

József Wildt International Mathematical Competition

The Edition XXIXth, 2019¹

The solution of the problems W.1 - W.70 must be mailed before 26. October 2019, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania, E-mail: benczemihaly@gmail.com; benczemihaly@yahoo.com

W1. The Pell numbers P_n satisfy $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Find

$$\sum_{n=1}^{\infty} \left(\arctan \frac{1}{P_{2n}} + \arctan \frac{1}{P_{2n+2}} \right) \arctan \frac{2}{P_{2n+1}}.$$

Ángel Plaza

W2. If $0 < a \leq c \leq b$ then:

$$\frac{(b^{30} - a^{30})(b^{30} - c^{30})}{36b^{10}} \leq \frac{(b^{25} - a^{25})(b^{25} - c^{25})}{25} \leq \frac{(b^{30} - a^{30})(b^{30} - c^{30})}{36(ac)^5}$$

Daniel Sitaru

W3. Compute

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x + 1 - x^2}{(1 + x \sin x) \sqrt{1 - x^2}} dx$$

D.M. Bătineţu-Giurgiu and Stanciu Neculai

W4. If $x, y, z, t > 1$ then:

$$\left(\log_{zxt} x \right)^2 + \left(\log_{xyt} y \right)^2 + \left(\log_{xyz} z \right)^2 + \left(\log_{yzt} t \right)^2 > \frac{1}{4}$$

Daniel Sitaru

W5. Let $n \geq 1$. Find a set of distincts real numbers $(x_j)_{1 \leq j \leq n}$ such that for any bijections $f : \{1; 2; \dots; n\}^2 \rightarrow \{1; 2; \dots; n\}^2$ the matrix $(x_{f(i,j)})_{1 \leq i,j \leq n}$ is invertible.

Moubinool Omarjee

W6. Compute

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(1 + \ln x) \cos x + x \sin x \ln x}{\cos^2 x + x^2 \ln^2 x} dx$$

D.M. Bătineţu-Giurgiu and Stanciu Neculai

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W7. If

$$\Omega_n = \sum_{k=1}^n \left(\int_{-\frac{1}{k}}^{\frac{1}{k}} (2x^{10} + 3x^8 + 1) \cdot \cos^{-1}(kx) dx \right)$$

then find:

$$\Omega = \lim_{n \rightarrow \infty} (\Omega_n - \pi \cdot H_n)$$

Daniel Sitaru

W8. Let $(a_n)_{n \geq 1}$ be a positive real sequence given by $a_n = \sum_{k=1}^n \frac{1}{k}$. Compute

$$\lim_{n \rightarrow \infty} e^{-2a_n} \sum_{k=1}^n \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!} \right)^2 \right]$$

where we denote by $[x]$ the integer part of x .

D.M. Bătinețu-Giurgiu and Stanciu Neculai

W9. Let $\alpha > 0$ be a real number. Compute the limit of the sequence $\{x_n\}_{n \geq 1}$ defined by

$$x_n = \begin{cases} \sum_{k=1}^n \sinh\left(\frac{k}{n^2}\right), & n > \frac{1}{\alpha}; \\ 0, & n \leq \frac{1}{\alpha}. \end{cases}$$

José Luis Díaz-Barrero

W10. If $si(x) = -\int_x^\infty \left(\frac{\sin t}{t}\right) dt; x > 0$ then:

$$\int_e^{e^2} \left(\frac{1}{x} (si(e^4 x) - si(e^3 x))\right) dx = \int_3^{e^4} \left(\frac{1}{x} (si(e^2 x) - si(ex))\right) dx$$

Daniel Sitaru

W11. Let $(s_n)_{n \geq 1}$ be a sequence given by $s_n = -2\sqrt{n} + \sum_{k=1}^n \frac{1}{\sqrt{k}}$ with $\lim_{n \rightarrow \infty} s_n = s = \text{Ioachimescu}$ constant and $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$ be a positive real sequences such that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{na_n} = a \in R_+^*, \lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n \sqrt{n}} = b \in R_+^*$$

Compute

$$\lim_{n \rightarrow \infty} (1 + e^{s_n} - e^{s_{n+1}})^{\sqrt[n]{a_n b_n}}$$

D.M. Bătinețu-Giurgiu and Stanciu Neculai

W12. If $0 < a < b$ then:

$$\frac{\int_a^{\frac{a+b}{2}} (\tan^{-1} t) dt}{\int_a^b (\tan^{-1} t) dt} < \frac{1}{2}$$

Daniel Sitaru

W13. Let a, b and c be complex numbers such that $abc = 1$. Find the value of the cubic root of

$$\begin{vmatrix} b + n^3c & n(c - b) & n^2(b - c) \\ n^2(c - a) & c + n^3a & n(a - c) \\ n(b - a) & n^2(a - b) & a + n^3b \end{vmatrix}$$

José Luis Díaz-Barrero

W14. If $a, b, c > 0; ab + bc + ca = 3$ then:

$$4(\tan^{-1} 2)(\tan^{-1}(\sqrt[3]{abc})) \leq \pi \tan^{-1}(1 + \sqrt[3]{abc})$$

Daniel Sitaru

W15. It is possible to partition the set $\{100, 101, \dots, 1000\}$ into two subsets so that for any two distinct elements x and y belonging to the same subset $\sqrt[3]{x+y}$ is irrational?

José Luis Díaz-Barrero

W16. If $f : [a, b] \rightarrow (0, \infty); 0 < a \leq b; f$ derivable; f' continuous then:

$$\int_a^b \frac{f'(x)\sqrt{f(x)}}{f^3(x)+1} dx \leq \tan^{-1}\left(\frac{f(b)-f(a)}{1+f(a)f(b)}\right)$$

Daniel Sitaru

W17. Let $f_n = \left(1 + \frac{1}{n}\right)^n ((2n-1)!!F_n)^{1/n}$. Find $\lim_{n \rightarrow \infty} (f_{n+1} - f_n)$ where F_n denotes the n th Fibonacci number (given by $F_0 = 0, F_1 = 1$, and by $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$).

Ángel Plaza

W18. Let $\{c_k\}_{k \geq 1}$ be a sequence with $0 \leq c_k \leq 1, c_1 \neq 0, \alpha > 1$. Let $C_n = c_1 + \dots + c_n$. Prove

$$\lim_{n \rightarrow \infty} \frac{C_1^\alpha + \dots + C_n^\alpha}{(C_1 + \dots + C_n)^\alpha} = 0$$

Perfetti Paolo

W19. Let $\{F_n\}_{n \in \mathbb{Z}}$ and $\{L_n\}_{n \in \mathbb{Z}}$ denote the Fibonacci and Lucas numbers, respectively, given by

$$F_{n+1} = F_n + F_{n-1} \text{ and } L_{n+1} = L_n + L_{n-1} \text{ for all } n \geq 1,$$

with $F_0 = 0, F_1 = 1, L_0 = 2$, and $L_1 = 1$. Prove that for integers $n \geq 1$ and $j \geq 0$

$$(i) \sum_{k=1}^n F_{k \pm j} L_{k \mp j} = F_{2n+1} - 1 + \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{\pm j} F_{\pm 2j}, & \text{if } n \text{ is odd} \end{cases}.$$

$$(ii) \sum_{k=1}^n F_{k+j} F_{k-j} L_{k+j} L_{k-j} = \frac{F_{4n+2} - 1 - nL_{4j}}{5}.$$

Ángel Plaza

W20. i). Let G be a $(4, 4)$ unoriented graph, 2-regular, containing a cycle with the length 3. Find the characteristic polynomial $P_G(\lambda)$, its spectrum $Spec(G)$ and draw the graph G .

ii). Let G' be another 2-regular graph, having its characteristic polynomial $P_{G'}(\lambda) = \lambda^4 - 4\lambda^2 + \alpha$, $\alpha \in \mathbb{R}$. Find the spectrum $Spec(G')$ and draw the graph G' .

iii). Are the graphs G and G' cospectral or isomorphic?

Laurențiu Modan

W21. Let f be a continuously differentiable function on $[0, 1]$ and $m \in \mathbb{N}$. Let $A = f(1)$ and let $B = \int_0^1 x^{-\frac{1}{m}} f(x) dx$. Calculate

$$\lim_{n \rightarrow \infty} n \left(\int_0^1 f(x) dx - \sum_{k=1}^n \left(\frac{k^m}{n^m} - \frac{(k-1)^m}{n^m} \right) f\left(\frac{(k-1)^m}{n^m}\right) \right)$$

in terms of A and B .

Li Yin

W22. Let A and B the series:

$$A = \sum_{n>0} \frac{C_{2n}^1}{C_{2n}^0 + C_{2n}^1 + \dots + C_{2n}^{2n}}, \quad B = \sum_{n>0} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + \frac{5}{2})}.$$

Study if $\frac{A}{B}$ is irrational number.

Laurențiu Modan

W23. If b, c are the legs, and a is the hypotenuse of a right triangle, prove that

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 5 + 3\sqrt{2}$$

Ovidiu Pop

W24. If $a, b, c > 0$, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{a+b}{a+b+2c} + \frac{b+c}{b+c+2a} + \frac{c+a}{c+a+2b}$$

Ovidiu Pop

W25. Let $x_i, y_i, z_i, \omega_i \in R^+, i = 1, 2, \dots, n$, such that

$$\sum_{i=1}^n x_i = nx, \sum_{i=1}^n y_i = ny, \sum_{i=1}^n \omega_i = n\omega$$

$$\Gamma(z_i) \geq \Gamma(\omega_i), \sum_{i=1}^n \Gamma(z_i) = n\Gamma^*(z).$$

Then

$$\sum_{i=1}^n \frac{(\Gamma(x_i) + \Gamma(y_i))^2}{\Gamma(z_i) - \Gamma(\omega_i)} \geq n \frac{(\Gamma(x) + \Gamma(y))^2}{\Gamma^*(z) - \Gamma(\omega)}.$$

Li Yin

W26. Let $n \in \mathbb{N}, n \geq 2, a_1, a_2, \dots, a_n \in R$ and $a_n = \max\{a_1, a_2, \dots, a_n\}$

a). If $t_k, t'_k \in R, k \in \{1, 2, \dots, n\}, t_k \leq t'_k$, for any $k \in \{1, 2, \dots, n-1\}$ and

$$\sum_{k=1}^n t_k = \sum_{k=1}^n t'_k$$

prove that

$$\sum_{k=1}^n t_k a_k \geq \sum_{k=1}^n t'_k a_k$$

b). If $b_k, c_k \in R_+, k \in \{1, 2, \dots, n\}, b_k \leq c_k$ for any $k \in \{1, 2, \dots, n-1\}$ and

$$b_1 \cdot b_2 \cdot \dots \cdot b_n = c_1 \cdot c_2 \cdot \dots \cdot c_n$$

prove that

$$\prod_{k=1}^n b_k^{a_k} \geq \prod_{k=1}^n c_k^{a_k}$$

Ovidiu Pop and Petru Braica

W27. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(-x) + \int_0^x tf(x-t)dt = x, \quad \forall x \in \mathbb{R}.$$

Ovidiu Furdui and Alina Sîntămărian

W28. In a room, we have 2019 aligned switches, connected to 2019 light bulbs, all initially switched on. Then, 2019 people enter the room one by one, performing the operation: The first, uses all the switches; the second, every second switch; the third, every third switch, and so on. How many lightbulbs remain switched on, after all the people entered ?

Ovidiu Bagdasar

W29. Prove that

$$\int_0^\infty e^{3t} \frac{4e^{4t}(3t-1) + 2e^{2t}(15t-17) + 18(1-t)}{(1+e^{4t}-e^{2t})^2} dt = 12 \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2} - 10$$

Perfetti Paolo

W30. a). Prove that

$$\lim_{n \rightarrow \infty} \left(n + \frac{1}{4} - \zeta(3) - \zeta(5) - \dots - \zeta(2n+1) \right) = 0.$$

b). Calculate

$$\sum_{n=1}^\infty \left(n + \frac{1}{4} - \zeta(3) - \zeta(5) - \dots - \zeta(2n+1) \right).$$

Ovidiu Furdui and Alina Sîntămărian

W31. Let $a, b \in \Gamma, a < b$ and the differentiable function $f : [a, b] \rightarrow \Gamma$, such that $f(a) = a$ and $f(b) = b$. Prove that

$$\int_a^b (f'(x))^2 dx \geq b - a$$

Dorin Mărghidanu

W32. Let u_k, v_k, a_k and b_k be non-negative real sequences such as $u_k > a_k$ and $v_k > b_k$, where $k = 1, 2, \dots, n$. If $0 < m_1 \leq u_k \leq M_1$ and $0 < m_2 \leq v_k \leq M_2$, then

$$\sum_{k=1}^n (\ell u_k v_k - a_k b_k) \geq \left(\sum_{k=1}^n (u_k^2 - a_k^2) \right)^{1/2} \left(\sum_{k=1}^n (v_k^2 - b_k^2) \right)^{1/2}, \quad (1.1)$$

where

$$\ell = \frac{M_1 M_2 + m_1 m_2}{2(m_1 M_1 m_2 M_2)^{1/2}}. \quad (1.2)$$

Chang-Jian Zhao and Mihály Bencze

W33. Let $0 < \frac{1}{q} \leq \frac{1}{p} < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let u_k, v_k, a_k and b_k be non-negative real sequences such as $u_k^2 > a_k^p$ and $v_k > b_k^q$, where $k = 1, 2, \dots, n$. If $0 < m_1 \leq u_k \leq M_1$ and $0 < m_2 \leq v_k \leq M_2$, then

$$\begin{aligned} & \left(\sum_{k=1}^n (\ell^p (u_k + v_k)^2 - (a_k + b_k)^p) \right)^{1/p} \geq \\ & \geq \left(\sum_{k=1}^n (u_k^2 - a_k^p) \right)^{1/p} + \left(\sum_{k=1}^n (v_k^2 - b_k^p) \right)^{1/p}, \end{aligned} \quad (1.5)$$

where ℓ is as in (1.2).

Chang-Jian Zhao and Mihály Bencze

W34. Let a, b, c be positive real numbers and let m, n ($m \geq n$) be positive integers. Prove that

$$\begin{aligned} & \frac{a^{n-1} b^{n-1} c^{m-n-1}}{a^{m+n} + b^{m+n} + a^n b^n c^{m-n}} + \frac{b^{n-1} c^{n-1} a^{m-n-1}}{b^{m+n} + c^{m+n} + b^n c^n a^{m-n}} + \\ & + \frac{c^{n-1} a^{n-1} b^{m-n-1}}{c^{m+n} + a^{m+n} + a^n a^n b^{m-n}} \leq \frac{1}{abc} \end{aligned}$$

Dorin Mărghidanu and Kunihiko Chikaya

W35. Calculate

$$\lim_{n \rightarrow \infty} \frac{n! \left(1 + \frac{1}{n}\right)^{n^2+n}}{n^{n+1/2}}.$$

Arkady Alt

W36. For any $a, b, c > 0$ and for any $n \in \mathbb{N}^*$, prove the inequality

$$(a-b) \left(\frac{a}{b}\right)^n + (b-c) \left(\frac{b}{c}\right)^n + (c-a) \left(\frac{c}{a}\right)^n \geq (a-b) \frac{a}{b} + (b-c) \frac{b}{c} + (c-a) \frac{c}{a}$$

Dorin Mărghidanu

W37. For real $a > 1$ find

$$\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{k=2}^n (a - a^{1/k})}.$$

Arkady Alt

W38. Let a, b, c be the sides of an acute triangle $\triangle ABC$, then for any $x, y, z \geq 0$, such that

$$xy + yz + zx = 1,$$

holds inequality:

$$a^2 x + b^2 y + c^2 z \geq 4F,$$

where F is the area of the triangle $\triangle ABC$.

Arkady Alt

W39. Let u, v, w complex numbers such that:

$$u + v + w = 1, \quad u^2 + v^2 + w^2 = 3, \quad uvw = 1$$

Prove that

- u, v, w are distinct numbers two by two
- if $S^{(k)} := u^k + v^k + w^k$, then $S^{(k)}$ is an odd natural number
- the expression

$$\frac{u^{2n+1} - v^{2n+1}}{u - v} + \frac{v^{2n+1} - w^{2n+1}}{v - w} + \frac{w^{2n+1} - u^{2n+1}}{w - u}$$

is an integer number.

Dorin Mărghidanu

W40. Let f_n be n -th Fibonacci number defined by recurrence $f_{n+1} - f_n - f_{n-1} = 0$, $n \in \mathbb{N}$ and initial conditions $f_0 = 0, f_1 = 1$. Prove that for any $n \in \mathbb{N}$

$$(n-1)(n+1)(2nf_{n+1} - (n+6)f_n)$$

is divisible by 150 for any $n \in \mathbb{N}$.

Arkady Alt

W41. For $n \in \mathbb{N}$, consider in \mathbb{R}^3 the regular tetrahedron with vertices $O(0, 0, 0)$, $A(n, 9n, 4n)$, $B(9n, 4n, n)$ and $C(4n, n, 9n)$. Show that the number N of points (x, y, z) , $(x, y, z \in \mathbb{Z})$ inside or on the boundary of the tetrahedron $OABC$ is given by

$$N = \frac{343n^3}{3} + \frac{35}{2}n^2 + \frac{7}{6}n + 1.$$

Eugen J. Ionascu

W42. For p, q, l strictly positive real numbers, consider the following problem: for $y \geq 0$ fixed, determine the values $x \geq 0$ such that $x^p - lx^q \leq y$. Denote by $S(y)$ the set of solutions of this problem. Prove that if one has $p < q$, $\varepsilon \in (0, l^{\frac{1}{p-q}})$, $0 \leq x \leq \varepsilon$ and $x \in S(y)$, then

$$x \leq ky^\delta, \quad \text{where } k = \varepsilon(\varepsilon^p - l\varepsilon^q)^{-\frac{1}{p}} \text{ and } \delta = \frac{1}{p}.$$

József Kolumbán

W43. Consider the sequence of polynomials $P_0(x) = 2$, $P_1(x) = x$ and $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$ for $n \geq 2$. Let x_n be the greatest zero of P_n in the interval $|x| \leq 2$. Show that

$$\lim_{n \rightarrow \infty} n^2 \left(4 - 2\pi + n^2 \int_{x_n}^2 P_n(x) dx \right) = 2\pi - 4 - \frac{\pi^3}{12}.$$

Eugen J. Ionascu

W44. We consider a natural number $n, n \geq 2$ and the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ n & 1 & 2 & \dots & n-1 \\ n-1 & n & 1 & \dots & n-2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 3 & 4 & \dots & 1 \end{pmatrix}$$

Show that:

$$\begin{aligned} & \varepsilon^n \det(I_n - A^{2n}) + \varepsilon^{n-1} \det(\varepsilon \cdot I_n - A^{2n}) + \varepsilon^{n-2} \det(\varepsilon^2 \cdot I_n - A^{2n}) + \dots \\ & \quad + \det(\varepsilon^n \cdot I_n - A^{2n}) = \\ & = n(-1)^{n-1} \left[\frac{n^n(n+1)}{2} \right]^{2n^2-4n} \left(1 + (n+1)^{2n} \left(2n + (-1)^n \binom{2n}{n} \right) \right) \end{aligned}$$

where $\varepsilon \in C \setminus R$, $\varepsilon^{n+1} = 1$.

Stănescu Florin

W45. Consider the complex numbers a_1, a_2, \dots, a_n , $n \geq 2$. Which have the following properties:

- a). $|a_i| = 1, (\forall) i = \overline{1, n}$; b). $\sum_{k=1}^n \arg a_k \leq \pi$

Show that the inequality

$$\begin{aligned} & \left(n^2 \cot\left(\frac{\pi}{2n}\right) \right)^{-1} \left| \sum_{k=0}^n (-1)^k [3n^2 - (8k+5)n + 4k(k+1)] \sigma_k \right| \geq \\ & \geq \sqrt{\left(1 + \frac{1}{n}\right)^2 \cot^2\left(\frac{\pi}{2n}\right) + 16} \left| \sum_{k=0}^n (-1)^k \cdot \sigma_k \right|, \end{aligned}$$

where $\sigma_0 = 1, \sigma_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1} a_{i_2} \dots a_{i_k}, (\forall) k = \overline{1, n}$.

Stănescu Florin

W46. Let $x, y, z > 0$ such that $x^2 + y^2 + z^2 = 3$. Then

$$x^3 \arctan \frac{1}{x} + y^3 \arctan \frac{1}{y} + z^3 \arctan \frac{1}{z} < \frac{\pi\sqrt{3}}{2}$$

Marian Cucoaneş and Marius Drăgan

W47. a). If $a, b, c, d > 0$, show inequality:

$$\arctan^2 \left(\frac{ad-bc}{ac+bd} \right) \geq 2 \left(1 - \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}} \right)$$

b). Calculate

$$\lim_{n \rightarrow \infty} n^\alpha \left(n - \sum_{k=1}^n \frac{n^2 + k^2 - k}{\sqrt{(n^2 + k^2)(n^2 + (k-1)^2)}} \right),$$

where $\alpha \in R$.

Stănescu Florin

W48. Let $f : (0, +\infty) \rightarrow R$ a convex function and $\alpha, \beta, \gamma > 0$. Then

$$\begin{aligned} & \frac{1}{6\alpha} \int_0^{6\alpha} f(x) dx + \frac{1}{6\beta} \int_0^{6\beta} f(x) dx + \frac{1}{6\gamma} \int_0^{6\gamma} f(x) dx \geq \\ & \geq \frac{1}{3\alpha + 2\beta + \gamma} \int_0^{3\alpha + 2\beta + \gamma} f(x) dx + \frac{1}{\alpha + 3\beta + 2\gamma} \int_0^{\alpha + 3\beta + 2\gamma} f(x) dx + \end{aligned}$$

$$+\frac{1}{2\alpha + \beta + 3\gamma} \int_0^{2\alpha + \beta + 3\gamma} f(x) dx \quad (1)$$

Marius Drăgan

W49. Let $a, b, c \in (0, +\infty)$. Then the following inequality is true:

$$\begin{aligned} & \sqrt{(a+b)(b+c)} + \sqrt{(b+c)(c+a)} + \sqrt{(c+a)(a+b)} + a + b + c \leq \\ & \leq (ab + bc + ca) \left(\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \right) \end{aligned}$$

Mihály Bencze and Marius Drăgan

W50. Let $x, y, z > 0$, $\lambda \in (-\infty, 0) \cup (1, +\infty)$ such that $x + y + z = 1$. Then

$$\sum x^\lambda y^\lambda \sum \frac{1}{(x+y)^{2\lambda}} \geq 9 \left(\frac{1}{4} - \frac{1}{9} \sum \frac{1}{(x+1)^2} \right)^\lambda$$

Marius Drăgan and Sorin Rădulescu

W51. Let a, b, c, d, e be real strictly positive real numbers such that $abcde = 1$. Then is true the following inequality:

$$\frac{de}{a(b+1)} + \frac{ea}{b(c+1)} + \frac{ab}{c(d+1)} + \frac{bc}{d(e+1)} + \frac{cd}{e(a+1)} \geq \frac{5}{2}$$

Mihály Bencze and Marius Drăgan

W52. Let $f : R \rightarrow R$ a periodic and continue function with period T and $F : R \rightarrow R$ antiderivative of f . Then

$$\int_0^T \left[F(nx) - F(x) - f(x) \frac{(n-1)T}{2} \right] dx = 0$$

Marius Drăgan and Mihály Bencze

W53. Compute

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{n+k+1\sqrt{n+1} - n+k\sqrt{n}}{n+k\sqrt{n+1} - n+k\sqrt{n}}$$

Marius Drăgan

W54. Let x_1, x_2, \dots, x_n be a positive numbers, $k \geq 1$. Then the following inequality is true:

$$(x_1^k + x_2^k + \dots + x_n^k)^{k+1} \geq (x_1^{k+1} + x_2^{k+1} + \dots + x_n^{k+1})^k + 2 \left(\sum_{1 \leq i < j \leq n} x_i^k x_j \right)^k$$

Marius Drăgan

W55. Let $f, g, h : [a, b] \rightarrow R$ be n positive numbers such that $\sum_{i=1}^n \sqrt{a_i} = \sqrt{n}$. Then

$$\prod_{i=1}^{n-1} \left(1 + \frac{1}{a_i}\right)^{a_{i+1}} \left(1 + \frac{1}{a_n}\right)^{a_1} \geq 1 + \frac{n}{\sum_{i=1}^n a_i}$$

Marius Drăgan

W56. Let $f, g, h : [a, b] \rightarrow R$, three integrable functions such that:

$$\int_a^b f g dx = \int_a^b g h dx = \int_a^b h f dx = \int_a^b g^2 dx \int_a^b h^2 dx = 1$$

Then

$$\int_a^b g^2 dx = \int_a^b h^2 dx = 1$$

Marius Drăgan and Sorin Rădulescu

W57. Let be $x_1 = \frac{1}{n+1\sqrt{n!}}$ and $x_2 = \frac{1}{n+1\sqrt{(n-1)!}}$ for all $n \in N^*$ and $f : \left(\frac{1}{n+1\sqrt{(n+1)!}}, 1\right] \rightarrow R$ where

$$f(x) = \frac{n+1}{x \ln(n+1)! + (n+1) \ln(x^x)}$$

Prove that the sequence $(a_n)_{n \geq 1}$ when $a_n = \int_{x_1}^{x_2} f(x) dx$ is convergent and compute $\lim_{n \rightarrow \infty} a_n$.

Ionel Tudor

W58. In the $[ABCD]$ tetrahedron having all the faces acute angled triangles, is denoted by r_X, R_X the radius lengths of the circle inscribed and circumscribed respectively on the face opposite to the $X \in \{A, B, C, D\}$ peak, and with R the length of the radius of the sphere circumscribed to the tetrahedron. Show that inequality occurs

$$8R^2 \geq (r_A + R_A)^2 + (r_B + R_B)^2 + (r_C + R_C)^2 + (r_D + R_D)^2$$

Marius Olteanu

W59. In the any $[ABCD]$ tetrahedron we denote with α, β, γ the measures, in radians, of the angles of the three pairs of opposite edges and with r, R the lengths of the rays of the sphere inscribed and respectively circumscribed the tetrahedron. Demonstrate inequality

$$\left(\frac{3r}{R}\right)^3 \leq \sin \frac{\alpha + \beta + \gamma}{3}$$

(A refinement of inequality $R \geq 3r$).

Marius Olteanu

W60. In all tetrahedron ABCD holds

$$1). (n(n+2))^{\frac{1}{n}} \sum \left(\frac{(h_a - r)^2}{(h_a^n - r^n)(h_a^{n+2} - r^{n+2})} \right)^{\frac{1}{n}} \leq \frac{1}{r^2}$$

2). $(n(n+2))^{\frac{1}{n}} \sum \left(\frac{(2r_a-r)^2}{((2r_a)^{n-r^n})(2r_a)^{n+2-r^n+2}} \right)^{\frac{1}{n}} \leq \frac{1}{r^2}$
for all $n \in N^*$.

Mihály Bencze

W61. If $a, b, c \in R$ then

$$\sum \sqrt{(a+c)^2 b^2 + a^2 c^2} + \sqrt{5} \left| \sum ab \right| \geq \sum \sqrt{(ab+2bc+ca)^2 + (b+c)^2 a^2}.$$

Mihály Bencze

W62. Prove that

$$\int_0^{\frac{\pi}{2}} (\cos x)^{1+\sqrt{2n+1}} dx \leq \frac{2^{n-1} n! \sqrt{\pi}}{\sqrt{2} (2n+1)!}$$

for all $n \in N^*$.

Mihály Bencze

W63. If $b_k \geq a_k \geq 0$ ($k = 1, 2, 3$) and $\alpha \geq 1$ then

$$\begin{aligned} & (\alpha+3) \sum_{cyclic} (b_1 - a_1) \cdot \\ & \cdot \left((b_2 + b_3)^{\alpha+2} + (a_2 + a_3)^{\alpha+2} - (a_2 + b_3)^{\alpha+1} - (a_3 + b_2)^{\alpha+1} \right) \leq \\ & \leq (\alpha+2)(\alpha+3) \sum_{cyclic} (b_1 - a_1)(b_2 - a_2)(b_3^{\alpha+1} - a_3^{\alpha+1}) + (b_3 + b_2 + a_1)^{\alpha+3} + \\ & + (b_3 + a_2 + a_1)^{\alpha+3} + (a_3 + b_2 + a_1)^{\alpha+3} + (a_3 + a_2 + b_2)^{\alpha+3} - (b_3 + b_2 + a_1)^{\alpha+3} - \\ & - (b_3 + a_2 + a_1)^{\alpha+3} - (a_3 + b_2 + b_1)^{\alpha+3} - (a_3 + a_2 + a_1)^{\alpha+3} \end{aligned}$$

Mihály Bencze

W64. Prove that exist different natural numbers x, y, z, t for which

$$256 \cdot 2019^{180n+1} = 2 \cdot x^9 - 2 \cdot y^6 + z^5 - t^4$$

for all $n \in N$.

Mihály Bencze and Chang-Jian Zhao

W65. If $a, b, c \geq 1; y \geq x \geq 1; p, q, r > 0$ then

$$\begin{aligned} & \left(\frac{1+y(a^p b^q c^r)^{\frac{1}{p+q+r}}}{1+x(a^p b^q c^r)^{\frac{1}{p+q+r}}} \right)^{\frac{p+q+r}{(a^p b^q c^r)^{\frac{1}{p+q+r}}}} \left(\frac{1+ya}{1+xa} \right)^{\frac{p}{a}} \cdot \left(\frac{1+yb}{1+xb} \right)^{\frac{q}{b}} \left(\frac{1+yc}{1+xc} \right)^{\frac{r}{c}} \geq \\ & \geq \prod \left(\frac{1+y(a^p b^q)^{\frac{1}{p+q}}}{1+x(a^p b^q)^{\frac{1}{p+q}}} \right)^{\frac{p+q}{(a^p b^q)^{\frac{1}{p+q}}}} \end{aligned}$$

Mihály Bencze

W66. If $0 < a \leq b$ then

$$\frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2(b^2 - a^2)\sqrt{3}}{(a^2 + 2)(b^2 + 2)} \leq \int_a^b \frac{(x^2 + 1)(x^2 + x + 1) dx}{(x^3 + x^2 + 1)(x^3 + x + 1)} \leq \frac{4}{\sqrt{3}} \operatorname{arctg} \frac{(b - a)\sqrt{3}}{a + b + 2(1 + ab)}$$

Mihály Bencze

W67. Denote T the Toricelli point of the triangle ABC. Prove that

$$AB^2 \cdot BC^2 \cdot CA^2 \geq 3(TA^2 \cdot TB + TB^2 \cdot TC + TC^2 \cdot TA) \cdot (TA \cdot TB^2 + TB \cdot TC^2 + TC \cdot TA^2)$$

Mihály Bencze

W68. In all tetrahedron ABCD holds

$$1). \sum \frac{h_a - r}{h_a + r} \geq \sum \frac{h_a^t - r^t}{(h_a + r)^{t+1}} \quad 2). \sum \frac{2r_a - r}{2r_a + r} \geq \sum \frac{(2r_a)^t - r^t}{(2r_a + r)^{t+1}}$$

for all $t \in [0, 1]$.

Mihály Bencze

W69. Denote $\overline{w}_a, \overline{w}_b, \overline{w}_c$ the external angle-bisectors in triangle ABC, prove that

$$\sum \frac{1}{w_a} \leq \sqrt{\frac{(s^2 - r^2 - 4Rr)(8R^2 - s^2 - r^2 - 2Rr)}{8s^2 R^2 r}}$$

Mihály Bencze

W70. If $x \in (0, \frac{\pi}{2})$ then

$$\left(\frac{\sin(\frac{\pi}{2} \sin x)}{\sin x} \right)^2 + \left(\frac{\sin(\frac{\pi}{2} \cos x)}{\cos x} \right)^2 \geq 3.$$

Mihály Bencze