in a triangle ABC we have the inequality

$$\sum_{cyclic} \sec\left(\frac{A}{2}\right) + \sqrt{3} \ge 2 \sum_{cyclic} tg\left(\frac{\pi + A}{8}\right) + \frac{4R + r}{s}.$$

Stănescu Florin

W38. Find sum of series

$$\sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \frac{k^3 - k^2\ell + \ell^3}{k^3\ell^4(\ell - k)}.$$

Pál Péter Dályay

W39. Let p be a positive integer, and let m be an odd positive integer. Find the maximal power of 2 that divides sum $S_{2p}(m) = \sum\limits_{k=1}^{m} k^{2p}$-t!

Pál Péter Dályay

W40. Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$(x^4 + y^4 + z^4)^2 \geq 3(x^5 + y^5 + z^5).$$

Pál Péter Dályay

W41. Prove that $\left(\frac{x^2}{a} + \frac{y^2}{b}\right)\sqrt{\frac{\frac{b^2}{x^2}+\frac{a^2}{y^2}}{2}} \geq \sqrt{2\left(x^2+y^2\right)}$ for any $a, b, x, y \in R_+^*$.

Ovidiu Pop

W42. Let a, b, c be a real numbers , with the property $0 < a \leq b \leq c$.
1). Prove that $\frac{3a-2b+5c}{a} + \frac{3b-2c+5a}{b} + \frac{3c-2a+5b}{c} \geq 18$
2). If $c < \frac{5a+3b}{2}$ and $b > \frac{5a+2c}{7}$ then $\frac{a}{3a-2b+5c} + \frac{b}{3b-2c+5a} + \frac{c}{3c-2a+5b} \geq \frac{1}{2}$

Ovidiu Pop

W43. Let be $A, B \in M_n(C)$ and $n_i, m_i \in N^*$ $i \in \{1, 2, ..., k\}$ such that $(n_1, n_2, ..., n_k) = (m_1, m_2, ..., m_k) = 1$ and $A^{m_1}B^{n_1} = A^{m_2}B^{n_2} = ... = A^{m_k}B^{n_k} = I_n$, then exist $r \in Z$ such that $A^r = B^r = I_n$.

Marius Drăgan and Mihály Bencze

W44. Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be two sequences of real numbers such that $\lim\limits_{n\to\infty} \frac{y_n}{n} = 0$, $\lim\limits_{n\to\infty} \frac{y_n^2}{n} = \beta \in R$, $\lim\limits_{n\to\infty} x_n = 0$, $\lim\limits_{n\to\infty}\left(\sum\limits_{k=1}^{n} x_k - y_n\right) = \alpha \in R$. Prove that

$$\lim_{n\to\infty}\left(n\left(\left(1+\frac{x_1}{n}\right)\left(1+\frac{x_2}{n}\right)...\left(1+\frac{x_n}{n}\right)-1\right)-y_n\right) = \alpha + \frac{\beta}{2}.$$

Marius Drăgan and Mihály Bencze

W45. Let be $p_i \in [0, 1]$ $(i = 1, 2, ..., k)$ such that $p_1 + p_2 + ... + p_k = 1$, $n > 1$ real number. Prove that:

$$\sum_{i=1}^{k} p_i^n \geq \left(\sum_{i=1}^{k} p_i^2\right)^n +$$

$$+\left(p_1p_2 + p_2p_3 + ... + p_{k-1}p_k\right)^n + ... + \left(p_1p_k + p_2p_1 + ... + p_kp_{k-1}\right)^n.$$

Marius Drăgan and Mihály Bencze

W46. If $k \geq 1$ then

$$2\left(\frac{5}{26}\right)^k + 2\left(\frac{5}{13}\right)^k + 3\left(\frac{15}{26}\right)^k + 5\left(\frac{25}{26}\right)^k + 8\left(\frac{25}{13}\right)^k \geq 20.$$

Generalization.

Marius Drăgan

W47. Compute
i). $\lim\limits_{n\to\infty}\int\limits_0^1 \{x\}\{2x\}...\{nx\}\,dx$
ii). $\int\limits_a^b \{nx\}^n\,dx$, $a < b$

Marius Drăgan

W48. Let $A \in M_n(R)$ be such that $a_{ij} > 0$, $i \neq j$ with the sum of elements from every $n-1$ lines are zero and the sum from the line n is nonzero. Then $\det A \neq 0$.

Liviu Bordianu and Marius Drăgan

W49. In all triangle ABC hold

$$\left(\sum m_a\right)^2 \leq 3s^2 + \frac{9}{4}\min\left\{(a-b)^2; (b-c)^2; (c-a)^2\right\}.$$

Marius Drăgan

W50. In all triangle ABC hold

$$\sum \frac{a(h_a-2r)}{h_a+2r} \leq \frac{a+b+c}{5} \leq \sum \frac{(b+c-a)(h_a-2r)}{h_a+2r}.$$

Mihály Bencze

W51. Let $f: R \to R$ be a continuous convex functon and $a_k, \lambda_k > 0$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{k=1}^{n} \lambda_k = 1$. Prove that

$$\sum_{k=1}^{n} \frac{\lambda_k}{a_k} \int_0^{a_k} f(x)\,dx \geq \frac{1}{\sum\limits_{k=1}^{n} \lambda_k a_k} \int_0^{\sum\limits_{k=1}^{n} \lambda_k a_k} f(x)\,dx.$$

Mihály Bencze

W52. Prove that

$$\sum_{k=1}^{n} \sum_{1 \leq i_1 < ... < i_k \leq n} \frac{4^k}{(i_1+3)(i_2+3)...(i_k+3)} = \frac{n(n+11)\left(n^2+11n+58\right)}{840}.$$

Mihály Bencze

W53. If F_k denote the k^{th} Fibonacci number $(F_0 = F_1 = 1,\ F_{n+2} = F_{n+1} + F_n)$ then

$$\prod_{k=1}^{n} \frac{e^{F_{k+1}} - e^{F_k}}{F_{k-1}} \geq e^{\frac{1}{2}(F_{n+4}-5)}.$$

Mihály Bencze

W54. If $a_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} \frac{a_1}{\sqrt{a_2}} arctg \frac{1}{\sqrt{a_2}} \geq \frac{\left(\sum\limits_{k=1}^{n} a_k\right)^{\frac{3}{2}}}{\sqrt{\sum\limits_{cyclic} a_1 a_2}} arctg \sqrt{\frac{\sum\limits_{k=1}^{n} a_k}{\sum\limits_{cyclic} a_1 a_2}}.$$

Mihály Bencze

W55. In all acute triangle ABC holds

$$\left(\sum \sqrt{\sin A} + \sum \sqrt{\cos A}\right)^2 \leq 3\sqrt{2}\left(3 + \sum \frac{\sin 2A}{\cos\left(\frac{\pi}{4} - A\right)}\right).$$

Mihály Bencze

W56. If $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for all $n \geq 2$, then compute

$$\sum_{k=1}^{\infty} arctg \frac{P_{k+2}^2 - P_k^2}{2\left(P_k P_{k+1}^2 P_{k+2} - 1\right)} arctg \frac{2P_{k+1}^2}{P_k P_{k+1}^2 P_{k+2} + 1}.$$

Mihály Bencze

W57. Prove that if $a, b, c, d \in [1, \infty)$ then:

$$\frac{3}{a+1} + \frac{3}{b+1} + \frac{2}{c+1} + \frac{1}{d+1} <$$

$$< 6 + \frac{1}{1+a+b} + \frac{1}{1+a+b+c} + \frac{1}{1+a+b+c+dw}$$

Daniel Sitaru

W58. 1. In any triangle ABC, we have the following inequality:

$$am_a + bm_b + cm_c \geq a^2 + b^2 + c^2 - ab - bc - ca + 6\triangle,$$

where a, b, c are the lengths of the sides: $m_a.m_b, m_c, m_c$ - the lengths of the medians and $\triangle-$ the area of the triangle ABC.
2. In any triangle ABC, the following inequality holds:

$$\sqrt{\frac{m_a - h_a}{a}} + \sqrt{\frac{m_b - h_b}{b}} + \sqrt{\frac{m_c - h_c}{c}} \geq \frac{\sqrt{2}}{2}\left(\left|\frac{b-c}{a}\right| + \left|\frac{c-a}{b}\right| + \left|\frac{a-b}{c}\right|\right).$$

Nicuşor Minculete

W59. How many squares can you draw on a finite lattice board defined by

$$B = \{(x, y) \in N \times N : 0 \leq x, y \leq 2017\},$$

in such a way that all vertices have integer coordinates ?

Ovidiu Bagdasar

W60. Find the number of segments having integer length that can be drawn between points of the finite lattice board defined by

$$B = \{(x, y) \in N \times N : 0 \leq x, y \leq 2017\}.$$

Ovidiu Bagdasar

W61. Let $x, y, z > 0$ be real numbers. Prove that the following inequality holds:

$$x^3y^3 + y^3z^3 + z^3x^3 \geq xyz(x^2y + y^2z + z^2x).$$

Ovidiu Bagdasar

W62. If $A > 0, B > 0, A + B \leq \pi$, and $0 \leq \lambda \leq \frac{1}{2}$, then

$$\cos^2 \lambda A + \sin^2 \lambda B - 2\cos \lambda A \cdot \sin \lambda B \cdot \sin \lambda\pi - \cos^2 \lambda\pi \leq \sin^4\left(\frac{(2\lambda+1)\pi}{4}\right)$$

Chang-Jian Zhao and Mihály Bencze