

Weak Liouville-Arnol'd Theorems and Their Implications

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Received: 18 May 2011 / Accepted: 9 April 2012
Published online: 26 August 2012 – © Springer-Verlag 2012

Abstract: This paper studies the existence of invariant smooth Lagrangian graphs for Tonelli Hamiltonian systems with symmetries. In particular, we consider Tonelli Hamiltonians with n independent but not necessarily involutive constants of motion and obtain two theorems reminiscent of the Liouville-Arnol'd theorem. Moreover, we also obtain results on the structure of the configuration spaces of such systems that are reminiscent of results on the configuration space of completely integrable Tonelli Hamiltonians.

1. Introduction

In the study of Hamiltonian systems, a special role is played by *invariant Lagrangian manifolds*. These objects arise quite naturally in many physical and geometric problems and share a deep relation with the dynamics of the system and with the Hamiltonian itself. Our concern in this paper is with Hamiltonian systems that possess invariant Lagrangian *graphs*, or more precisely, with conditions that imply the existence of such graphs. Specifically, we address the following question:

Question I. *When does a Hamiltonian system possess an invariant smooth Lagrangian graph?*

It is natural to expect that “sufficiently” symmetric systems ought to possess an abundance of invariant Lagrangian graphs. Inspired by the results in [33], this paper demonstrates, with two different notions of symmetry, conditions that imply the existence of such graphs. This approach to Question I leads us to two theorems which in important aspects mirror the classical theorem of Liouville-Arnol'd. While on the one hand the first of these theorems can be seen as a (non-trivial) generalisation of the main theorem in [33] (see Remark 1.1 for more details), on the other hand our present analysis extends well beyond, providing a much deeper insight into the nature and the properties of the so-called *weakly-integrable systems*. There is a large literature on the structure of the configuration space of a completely integrable Tonelli Hamiltonian,

see *inter alia* [9,20,37,38]; in pursuing the analogy between the present paper’s *weak* Liouville-Arnol’d theorems and the classical theorem, we have proven two results on the topological structure of the configuration spaces of weakly-integrable systems. Indeed, we believe that the following is an interesting question

Question II. *If a Hamiltonian system possesses an invariant smooth Lagrangian graph, what is true of its configuration space?*

To address each of these questions, we use two notions of “symmetry” in this paper. Let us introduce those:

Classical symmetries. Let us recall some terminology used to describe classical symmetries. The cotangent bundle, T^*M , of a smooth manifold M is equipped with a canonical Poisson structure $\{\cdot, \cdot\}$. Given a smooth function H , the vector field $X_H = \{H, \cdot\}$ is a *Hamiltonian system* with Hamiltonian H . The skew-symmetry of $\{\cdot, \cdot\}$ implies that if $\{H, F\} \equiv 0$, then the vector field X_H is tangent to the level sets of F ; and, the Jacobi identity implies it commutes with X_F . In such a situation, these Hamiltonians are said to *Poisson-commute*, or be *in involution*, and F is said to be a *constant of motion*, or *first integral*. The Liouville-Arnol’d theorem describes the situation when H has n independent, Poisson commuting integrals.

Theorem (Liouville-Arnol’d). *Let (V, ω) be a symplectic manifold with $\dim V = 2n$ and let $H : V \rightarrow \mathbb{R}$ be a proper Hamiltonian. Suppose that there exists n integrals of motion $F_1, \dots, F_n : V \rightarrow \mathbb{R}$ such that:*

- i) F_1, \dots, F_n are C^2 and functionally independent almost everywhere on V ;
- ii) F_1, \dots, F_n are pairwise in involution, i.e. $\{F_i, F_j\} = 0$ for all $i, j = 1, \dots, n$.

Suppose the non-empty regular level set $\Lambda_a := \{F_1 = a_1, \dots, F_n = a_n\}$ is connected. Then Λ_a is an n -torus, \mathbb{T}^n and there is a neighbourhood \mathcal{O} of $0 \in H^1(\Lambda_a; \mathbb{R})$ such that for each $c' \in \mathcal{O}$ there is a unique smooth Lagrangian $\Lambda_{c'}$ that is a graph over Λ_a with cohomology class c' . Moreover, the flow of $X_H|_{\Lambda_{c'}}$ is a rigid rotation.

Remark. There are numerous proofs of this theorem in its modern formulation, see *inter alia* [3,5,13,22,25]. The map $F := (F_1, \dots, F_n)$ is referred to as an *integral map*, *first-integral map* or a *momentum map*. The invariance of the level set Λ_a simply follows from F being an integral of motion; the fact that it is a Lagrangian torus and that the Hamiltonian flow is conjugate to a rigid rotation, strongly relies on these integrals being pairwise in involution and independent.

Inspired by the Liouville-Arnol’d theorem, we address Question I in the case of systems that possess a sufficiently large number of symmetries. Let us recall the definition of a *weakly integrable* system.

Definition 1.1 (Weak integrability [33]). *Let $H \in C^2(T^*M)$. If there is a C^2 map $F : T^*M \rightarrow \mathbb{R}^n$ whose singular set is nowhere dense, and F Poisson-commutes with H , then we say that H is **weakly integrable**.*

1.1. Results. Recall that a Hamiltonian $H \in C^2(T^*M)$ is *Tonelli* if it is fibrewise strictly convex and enjoys fibrewise superlinear growth.¹ This paper’s first result is

¹ Section 2 provides a synopsis of Mather theory and Fathi’s weak KAM theory.

Theorem 1.1 (Weak Liouville-Arnol'd). *Let M be a closed manifold of dimension n and $H : T^*M \rightarrow \mathbb{R}$ a weakly integrable Tonelli Hamiltonian with integral map $F : T^*M \rightarrow \mathbb{R}^n$. If for some cohomology class $c \in H^1(M; \mathbb{R})$ the corresponding Aubry set $\mathcal{A}_c^* \subset \text{Reg } F$, then there exists an open neighborhood \mathcal{O} of c in $H^1(M; \mathbb{R})$ such that the following holds:*

- i) *For each $c' \in \mathcal{O}$ there exists a smooth invariant Lagrangian graph $\Lambda_{c'}$ of cohomology class c' , which admits the structure of a smooth \mathbb{T}^d -bundle over a base B^{n-d} that is parallelisable, for some $d > 0$.*
- ii) *The motion on each $\Lambda_{c'}$ is Schwartzman strictly ergodic (see [16]), i.e. all invariant probability measures have the same rotation vector and the union of their supports equals $\Lambda_{c'}$. In particular, all orbits are conjugate by a smooth diffeomorphism isotopic to the identity.*
- iii) *Mather's α -function $\alpha_H : H^1(M; \mathbb{R}) \rightarrow \mathbb{R}$ is differentiable at all $c' \in \mathcal{O}$ and its convex conjugate $\beta_H : H_1(M; \mathbb{R}) \rightarrow \mathbb{R}$ is differentiable at all rotation vectors $h \in \partial\alpha_H(\mathcal{O})$, where $\partial\alpha_H(\mathcal{O})$ denotes the set of subderivatives of α_H at some element of \mathcal{O} .*

Remark. i) We named this theorem as such because it drops the involutivity hypothesis of the classical theorem and still obtains results that are quite analogous. ii) The theorem remains true if one replaces the hypothesis $\mathcal{A}_c^* \subset \text{Reg } F$ with $\mathcal{M}_c^* \subset \text{Reg } F$, where \mathcal{M}_c^* denotes the Mather set. iii) We conjecture that weak integrability implies that $\dim H^1(M; \mathbb{R}) \leq \dim M$ with equality if and only if M is a torus even without the a priori assumption $\mathcal{A}_c^* \subset \text{Reg } F$.

Remark 1.1. This theorem extends and improves the main result in [33] in many non-trivial respects.

- i) First of all, we provide a description of the topological structure of these invariant Lagrangian graphs $\Lambda_{c'}$, showing that they admit the structure of a smooth \mathbb{T}^d -bundle over a parallelisable base.
- ii) Then, we prove that each $\Lambda_{c'}$ has a well-defined rotation vector, which implies the differentiability of Mather's α function.
- iii) Moreover, we prove that the flow of $X_H|_{\Lambda_{c'}}$ is a rotation on the \mathbb{T}^d fibres of $\Lambda_{c'}$ with rotation vector $h_{c'} = \partial\alpha_H(c')$, where $\partial\alpha_H(c')$ is the derivative of α_H at c' . This is analogous to what happens in the classical Liouville-Arnol'd theorem, where the rotation vector is the derivative of H at c' .

Let us then pursue the analogy with the Liouville-Arnol'd theorem and complete integrability and turn now to the implications of Theorem 1.1 for the topology of the configuration space M . Recall that a smooth manifold is *irreducible* if, when written as a connect sum, one of the summands is a standard sphere. In 3-manifold topology, a central role is played by those closed 3-manifolds which contain a non-separating *incompressible* surface, or dually, which have non-vanishing first Betti number. Such manifolds are called *Haken*; it is an outstanding conjecture that every irreducible 3-manifold with infinite fundamental group has a finite covering that is Haken [18, Questions 1.1–1.3]. This conjecture is implied by the virtually fibred conjecture [1]. Given the proof of the geometrisation conjecture, the virtual Haken conjecture is proven for all cases but hyperbolic 3-manifolds. Thurston and Dunfield have shown there is good reason to believe the conjecture is true in this case [14].

Theorem 1.2. *Assume the hypotheses of Theorem 1.1. Then M is diffeomorphic to a trivial \mathbb{T}^d -bundle over a parallelisable base B such that all finite covering spaces of B have zero first Betti number. Therefore*

- i) $\dim M \leq 3$ implies that M is diffeomorphic to a torus;
- ii) $\dim M = 4$ implies, assuming the virtual Haken conjecture, that M is diffeomorphic to either \mathbb{T}^4 or $\mathbb{T}^1 \times E$, where E is an orientable 3-manifold finitely covered by S^3 .
- iii) If $\dim H^1(M; \mathbb{R}) \geq \dim M$, then $\dim H^1(M; \mathbb{R}) = \dim M$ and M is diffeomorphic to $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

Non-classical symmetries. In the second part of this article, we investigate the case in which the system's symmetries are not classical and do not come from conserved quantities, but are induced by invariance under the action of an amenable Lie group on the universal cover of the manifold. This action need not descend to the quotient and is generally only evident in statistical properties of orbits. In particular, these symmetries may only manifest themselves in the structure of the action-minimizing sets.

Recall that a topological group is *amenable* if it admits a left-invariant, finitely additive, Borel probability measure. Due to the Levi decomposition, an amenable Lie group is a semi-direct product of its solvable radical and a compact subgroup. A solvable Lie group is said to be *exponential* or *type (E)* if the exponential map of the Lie algebra is surjective; we will say an amenable Lie group is of type (E) if its radical is of type (E). For each bi-invariant 1-form ϕ on the simply-connected amenable Lie group G with lattice subgroup Γ , let $\Lambda_c = \Gamma \cdot \text{graph}(\phi) \subset T^*(\Gamma \backslash G)$ be the Lagrangian graph of cohomology class c . The union of such graphs is a submanifold $\mathcal{M} \subset T^*(\Gamma \backslash G)$ naturally diffeomorphic to $H^1(\Gamma \backslash G; \mathbb{R}) \times \Gamma \backslash G$ and this diffeomorphism sends Λ_c to $\{c\} \times \Gamma \backslash G$.

Theorem 1.3. *Let G be a simply-connected amenable Lie group and let $\Gamma \triangleleft G$ be a lattice subgroup, $M = \Gamma \backslash G$ and H be induced by a left-invariant C^r Tonelli Hamiltonian on T^*G . Then*

- i) for all $c \in H^1(M; \mathbb{R})$, the Mather set $\mathcal{M}_c^*(H)$ equals the Lagrangian graph Λ_c ;
- ii) the flow of $X_H|_{\Lambda_c}$ is a right-translation by a 1-parameter subgroup of G ;
- iii) the motion on Λ_c is Schwartzman strictly ergodic;
- iv) Mather's α function $\alpha_H : H^1(M; \mathbb{R}) \rightarrow \mathbb{R}$ is C^r .

Remark 1.2. (i–ii) provide analogues to the Lagrangian tori and action-angle coordinates in the classical Liouville-Arnol'd theorem. However, there are some oddities: for example, it is possible that these right-translations have positive topological entropy. Indeed, this is *exactly* what happens in the *Sol* 3-manifold examples of Bolsinov-Taimanov [8] (in that example, all such Tonelli Hamiltonians are also completely integrable). Moreover, in this case we can prove that the *frequency map* $\partial \alpha_H$ has the same regularity as the Hamiltonian vector field (see iv). Whether or not the same property holds in Theorem 1.1 remains an open question. Observe that this problem is strictly related to the regularity of the family $\{\Lambda_{c'}\}$ as a function of c' .

To prove this theorem we introduce a generalised notion of rotation vector and a novel averaging procedure (see Sect. 4), which are likely to be of independent interest.

Finally, under some additional assumptions we can complete Theorem 1.3 and prove the following implications for the topology of the configuration space.

Theorem 1.4. *Assume the hypotheses of Theorem 1.3. Assume additionally that G is of type (E). If H is weakly integrable with integral map $F : T^*M \rightarrow \mathbb{R}^n$ and there is a C^1 Lagrangian graph $\Lambda \subset H^{-1}(h)$ and $\Lambda \cap \text{Reg } F \neq \emptyset$, then M is finitely covered by a compact reductive Lie group with a non-trivial centre.*

1.2. Methodological remarks. From a superficial perspective, Theorems 1.1 and 1.3 appear quite distinct. However, they are quite intimately related. In trying to weaken the Liouville-Arnol'd theorem, one must find a substitute for its involutivity hypothesis. Our substitute is to apply Mather theory and Fathi's weak KAM theory to systems with symmetry. It is natural to wonder if non-classical symmetries might also leave traces of their existence in the form of invariant Lagrangian graphs—complete solutions of the Hamilton-Jacobi equation. Theorem 1.3 shows that certain types of symmetry, that need not be associated with conserved quantities, do manifest themselves in this fashion.

2. Action-Minimizing Sets and Integrals of Motion

In the study of weakly integrable systems, or more generally of convex and superlinear Hamiltonian systems, the main idea behind dropping the hypothesis on the involution of the integrals of motion consists in studying the relationship between the existence of integrals of motion and the structure of some invariant sets obtained by *action-minimizing methods*, which are generally called *Mather*, *Aubry* and *Mañé sets*.

In this section we want to provide a brief description of this theory, originally developed by John Mather, and the main properties of these sets. We refer the reader to [15, 21, 23, 24, 32] for more exhaustive presentations of this material. Roughly speaking these action-minimizing sets represent a generalization of invariant Lagrangian graphs, in the sense that, although they are not necessarily submanifolds, nor even connected, they still enjoy many similar properties. What is crucial for our study of weakly integrable systems is that these sets have an intrinsic Lagrangian structure, which implies many of their symplectic properties, including a forced *local involution* of the integrals of motion, as noticed in [33].

More specifically, we are interested in studying the existence of *action-minimizing invariant probability measures* and *action-minimizing orbits* in the following setting.

Let $H : T^*M \rightarrow \mathbb{R}$ be a C^2 Hamiltonian, which is strictly convex and uniformly superlinear in the fibres. H is called a *Tonelli Hamiltonian*. This Hamiltonian defines a vector field on T^*M , known as the *Hamiltonian vector field*, that can be defined as the unique vector field X_H such that $\omega(X_H, \cdot) = dH$, where ω is the canonical symplectic form on T^*M . We call the associated flow the *Hamiltonian flow* and denote it by Φ_H^t .

To any Tonelli Hamiltonian system one can also associate an equivalent dynamical system in the tangent bundle TM , called the *Lagrangian system*. Let us consider the associated *Tonelli Lagrangian* $L : TM \rightarrow \mathbb{R}$, defined as $L(x, v) := \max_{p \in T_x^*M} (\langle p, v \rangle - H(x, p))$. It is possible to check that L is also strictly convex and uniformly superlinear in the fibres. In particular this Lagrangian defines a flow on TM , known as the *Euler-Lagrange flow* and denoted by Φ_L^t , which can be obtained by integrating the so-called *Euler-Lagrange equations*:

$$\frac{d}{dt} \frac{\partial L}{\partial v}(x, v) = \frac{\partial L}{\partial x}(x, v).$$

The Hamiltonian and Lagrangian flows are totally equivalent from a dynamical system point of view, in the sense that there exists a conjugation between the two. In other

words, there exists a diffeomorphism $\mathcal{L}_L : TM \longrightarrow T^*M$, called the *Legendre transform*, defined by $\mathcal{L}_L(x, v) = (x, \frac{\partial L}{\partial v}(x, v))$, such that $\Phi_H^t = \mathcal{L} \circ \Phi_L^t \circ \mathcal{L}^{-1}$.

In classical mechanics, a special role in the study of Hamiltonian dynamics is represented by invariant Lagrangian graphs, *i.e.* graphs of the form $\Lambda := \{(x, \eta(x)) : x \in M\}$ that are Lagrangian (*i.e.* $\omega|_\Lambda \equiv 0$) and invariant under the Hamiltonian flow Φ_H^t . Recall that being a Lagrangian graph in T^*M is equivalent to say that η is a closed 1-form ([10, Sect. 3.2]). These graphs satisfy many interesting properties, but unfortunately they are quite rare. The theory that we are going to describe aims to provide a generalisation of these graphs; namely, we shall construct several compact invariant subsets of the phase space, which are not necessarily submanifolds, but that are contained in Lipschitz Lagrangian graphs and enjoy similar interesting properties.

Let us start by recalling that the Euler-Lagrange flow Φ_L^t can be also characterised in a more variational way, introducing the so-called *Lagrangian action*. Given an absolutely continuous curve $\gamma : [a, b] \longrightarrow M$, we define its action as $A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt$. It is a classical result that a curve $\gamma : [a, b] \longrightarrow M$ is a solution of the Euler-Lagrange equations if and only if it is a critical point of A_L , restricted to the set of all curves connecting $\gamma(a)$ to $\gamma(b)$ in time $b - a$. However, in general, these extrema are not minima (except if their time-length $b - a$ is very small). Whence the idea of considering minimizing objects and seeing if - whenever they exist - they enjoy special properties or possess a more distinguished structure.

Mather's approach is indeed based on this idea and is concerned with the study of invariant probability measures and orbits that minimize the Lagrangian action (by action of a measure, we mean the collective average action of the orbits in its support, *i.e.* the integral of the Lagrangian against the measure). It is quite easy to prove (see [16, Lem. 3.1] and [32, Sect. 3]) that invariant probability measures (resp. Hamiltonian orbits) contained in an invariant Lagrangian graph Λ (actually its pull-back using \mathcal{L}) minimize the Lagrangian action of $L - \eta$, which we shall denote $A_{L-\eta}$, over the set $\mathfrak{M}(L)$ of all invariant probability measures for Φ_L^t (resp. over the set of all curves with the same end-points and defined for the same time interval). This idea of changing Lagrangian (which is at the same time a necessity) plays an important role as it allows one to magnify some motions rather than others. For instance, consider the case of an integrable system: one cannot expect to recover all these motions (which foliate the whole phase space) by just minimizing the same Lagrangian action! What is important to point out is that even if we modify L , because of the closedness of η we do not change the associated Euler-Lagrange flow, *i.e.* $L - \eta$ has the same Euler-Lagrange flow as L (see [23, p. 177] or [32, Lem. 4.6]). This is a crucial step in Mather's approach in [23]: consider a family of modified Tonelli Lagrangians given by $L_\eta(x, v) = L(x, v) - \langle \eta(x), v \rangle$, where η is a closed 1-form on M . These Lagrangians have the same Euler-Lagrange flow as L , but different action-minimizing orbits and measures. Moreover, these action-minimizing objects depend only on the cohomology class of η [23, Lem. p. 176].

Hence, for each $c \in H^1(M; \mathbb{R})$, if we choose η_c to be any smooth closed 1-form on M with cohomology class $[\eta_c] = c$, we can study action-minimizing invariant probability measures (or orbits) for $L_{\eta_c} := L - \eta_c$. In particular, this allows one to define several compact invariant subsets of TM :

- $\widetilde{\mathcal{M}}_c(L)$, the *Mather set of cohomology class c* , given by the union of the supports of all invariant probability measures that minimize the action of L_{η_c} (*c -action minimizing measure* or *Mather's measures of cohomology class c*). See [23].

- $\tilde{\mathcal{N}}_c(L)$, the *Mañé set of cohomology class c* , given by the union of all orbits that minimize the action of L_{η_c} on the finite time interval $[a, b]$, for any $a < b$. These orbits are called *c -global minimizers* or *c -semi static curves*. [21,23,24].
- $\tilde{\mathcal{A}}_c(L)$, the *Aubry set of cohomology class c* , given by the union of the so called *c -regular minimizers* of L_{η_c} (or *c -static curves*). These are a special kind of c -global minimizers that, roughly speaking, do not only minimize the Lagrangian action to go from the starting point to the end-point, but that - up to a change of sign - also minimize the action to go backwards, *i.e.* from the end-point to the starting one. A precise definition would require a longer discussion. Since we are not using this definition in the following, we refer the interested reader to [21,24,32].

Remark 2.1. i) These sets are non-empty, compact, invariant and moreover they satisfy the following inclusions:

$$\tilde{\mathcal{M}}_c(L) \subseteq \tilde{\mathcal{A}}_c(L) \subseteq \tilde{\mathcal{N}}_c(L) \subseteq TM.$$

- ii) The most important feature of the Mather set and the Aubry set is the so-called *graph property*, namely they are contained in Lipschitz graphs over M (*Mather's graph theorem* [23, Thm. 2]). More specifically, if $\pi : TM \rightarrow M$ denotes the canonical projection along the fibres, then $\pi|_{\tilde{\mathcal{A}}_c(L)}$ is injective and its inverse $(\pi|_{\tilde{\mathcal{A}}_c(L)})^{-1} : \pi(\tilde{\mathcal{A}}_c(L)) \rightarrow \tilde{\mathcal{A}}_c(L)$ is Lipschitz. The same is true for the Mather set (it follows from the above inclusion). Observe that in general the Mañé set does not necessarily satisfy the graph property.
- iii) As we have mentioned above, when there is an invariant Lagrangian graph Λ of cohomology class c (*i.e.* it is the graph of a closed 1-form of cohomology class c), then $\tilde{\mathcal{N}}_c(L) = \mathcal{L}^{-1}(\Lambda)$. A priori $\tilde{\mathcal{A}}_c(L) \subseteq \mathcal{L}_L^{-1}(\Lambda)$ and $\tilde{\mathcal{M}}_c(L) \subseteq \mathcal{L}_L^{-1}(\Lambda)$. In particular $\tilde{\mathcal{M}}_c(L) = \mathcal{L}_L^{-1}(\Lambda)$ if and only if the whole Lagrangian graph is the support of an invariant probability measure (*i.e.* the motion on it is recurrent).
- iv) Similarly to what happens for invariant Lagrangian graphs, the energy $E(x, v) = \langle \frac{\partial L}{\partial v}(x, v), v \rangle - L(x, v)$ (*i.e.* the pull-back of the Hamiltonian to TM using the Legendre transform) is constant on these sets, *i.e.* for any $c \in H^1(M; \mathbb{R})$ the corresponding sets lie in the same energy level $\alpha_H(c)$. Moreover, Carneiro [11] proved a characterization of this energy value in terms of the minimal Lagrangian action of $L - \eta_c$. More specifically:

$$\alpha_H(c) = - \min_{\mu \in \mathfrak{M}(L)} A_{L-\eta_c}(\mu).$$

This defines a function $\alpha_H : H^1(M; \mathbb{R}) \rightarrow \mathbb{R}$ that is generally called *Mather's α -function* or *effective Hamiltonian* (see also [23, p. 177]).

- v) It is possible to show that Mather's α -function is convex and superlinear [23, Thm. 1]. In particular, one can consider its convex conjugate, using Fenchel duality, which is a function on the dual space $(H^1(M; \mathbb{R}))^* \simeq H_1(M; \mathbb{R})$ and is given by:

$$\begin{aligned} \beta_H : H_1(M; \mathbb{R}) &\longrightarrow \mathbb{R} \\ h &\longmapsto \max_{c \in H^1(M; \mathbb{R})} (\langle c, h \rangle - \alpha_H(c)). \end{aligned}$$

This function is also convex and superlinear and is usually called *Mather's β -function*, or *effective Lagrangian*. It has also a meaning in terms of the minimal

Lagrangian action. In fact, one can interpret elements in $H_1(M; \mathbb{R})$ as rotation vectors of invariant probability measures [23, p. 177] (or ‘Schwartzman asymptotic cycles’ [30]). In particular $\beta_H(h)$ represents the minimal Lagrangian action of L over the set of all invariant probability measures with rotation vector h . Observe that in this case we do not need to modify the Lagrangian, since the constraint on the rotation vector will play somehow the role of the previous modification (it is in some sense the same idea as with Lagrange multipliers and constrained extrema of a function). We refer the reader to [23, 32] for a more detailed discussion on the relation between these two different kinds of action-minimizing processes.

Using the duality between Lagrangian and Hamiltonian, via the Legendre transform introduced above, one can define the analogue of the Mather, Aubry and Mañé sets in the cotangent bundle, simply considering

$$\mathcal{M}_c^*(H) = \mathcal{L}_L(\widetilde{\mathcal{M}}_c(L)), \quad \mathcal{A}_c^*(H) = \mathcal{L}_L(\widetilde{\mathcal{A}}_c(L)) \quad \text{and} \quad \mathcal{N}_c^*(H) = \mathcal{L}_L(\widetilde{\mathcal{N}}_c(L)).$$

These sets continue to satisfy the properties mentioned above, including the graph theorem. Moreover, it follows from Carneiro’s result [11], that they are contained in the energy level $\{H(x, p) = \alpha_H(c)\}$. However, one could try to define these objects directly in the cotangent bundle. For any cohomology class c , let us fix a representative η_c . Observe that if $\Lambda := \{(x, \eta(x)) : x \in M\}$ is an invariant Lagrangian graph of cohomology class c , i.e. $\eta = \eta_c + du$ for some $u : M \rightarrow \mathbb{R}$, then $H(x, \eta_c + du(x)) = \text{const}$. Therefore, the Lagrangian graph is a solution (and of course a subsolution) of the Hamilton-Jacobi equation $H(x, \eta_c + du(x)) = k$, for some $k \in \mathbb{R}$. In general solutions of this equation, in the classical sense, do not exist. However Albert Fathi proved that it is always possible to find *weak solutions*, in the *viscosity sense*, and use them to recover the above results. This theory, that can be considered as the analytic counterpart of the variational approach discussed above, is nowadays called *weak KAM theory*. We refer the reader to [15] for a more complete and precise presentation.

It turns out that for a given cohomology class c these weak solutions can exist only in a specific energy level, that - quite surprisingly - coincides with Mather’s value $\alpha_H(c)$. This is also the least energy value for which the Hamilton-Jacobi equation can have subsolutions:

$$H(x, \eta_c + du(x)) \leq k, \tag{1}$$

where $u \in C^1(M)$. Observe that the existence of C^1 -subsolutions corresponding to $k = \alpha_H(c)$ is a non-trivial result due to Fathi and Siconolfi [17]. Moreover they proved that these subsolutions are dense in the set of Lipschitz subsolutions. We shall call these subsolutions, η_c -critical subsolutions. Patrick Bernard [6] improved this result proving the existence and the denseness of $C^{1,1}$ η_c -critical subsolutions, which is the best result that one can generally expect to find. The main problem in fact is represented by the Aubry set itself, that plays the role of a *non-removable intersection* (see also [27]). More specifically, for any η_c -critical subsolution u , the value of $\eta_c + d_x u$ is prescribed on $\pi(\mathcal{A}_c^*(H))$, where $\pi : T^*M \rightarrow M$ is the canonical projection. Therefore, if the Aubry set is not sufficiently smooth (it is at least Lipschitz), then these subsolutions cannot be smoother. However, on the other hand this obstacle provides a new characterization of the Aubry set in terms of these subsolutions. Namely, if one denotes by \mathcal{S}_{η_c} the set of $C^{1,1}$ η_c -critical subsolutions, then:

$$\mathcal{A}_c^*(H) = \bigcap_{u \in \mathcal{S}_{\eta_c}} \{(x, \eta_c + d_x u) : x \in M\}. \tag{2}$$

As we have already recalled, in T^*M , with the standard symplectic form, there is a 1-1 correspondence between Lagrangian graphs and closed 1-forms (see for instance [10, Sect. 3.2]). Therefore, we could interpret the graphs of the differentials of these critical subsolutions as Lipschitz Lagrangian graphs in T^*M . Therefore the Aubry set can be seen as the intersection of these distinguished Lagrangian graphs and it is exactly this property that provides to this set the intrinsic Lagrangian structure mentioned above and that will play a crucial role in our proof.

In [33], in fact, Sorrentino used this characterization to study the relation between the existence of integrals of motion and the size of the above action-minimizing sets. Let H be a Tonelli Hamiltonian on T^*M and let F be an integral of motion of H . If we denote by Φ_H and Φ_F the respective flows, then:

Proposition 2.1 (See Lem. 2.2 in [33]). *The Mather set $\mathcal{M}_c^*(H)$ and the Aubry set $\mathcal{A}_c^*(H)$ are invariant under the action of Φ_F^t , for each $t \in \mathbb{R}$ and for each $c \in H^1(M; \mathbb{R})$.*

Moreover one can study the implications of the existence of *independent* integrals of motion, i.e. integrals of motion whose differentials are linearly independent, as vectors, at each point of these sets. It follows from the above proposition that this relates to the size of the Mather and Aubry sets of H . In order to make clear what we mean by the ‘size’ of these sets, let us introduce some notion of tangent space. We call *generalised tangent space* to $\mathcal{M}_c^*(H)$ (resp. $\mathcal{A}_c^*(H)$) at a point (x, p) , the set of all vectors that are tangent to curves in $\mathcal{M}_c^*(H)$ (resp. $\mathcal{A}_c^*(H)$) at (x, p) . We denote it by $T_{(x,p)}^G \mathcal{M}_c^*(H)$ (resp. $T_{(x,p)}^G \mathcal{A}_c^*(H)$) and define its *rank* to be the largest number of linearly independent vectors that it contains. Then:

Proposition 2.2 (See Prop. 2.4 in [33]). *Let H be a Tonelli Hamiltonian on T^*M and suppose that there exist k independent integrals of motion on $\mathcal{M}_c^*(H)$ (resp. $\mathcal{A}_c^*(H)$). Then, $\text{rank } T_{(x,p)}^G \mathcal{M}_c^*(H) \geq k$ (resp. $\text{rank } T_{(x,p)}^G \mathcal{A}_c^*(H) \geq k$) at all points $(x, p) \in \mathcal{M}_c^*(H)$ (resp. $(x, p) \in \mathcal{A}_c^*(H)$).*

Remark 2.2. In particular, the existence of the maximum possible number of integrals of motion (i.e. $k = n$) implies that these sets are invariant smooth Lagrangian graphs (see [33, Rem. 3.5] or [33, Lem. 3.4 and Lem. 3.6]). In particular, smoothness is a consequence of the fact that these graphs lie in level sets of the integral map, which is non-degenerate.

However the most important peculiarity of these action-minimizing sets observed in [33], at least as far as we are concerned, is that they force the integrals of motion to Poisson-commute on them. In fact, using the characterization of the Aubry set in terms of critical subsolutions of Hamilton-Jacobi and its symplectic interpretation given above (see (2) and the subsequent comment), one can recover the involution property of the integrals of motion, at least locally.

Proposition 2.3 (See Prop. 2.7 in [33]). *Let H be a Tonelli Hamiltonian on T^*M and let F_1 and F_2 be two integrals of motion. Then for each $c \in H^1(M; \mathbb{R})$ we have that $\{F_1, F_2\}(x, \hat{\pi}_c^{-1}(x)) = 0$ for all $x \in \overline{\text{Int}(\mathcal{A}_c(H))}$, where $\hat{\pi}_c = \pi|_{\mathcal{A}_c^*(H)}$ and $\mathcal{A}_c(H) = \pi(\mathcal{A}_c^*(H))$.*

Remark 2.3. Observe that the above set $\overline{\text{Int}(\mathcal{A}_c(H))}$ may be empty. What the proposition says is that whenever it is non-empty, the integrals of motion are forced to Poisson-commute on it. In the cases that we shall be considering hereafter, $\mathcal{A}_c(H) = M$ and therefore it is not empty.

3. Proof of Theorem 1.1

Proposition 3.1. *Let $\Lambda \subset H^{-1}(h)$ be a C^1 Lagrangian graph. If H is a weakly integrable Tonelli Hamiltonian and $\Lambda \subset \text{Reg } F$, then M admits the structure of a smooth \mathbb{T}^d -bundle over a parallelisable base B^{n-d} for some $d > 0$.*

Proof (Proposition 3.1). Since Λ is a C^1 Lagrangian graph that lies in an energy surface of H , Λ is the graph of a C^1 closed 1-form λ with cohomology class c . It follows that λ solves the Hamilton-Jacobi equation and from (2) that $\mathcal{A}_c^*(H) \subseteq \Lambda$ (see also [33, Sect. 3]). Moreover, Proposition 2.2 and Remark 2.2 allow us to conclude that $\mathcal{A}_c^*(H) = \Lambda$. Therefore, Proposition 2.1 implies that each vector field X_{F_i} , $i = 1, \dots, n$ is tangent to Λ . Let $Y = X_H|_\Lambda$ and $Y_i = X_{F_i}|_\Lambda$. Since $\Lambda \subset \text{Reg } F$, $\{Y_i\}$ is a framing of $T\Lambda$.

Let ϕ^i (resp. ϕ) be the flow of Y_i (resp. Y). Let Γ be the group of diffeomorphisms generated by the flows ϕ^i and ϕ . The Stefan-Sussman orbit theorem implies that Λ is the orbit of Γ : $\Lambda = \left\{ \prod_{j=1}^m \phi_{t_j}^{i_j}(p) : t_j \in \mathbb{R}, m \in \mathbb{N} \right\}$ for any $p \in \Lambda$ [34–36]. Since H Poisson-commutes with each of the F_i , the vector field Y commutes with Y_i for all i . Therefore, the flow ϕ of Y commutes with each ϕ^i , i.e. ϕ lies in the centre Z of Γ .

Let $p \in \Lambda$ be a given point and $q \in \Lambda$ a second point. Let $\Phi = \prod_{j=1}^m \phi_{t_j}^{i_j}$ be an element in Γ satisfying $\Phi(p) = q$. If φ_t is a 1-parameter subgroup of Z , then $\varphi_t(q) = \Phi(\varphi_t(p))$ for all $t \in \mathbb{R}$. Therefore, each orbit of φ is conjugate by a smooth conjugacy isotopic to the identity. We have seen that $\phi_t \in Z$ for all t , and the above shows that each orbit of ϕ_t (indeed, of Z) is conjugate.

Define a smooth Riemannian metric g on Λ by defining $\{Y_i\}$ to be an orthonormal framing of $T\Lambda$. Then, we see that each element in Z preserves g . Therefore Z is a group of isometries of a compact Riemannian manifold. The closure of Z in the group of C^1 diffeomorphisms of Λ , \bar{Z} , is therefore a compact connected abelian Lie group by the Montgomery-Zippin theorem [26]. Therefore, \bar{Z} is a d -dimensional torus for some $d > 0$ (since it contains the 1-parameter group ϕ_t).

Since Z centralises Γ , so does its closure \bar{Z} . Therefore, each orbit of \bar{Z} is conjugate. It follows that \bar{Z} acts freely on Λ . This gives Λ the structure of a principal \mathbb{T}^d -bundle.

Finally, let $p \in \Lambda$ be given. Possibly after a linear change of basis, we can suppose that Y_i , $i = 1, \dots, d$, is a basis of the tangent space to the \mathbb{T}^d -orbit through p , and Y_i , $i = d+1, \dots, n$ is a basis of the orthogonal complement. Therefore, Y_i , $i = d+1, \dots, n$ is a basis of the orthogonal complement to the fibre at all points on Λ . Since each vector field Y_i is \mathbb{T}^d -invariant, it descends to $B = \Lambda/\mathbb{T}^d$. Therefore, the vector fields on B induced by Y_i , $i = d+1, \dots, n$, frame TB . \square

Remark 3.1. A few remarks are in order. First, there is a $\xi \in \mathfrak{t} = \text{Lie } \mathbb{T}^d$ such that $\exp(t\xi) \cdot p = \phi_t(p)$ for all $t \in \mathbb{R}$ and $p \in \Lambda$. This follows from the fact that $\{\phi_t\} \subset \bar{Z}$ is a 1-parameter subgroup. Therefore, there is a torus T of dimension $c \leq d$ which is the closure of $\{\exp(t\xi)\}$ in \mathbb{T}^d such that each orbit closure of ϕ is the orbit of T . Second, for almost all constants $(\alpha_i) \in \mathbb{R}^d$, the vector field $Y_\alpha = Y + \sum_{i=1}^d \alpha_i Y_i$ will have dense orbits in each \mathbb{T}^d orbit. Third, since each orbit of ϕ is conjugate by a diffeomorphism isotopic to 1, the asymptotic homology of Λ is unique (see [16, Prop. A.1]). Finally, if, as in Theorem 1.1, one has an upper semicontinuous family of such Lagrangian graphs $\Lambda_{c'}$, then the dimension d' of the torus is an upper semicontinuous function of c' .

Proof (Theorem 1.1). Since \mathcal{A}_c^* is contained in the set of regular points of F , it follows from Proposition 2.2 and Remark 2.2 that the Aubry set \mathcal{A}_c^* is a C^1 invariant Lagrangian

graph Λ_c of cohomology class c and that it coincides with the Mather set \mathcal{M}_c^* (see also [33, Lems. 3.4 & 3.6]). Therefore, Λ_c supports an invariant probability measure of full support. In particular, since all c -critical subsolutions of the Hamilton-Jacobi equation (1), with $k = \alpha_H(c)$, have the same differential on the (projected) Aubry set [15, Thm. 4.11.5], it follows that, up to constants, there exists a unique c -critical subsolution, which is indeed a solution. It follows then that the Mañé set $\mathcal{N}_c^* = \mathcal{A}_c^*$ (see [15, Def. 5.2.5]). We can use the upper semicontinuity of the Mañé set (see for instance [2, Prop. 13]) to deduce that the Mañé set corresponding to nearby cohomology classes must also lie in $\text{Reg } F$ (note in fact that in general the Aubry set is not upper semicontinuous [7]). Hence, there exists an open neighborhood \mathcal{O} of c in $H^1(M; \mathbb{R})$ such that $\mathcal{A}_{c'}^* \subseteq \mathcal{N}_{c'}^* \subset \text{Reg } F$ for all $c' \in \mathcal{O}$ and applying the same argument as above, we can conclude that each $\mathcal{A}_{c'}^*$ is a smooth invariant Lagrangian graph of cohomology class c' and that it coincides with the Mather set $\mathcal{M}_{c'}^*$.

At this point (i) and (ii) follow from Proposition 3.1 and Remark 3.1.

The proof of (iii) is the same as in [33, Cor. 3.8], but in this case we also know that these graphs are Schwartzman uniquely ergodic, *i.e.* all invariant probability measures on $\Lambda_{c'}$ have the same rotation vector $h_{c'} \in H_1(M; \mathbb{R})$ (see Remark 3.1). The differentiability of α_H follows then from [16, Cor. 3.6]. The differentiability of β_H follows the disjointness of these graphs (see for instance [16, Thm. 3.3] or [32, Rem. 4.26 (ii)]). \square

Proof (Theorem 1.2). Let d be the largest dimension of the torus fibre of Λ_c for $c \in \mathcal{O}$. The upper semicontinuity of this dimension implies that there is an open set on which the dimension of the fibre equals d ; without loss of generality, it can be supposed that this open set is \mathcal{O} . By (iii) of Theorem 1.1, Mather's α -function is differentiable on \mathcal{O} . Since α_H is a locally Lipschitz function, it is continuously differentiable on \mathcal{O} . Therefore, the map

$$c \longmapsto h = \partial\alpha_H(c), \quad \mathcal{O} \xrightarrow{\partial\alpha_H} H_1(M; \mathbb{R})$$

is continuous and one-to-one (by [16, Thm. 3.3]) and hence a homeomorphism onto its image.

Let $b_1(M) = \dim H_1(M; \mathbb{R})$ be the first Betti number of M . Since the rotation vector of $Y = X_H|_{\Lambda_c}$ is the image of a cycle in $H_1(\mathbb{T}^d; \mathbb{R})$, $d \geq b_1(M)$. To prove that $d = b_1(M)$, we need a few lemmata. (We follow the notation of the proof of Theorem 1.1.)

Lemma 3.1. $H_1(B; \mathbb{R}) = \{0\}$.

Proof. Let $c \in \mathcal{O}$. The rotation vector of Y projects to 0 in $H_1(B; \mathbb{R})$, since Y is tangent to the \mathbb{T}^d fibres of Λ_c . But the projection map $\pi_c : \Lambda_c \rightarrow B$ is surjective on H_1 and $\pi_{c,*}(\partial\alpha_H(\mathcal{O}))$ is open since $\pi_{c,*}$ is an open map. These facts imply $H_1(B; \mathbb{R})$ is trivial. \square

Lemma 3.2. For all $f \in F^*C^\infty(\mathbb{R}^n)$ and $c \in \mathcal{O}$ the rotation set of $Y_f = X_f|_{\Lambda_c}$ contains a unique point.

Proof. The proof of Theorem 1.1 shows that there is a \mathbb{T}^d -connection on Λ_c that permits one to decompose Y_f into a vertical component Y_v and a horizontal component Y_h . Because Λ_c is a level set of F , Y_v (resp. Y_h) is a linear combination of the basis $Y_i, i = 1, \dots, d$ (resp. $Y_i, i = d + 1, \dots, n$) with constant coefficients. Therefore, Y_v

and Y_h commute and the flow of Y_f is a product of their commuting flows: $\phi_t^f = \phi_t^v \circ \phi_t^h$ (and ϕ_t^v lies in the centre of Γ). Since $H_1(B; \mathbb{R}) = \{0\}$, the rotation set of Y_h is trivial, so the rotation set of Y_f equals that of Y_v , which is a singleton from the proof of Proposition 3.1. \square

Suppose now that $d > b_1(M)$. Let $G = H + F^*\alpha \in F^*C^\infty(\mathbb{R}^n)$ be a Tonelli Hamiltonian such that for a residual set of $c \in \mathcal{O}$, the vertical component of $Y_G = X_G|_{\Lambda_c}$ generates a dense 1-parameter subgroup of the torus fibre. It is straightforward to see that such G exist. Lemma 3.2 implies that $\alpha_G|_{\mathcal{O}}$ is differentiable and therefore $\partial\alpha_G|_{\mathcal{O}}$ is a homeomorphism onto its image. If $d > b_1(M)$, then there are distinct $c, c' \in \mathcal{O}$ such that the rotation vectors $\rho(\Lambda_c) = \rho(\Lambda_{c'})$, which contradicts the injectivity of $\partial\alpha_G$. Therefore, $d = b_1(M)$.

Let $\kappa : \check{M} \rightarrow M$ be a finite covering. It is claimed that $b_1(\check{M}) = b_1(M)$.

$$\begin{array}{ccccc}
 & & \check{\eta}_c = \kappa^* \eta_c & & \\
 & & \curvearrowright & & \\
 \check{\Lambda}_c & \xrightarrow{\quad} & T^*\check{M} & \twoheadrightarrow & \check{M} \\
 \mathcal{K}|_{\check{\Lambda}_c} \downarrow & & \downarrow \mathcal{K} & & \downarrow \kappa \\
 \Lambda_c & \xrightarrow{\quad} & T^*M & \twoheadrightarrow & M \\
 & & \curvearrowleft \eta_c & &
 \end{array} \tag{3}$$

Since the cotangent lift of κ , \mathcal{K} , is a local symplectomorphism, the Tonelli Hamiltonian $\check{H} = \mathcal{K}^*H$ is weakly integrable with the first-integral map $\check{F} = \mathcal{K}^*F$. Let $c \in \mathcal{O}$ be a cohomology class and η_c a solution to the Hamilton-Jacobi equation for H whose graph Λ_c equals the Mather set \mathcal{M}_c^* (diagram (3)). The pullback $\check{\eta}_c = \kappa^*\eta_c$ solves the Hamilton-Jacobi equation for \check{H} and its graph $\check{\Lambda}_c$ is an invariant C^1 Lagrangian graph. By Proposition 3.1, there is a $\check{d} > 0$ such that $\check{\Lambda}_c$ admits the structure of a principal $\mathbb{T}^{\check{d}}$ -bundle. This torus action is defined by \check{d} commuting vector fields $\check{Y}_i = X_{\check{F}_i}|_{\check{\Lambda}_c}$, $i = 1, \dots, \check{d}$ induced by the first-integral map \check{F} . Since \mathcal{K} is a local symplectomorphism, $\mathcal{K}|_{\Lambda_c}$ is a local diffeomorphism. This shows that the dimension \check{d} equals d . By the previous paragraph, weak integrability implies that $\check{d} = b_1(\check{M})$ so $b_1(\check{M}) = b_1(M)$.

Let us prove that M is a trivial principal \mathbb{T}^d -bundle. This argument is indebted to that of Sepe [31]. A principal \mathbb{T}^d -bundle is classified up to isomorphism by a classifying map

$$\begin{array}{ccccc}
 & & \mathbb{T}^d & & \\
 & \swarrow & \downarrow & \searrow & \\
 M = f^*E\mathbb{T}^d & \longrightarrow & E\mathbb{T}^d & \equiv & \prod_{i=1}^d S^\infty \\
 \pi_f \downarrow & & \downarrow \pi & & \downarrow \text{Hopf fib.} \\
 B & \xrightarrow{\quad f \quad} & B\mathbb{T}^d & \equiv & \prod_{i=1}^d \mathbb{C}P^\infty.
 \end{array} \tag{4}$$

The classifying map f is null homotopic if and only if the pullback bundle is trivial. Classical obstruction theory shows that the single obstruction to a null homotopy of f is a cohomology class – the *Chern class* – with the following description. The trivial

section $*$ $\mapsto * \times 0$ of $E\mathbb{T}^d$ restricted to its 0-skeleton extends over the 1-skeleton. The obstruction to extending this section over the 2-skeleton defines a cohomology class $\eta \in H^2(B\mathbb{T}^d; \pi_1(\mathbb{T}^d)) = H^2(B\mathbb{T}^d; H_1(\mathbb{T}^d))$. By naturality, the obstruction to extending the trivial section of $f^*E\mathbb{T}^d$ over the 2-skeleton is the cohomology class $\eta_f = f^*\eta \in H^2(B; H_1(\mathbb{T}^d))$ – called the *Chern class*.

In terms of the E_2 page of the Leray-Serre spectral sequence with \mathbb{Z} -coefficients for the bundle $\mathbb{T}^d \hookrightarrow M \rightarrow B$, one has the differential $d_2^{0,1} : E_2^{0,1} = H^1(\mathbb{T}^d) \rightarrow E_2^{2,0} = H^2(B)$. It has been shown above that the inclusion map $\mathbb{T}^d \hookrightarrow M$ is injective on H_1 , hence surjective on H^1 . Since a class in $E_2^{0,1}$ survives to a class in E_∞ if and only if it is in the kernel of $d_2^{0,1}$, the differential $d_2^{0,1}$ must therefore vanish. Since the differential $d_2^{2,0}$ vanishes, it follows that $H^2(B)$ survives to E_∞ .

On the other hand, for any cohomology class $\phi \in H^1(\mathbb{T}^d)$, the class $\eta \cup \phi = \langle \eta, \phi \rangle$ is a class in $H^2(B\mathbb{T}^d)$ which satisfies $\pi^*(\eta \cup \phi) = 0$ in $H^2(E\mathbb{T}^d)$. By naturality, the class $\eta_f \cup \phi \in H^2(B)$. This class, if non-zero, survives to E_∞ . On the other hand, $\pi_f^*(\eta_f \cup \phi) = 0$ in $H^2(M)$. This shows that $\eta_f \cup \phi = 0$ in $H^2(B)$. Since the class ϕ was arbitrary, it follows that η_f vanishes. Therefore $M = f^*E\mathbb{T}^d$ is a trivial principal \mathbb{T}^d -bundle.

Let us now prove (i–ii).

When $\dim M = 2$, it follows from (i) in Theorem 1.1 that M is orientable and has genus 0, therefore it must be \mathbb{T}^2 .

When $\dim M = 3$, one cannot have $d < 3$, since there are no parallelisable $(3 - d)$ -dimensional manifolds with trivial first Betti number. Therefore, $d = 3$ and $M = \mathbb{T}^3$.

When $\dim M = 4$, there are two options: $\dim B \leq 2$ or $\dim B = 3$. When $\dim B \leq 2$, B has the homotopy type of a point, hence it is a point, so $M = \mathbb{T}^4$. Assume that $\dim B = 3$. If $\pi_1(B)$ is a free product of irreducible finitely-presented groups G_i ($i = 0, \dots, g$), then Kneser's theorem [19] implies that $B = B_0 \# \dots \# B_g$, where B_i is a closed 3-manifold with $\pi_1(B_i) = G_i$. Since $H_1(B) = \bigoplus_i H_1(B_i)$, each homology group $H_1(B_i)$ is finite. According to [29, Prop. 2.1], if $H_1(B)$ is finite and $\pi_1(B_i)$ is not perfect for some i , then the universal abelian covering \hat{B} , or a 2-fold cover thereof, is a finite cover of B which has first Betti number at least 1. Thus, the only case to be resolved is that when $\pi_1(B_i)$ is perfect for all $i = 0, \dots, g$. By [29, Rem. at bottom of p. 570], Stallings' theorem implies that $G_i = [G_i, G_i]$ is isomorphic to π_1 of the Klein bottle – which is absurd. This proves that B is an irreducible 3-manifold. If $\pi_1(B)$ is infinite, then the virtual Haken conjecture implies that B has a finite covering with non-zero first Betti number. Therefore, $\pi_1(B)$ is finite and so by the proof of the Poincaré conjecture, B is finitely covered by S^3 .

Let us prove (iii). Let us denote $\Lambda_{c'} = \{(x, \lambda_{c'}(x)) : x \in M\}$ as usual. Observe that the map:

$$\begin{aligned} \Psi : \mathcal{O} \times M &\longrightarrow T^*M \\ (c', x) &\longmapsto \lambda_{c'}(x) \end{aligned}$$

is continuous. It is sufficient to show that if $c_n \rightarrow c'$ in \mathcal{O} , then λ_{c_n} converge uniformly to $\lambda_{c'}$. In fact, the sequence $\{\lambda_{c_n}\}_n$ is equicontinuous (it follows from Mather's graph theorem [23, Thm. 2]) and equibounded, therefore applying the Ascoli-Arzelà theorem we can conclude that – up to selecting a subsequence – λ_{c_n} converge uniformly to $\tilde{\lambda} = \eta_{c'} + du$, for some $u \in C^1(M)$. Observe that since $H(x, \lambda_{c_n}(x)) = \alpha_H(c_n)$ for all $x \in M$ and all n , and α_H is continuous, then $H(x, \tilde{\lambda}(x)) = \alpha_H(c')$ for all x . Therefore, u is a solution

of the Hamilton-Jacobi equation $H(x, \eta_{c'} + du) = \alpha_H(c')$. As we have observed in the beginning of this proof, for each $c' \in \mathcal{O}$ there is a unique solution of this equation, hence $\tilde{\lambda} = \lambda_{c'}$. This concludes the proof of the continuity of Ψ . Notice that this could be also deduced from the fact that Ψ is injective and semicontinuous.

If $\dim H^1(M; \mathbb{R}) \geq \dim M$, then the continuity of Ψ implies that these Lagrangian graphs $\Lambda_{c'}$ foliate an open neighborhood of Λ_c . It follows from Proposition 2.3 that the components of F commute in this open region. Therefore, each $\Lambda_{c'}$ is an n -dimensional manifold which is invariant under the action of n commuting vector fields, which are linearly independent at each point. It is a classical result that $\Lambda_{c'}$ is then diffeomorphic to an n -dimensional torus and that the motion on it is conjugate to a rotation (see for instance [3]). \square

4. Amenable Groups, Measures and Rotation Vectors

In this section it is assumed that X is a compact, path-connected, locally simply-connected metrizable space and (G, m_G) is a locally compact, simply-connected, metrizable, amenable topological group with Haar measure m_G . We will use d to denote a metric on both spaces; it will be assumed that the metric on G is *right*-invariant, without loss of generality. The space of m_G -essentially bounded measurable functions on G is denoted by $L^\infty(G)$. $L^\infty(G)^*$ has a distinguished subspace of functionals invariant under G 's left (resp. right) action; this subspace will be denoted by $L^\infty(G)_{G_-}^*$ (resp. $L^\infty(G)_{G_+}^*$). A functional $\nu \in L^\infty(G)^*$ which satisfies $\nu(1) = 1$ is called a *mean*. The set of left-invariant (resp. right-invariant) means is denoted by $\mathfrak{m}(G)^{G_-}$ (resp. $\mathfrak{m}(G)^{G_+}$); amenability of G implies that both $\mathfrak{m}(G)^{G_\pm}$ is non-empty, as is the intersection $\mathfrak{m}(G)$.

Let $\hat{\pi} : \hat{X} \rightarrow X$ be the universal abelian covering space of X , *i.e.* the regular covering space whose fundamental group is $[\pi_1 X, \pi_1 X]$ and on which $H_1(X; \mathbb{Z})$ (singular homology) acts as the group of deck transformations of $\hat{\pi}$.

Let $\phi : G \rightarrow X$ be a uniformly continuous map (it is not assumed that there is an action of G on X). The simple-connectedness of G implies that there is a lift $\hat{\phi}$ of ϕ to \hat{X} . It is well-known that the first singular cohomology group of X is naturally isomorphic to the group of homotopy classes of maps from X to S^1 , denoted by $[X, S^1]$. For each $f \in [X, S^1]$, let us construct the following commutative diagram:

$$\begin{array}{ccc}
 & \hat{X} & \xrightarrow{\hat{f}} \mathbb{R} \\
 \hat{\phi} \nearrow & \downarrow \hat{\pi} & \searrow \hat{f} \\
 G & \xrightarrow{\phi} X & \xrightarrow{f} S^1 \\
 & \searrow g & \downarrow p
 \end{array} \tag{5}$$

where $p(x) = x \bmod 1$ and \hat{f} is a lift of f to \hat{X} — the dotted diagonal line exists if and only if f is null-homotopic. Define the map

$$G \times G \xrightarrow{\zeta} \mathbb{R}^1 \qquad (s, t) \longmapsto g(st) - g(t). \tag{6}$$

A priori, ζ is a map into S^1 , but the simple-connectedness of G implies there is a unique lift of the map in (6) that is identically zero when $s = 1$ (the lift is trivially $\hat{g}(st) - \hat{g}(t)$). For a fixed $s \in G$, let $\zeta_s(t) = \zeta(s, t)$.

Lemma 4.1. *For each $s \in G$, $\zeta_s \in L^\infty(G)$.*

Proof. Since X is compact, f is uniformly continuous. Since ϕ is assumed to be uniformly continuous, g and therefore \hat{g} is uniformly continuous. Therefore, there is a $\delta > 0$ such that if $a, b \in G$ and $d(a, b) < \delta$ then $|\hat{g}(a) - \hat{g}(b)| < 1$. Let N be an integer exceeding $d(s, 1)/\delta$. Then the right-invariance of the metric d implies that for all $t \in G$, $d(st, t) = d(s, 1) < N\delta$, so by the triangle inequality, one concludes $|\hat{g}(st) - \hat{g}(t)| < N$. Thus $|\zeta_s(t)| < N$ for all $t \in G$. \square

Lemma 4.2. *Let $\nu \in m(G)^{G-}$ be a left-invariant mean on G . If $\hat{g} \in L^\infty(G)$, then $\langle \nu, \zeta_s \rangle = 0$ for all $s \in G$. In particular, if*

- (1) *f is null-homotopic; or*
- (2) *$\text{Im } \hat{\phi}$ is contained in a compact set,*

then $\langle \nu, \zeta_s \rangle$ vanishes for all $s \in G$.

Proof. If $\hat{g} \in L^\infty(G)$, then $\langle \nu, \zeta_s \rangle = \langle s_*\nu, \hat{g} \rangle - \langle \nu, \hat{g} \rangle = 0$ by left-invariance of ν . If f is null-homotopic, then the image of \hat{f} is a compact subset of \mathbb{R} , so $\hat{g} \in L^\infty(G)$; likewise, if $\text{Im } \hat{\phi}$ has compact closure. \square

Lemma 4.3. *Let $\phi, \phi' : G \rightarrow X$ be uniformly continuous maps. If there is a $K > 0$ such that their lifts $\hat{\phi}, \hat{\phi}' : G \rightarrow \hat{X}$ satisfy $d(\hat{\phi}(s), \hat{\phi}'(s)) < K$ for all $s \in G$, then $\langle \nu, \zeta_s - \zeta'_s \rangle$ vanishes for all $s \in G$ and $\nu \in m(G)^{G-}$.*

Proof. The proof of this lemma mirrors the preceding. By the assumption that $d(\hat{\phi}(t), \hat{\phi}'(t)) < K$ for all $t \in G$, one has that $\hat{h}(t) := \hat{f}\hat{\phi}(t) - \hat{f}\hat{\phi}'(t)$ lies in $L^\infty(G)$. Therefore, $\langle \nu, \zeta_s - \zeta'_s \rangle = \langle s_*\nu, \hat{h} \rangle - \langle \nu, \hat{h} \rangle = 0$ by left-invariance of the mean ν . \square

Lemma 4.4. *Let $\nu \in m(G)^{G-}$ be a left-invariant mean and $\phi : G \rightarrow X$ a uniformly continuous map. For each $s \in G$, the map*

$$f \mapsto \langle \nu, \zeta_s \rangle \tag{7}$$

(see (6)) induces a linear function $\rho_s(\nu) : H^1(X; \mathbb{R}) \rightarrow \mathbb{R}$. The function $\rho_s : m(G)^{G-} \rightarrow H_1(X; \mathbb{R})$ is affine and continuous in the weak- topology on $L^\infty(G)^{**}$.*

Proof. It suffices to show that this map is additive on $H^1(X; \mathbb{Z}) = [X, S^1]$, since it is extended by multiplicativity to a map on $H^1(X; \mathbb{R})$. First, let us show the map is well-defined on homotopy classes. Let f, f' be representatives of the homotopy class $[f]$. By compactness of $X \times [0, 1]$, there is an $N > 0$ such that $|\hat{f}(x) - \hat{f}'(x)| < N$ for all $x \in \hat{X}$. Therefore, $|\hat{g}(st) - \hat{g}'(st)| < N$ and $|\hat{g}(t) - \hat{g}'(t)| < N$ for all $t \in G$ (using the obvious notation), so both $s^*(\hat{g} - \hat{g}')$ and $\hat{g} - \hat{g}'$ are in $L^\infty(G)$. Thus, $\langle \nu, \zeta_s - \zeta'_s \rangle = \langle s_*\nu, \hat{g} - \hat{g}' \rangle - \langle \nu, \hat{g} - \hat{g}' \rangle = 0$ by left-invariance of ν . This proves the map (7) is well-defined on $[X, S^1]$.

To prove that the map (7) is additive, let $f, h : X \rightarrow S^1$ be representatives of the homotopy classes $[f], [h]$. The homotopy class $[f] + [h]$ is represented by $[f + h]$. From the diagram (5), it is clear that $\zeta^{f+h} = \zeta^f + \zeta^h$, where ζ^\bullet denotes ζ constructed with \bullet . This suffices to prove additivity, and that suffices to show that $\rho_s(\nu)$ is a linear form on $H^1(X; \mathbb{R})$.

Since the pairing defining $\rho_s(\nu)$ is the bilinear pairing between $L^\infty(G)^*$ and $L^\infty(G)$, it follows that ρ_s is an affine map that is continuous in the weak-* topology on linear maps $\text{Hom}(L^\infty(G)^*; H^1(X; \mathbb{R})^*)$. \square

Definition 4.1. Let $s \in G$. The set

$$R_s = \rho_s \left(\mathfrak{m}(G)^{G^-} \right) \tag{8}$$

is the rotation set of the left translation s .

Theorem 4.1. The map $\rho : G \rightarrow \text{Hom}(\mathfrak{m}(G)^{G^-}; H_1(X; \mathbb{R}))$ is continuous. For each $s \in G$, the rotation set R_s is a compact, convex subset of $H_1(X; \mathbb{R})$. The rotation-set map

$$s \mapsto R_s \tag{9}$$

is an upper semi-continuous set function.

Proof. If $s_n \rightarrow s$ in G , then for a fixed $f : X \rightarrow S^1$, one sees that $\zeta_{s_n} \rightarrow \zeta_s$ in $L^\infty(G) \cap C^0(G; \mathbb{R})$. Therefore, for any $v \in \mathfrak{m}(G)^{G^-}$, $\langle \rho_{s_n}(v), [f] \rangle \rightarrow \langle \rho_s(v), [f] \rangle$. This proves ρ is continuous in the weak-* topology.

Clearly $\mathfrak{m}(G)^{G^-}$ is convex. Since $\mathfrak{m}(G)^{G^-} \subset L^\infty(G)^*$ is a closed subset of the unit ball in $L^\infty(G)^*$, it is a compact set in the weak-* topology. Since ρ_s is continuous and affine, its image is compact and convex. \square

4.1. Examples. Let us compute some rotation sets.

4.1.1. Translations on tori. Let $X = \mathbb{T}^n$ and let $G = \mathbb{R}^n = \tilde{X}$ be the universal covering group acting in the tautological manner; the map ϕ is the orbit map of $\theta_0 \in \mathbb{T}^n$. A cohomology class $f \in [X, S^1]$ has a canonical representative, viz. $f(\theta) = \langle v, \theta \rangle \text{ mod } 1$, where $v \in \text{Hom}(\mathbb{Z}^n; \mathbb{Z})$. One arrives at the map $\tilde{g}(t) = \langle v, t + \theta_0 \rangle$ and $\zeta_s(t) = \langle v, s \rangle$ – which is independent of $t \in G$ – whence the mean of ζ_s equals $\langle v, s \rangle$ for any mean $v \in \mathfrak{m}(G)^G$. If one employs the tautological isomorphism between the real homology (resp. cohomology) group of \mathbb{T}^n and \mathbb{R}^n (resp. $\text{Hom}(\mathbb{R}^n; \mathbb{R})$), one obtains

$$\rho_s(v) = s$$

for all $s \in G$, $v \in \mathfrak{m}(G)^{G^-}$.

We note that this calculation computes the rotation vector/set of a subgroup, given a mean on the whole group. Lemma 4.6 below shows that there is no loss of generality.

4.1.2. Translations on quotients of contractible amenable Lie groups of type (E). Let G be a contractible, amenable Lie group of type (E) (hence a solvable Lie group of type (E)), $\Gamma \triangleleft G$ be a co-compact subgroup and $X = \Gamma \backslash G$. Let $g, g' \in G$ and let $\phi : G \rightarrow X$ be the map $\phi(t) = \Gamma g t^{-1} g'$. Let N be the commutator subgroup of G ; it is known that $\Gamma \cap N$ is a lattice in N , that the commutator subgroup of Γ is of finite index in $\Gamma \cap N$ and therefore ΓN is a closed subgroup of G [12, Lem. 3]. The map $F : X \rightarrow N \backslash X$ is therefore a submersion onto a torus whose dimension is the codimension of N in G . From the fact that the derived subgroup of Γ is of finite index in $\Gamma \cap N$, one sees that $[X, S^1] = F^*[N \backslash X, S^1]$.

Therefore, we have reduced the problem to the case of a translation on a torus, whence $\rho_s(v) = -Ns$ in the simply-connected abelian Lie group $N \backslash G$, $v \in \mathfrak{m}(G)^{G^-}$.

4.1.3. *Translations on quotients of amenable Lie groups of type (E).* The situation with simply-connected amenable Lie groups of type (E) is somewhat more complicated than the previous example, as exemplified by [12, Exs. 1 & 2]. These examples show how the first Bieberbach theorem may fail, but in these examples the Levi decomposition is trivial: the groups themselves are solvable and one might be led to believe that this is the only way that such pathological examples can arise.

Example. Let us give an example where the Levi decomposition is non-trivial and the first Bieberbach theorem fails. That is, let us give an example where $G = SK$ is a simply-connected amenable Lie group of type (E) where S is its solvable radical and K is a maximal compact subgroup, and $\Gamma < G$ is a lattice subgroup such that $\Gamma \cap S$ is not a lattice subgroup of S .

Let $k > 2$ be integers and let N be the nilpotent Lie group whose multiplication is defined by

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 \otimes y_2 - x_2 \otimes y_1))$$

where $x_i, y_i \in \mathbb{R}^k, z_i \in \mathbb{R}^k \otimes \mathbb{R}^k$. (10)

The cyclic group generated by

$$a = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
(11)

acts as a group of automorphisms of N , and this group is a discrete subgroup of a 1-parameter group of automorphisms A . Let $S = NA$, a solvable group of type (E). On the other hand, let K be the universal covering group of $SO_k \times SO_k$ (since $k > 2$, K is compact) and let K act on N via

$$\kappa \cdot g = (u \cdot x, v \cdot y, (u \otimes v) \cdot z)$$

where $g = (x, y, z) \in N, \kappa \in K \mapsto (u, v) \in SO_k \times SO_k$. (12)

This is an action by automorphisms of N and this action commutes with the action of A , so this action induces a natural action of K on S . This suffices to describe the group $G = SK$, an amenable Lie group of type (E).

The lattice subgroup Γ is described as follows. Let

$$N_{\mathbb{Z}} = \left\{ g = (x, y, z) \in N : x, y \in \mathbb{Z}^k, 2z \in \mathbb{Z}^k \otimes \mathbb{Z}^k \right\},$$

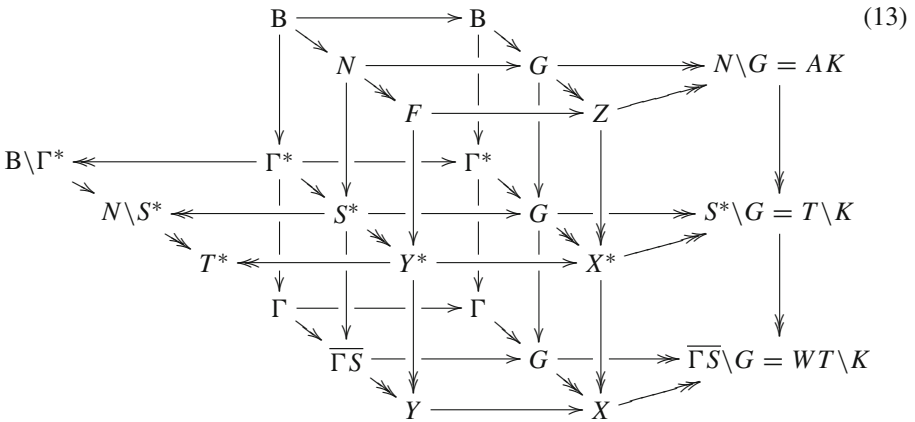
and observe that a preserves $N_{\mathbb{Z}}$. Let $b \in K$ and let $\gamma = ab$. The group Γ generated by γ and $N_{\mathbb{Z}}$ is discrete and co-compact in G for any choice of b . If b is of infinite order, then the intersection of Γ with S is just $N_{\mathbb{Z}}$ and is *not* a lattice in S . The projection of Γ to $K = S \backslash G$ is the group generated by b ; if b is chosen in general position, then the identity component of the closure is a maximal torus.

This example shows how the first Bieberbach theorem can fail for type (E) amenable Lie groups. However, the representation of K as a group of automorphisms of S is almost faithful, and this implies many of the nice properties mentioned in the previous paragraph. On the other hand, if one takes the amenable Lie group $G = \mathbb{C}^n \times SU_n$ with the lattice subgroup Γ generated by the set $\{(e_j, \rho_j), (ie_j, \rho_j) : j = 1, \dots, n\}$, where each ρ_j is a generic element in the maximal torus of diagonal matrices, then one sees that the intersection of Γ with S is trivial and the projection of Γ onto SU_n is dense in the maximal torus.

Let $G = SK$ be a simply-connected, amenable Lie group where S is its radical and K its maximal compact subgroup, and let $\Gamma < G$ be a lattice subgroup. Let us consider two cases in successive generality:

K is virtually a subgroup of $\text{Aut}(S)$. In this case, we suppose that the action of K on S by conjugation has a finite kernel. In this case, the machinery of [4, 12] is applicable.

Let S^* be the identity component of the closure of ΓS in G and let $\Gamma^* = S^* \cap \Gamma$. By [12, Lem. 3] and [4], one knows that S^* is a solvable subgroup containing S , Γ^* is of finite index in Γ , $S \setminus S^*$ is a torus subgroup, T , of K , the nilradical of S^* equals the nilradical R of S , $\Gamma \cap R$ is a lattice subgroup of R . Likewise, the derived subgroup of S , $N = [S, S] = [S^*, S^*]$, intersects Γ in a lattice subgroup of N . This information is summarised in the commutative diagram (13), where $B = \Gamma \cap N$, $F = B \setminus N$, $Z = B \setminus G$, $T^* = (B \setminus N) \setminus (N \setminus S^*)$ (a torus) and $A = S \setminus S^*$.



In diagram (13), all southeast sequences are fibrations with discrete fibre (covering spaces), all eastern sequences are fibrations, as are the backwards L sequences. In particular, X^* is a finite regular covering space of X which is fibred by the solvmanifold Y^* over the K -homogeneous space $T \setminus K$; the solvmanifold Y^* is itself fibred by the nilmanifold F over the torus T^* . Since S^* is the identity component of $\overline{\Gamma S}$, the group $W = \Gamma^* \setminus \Gamma$ permutes the components of $\overline{\Gamma S}$, which shows that $Y^* = Y$, so X is fibred by solvmanifolds, also.

Since $WT \setminus K$ has finite fundamental group, its first cohomology group over \mathbb{Z} vanishes. Therefore, the Leray-Serre spectral sequence for the fibring of X by Y shows that the restriction to a fibre induces an injection of $H^1(X; \mathbb{R})$ into $H^1(Y; \mathbb{R})$ (the image is the kernel of $d_2^{0,1}$ in Fig. 1). The fibring of Y by the nilmanifold F over the torus T^* is exactly as described in the previous example. In particular, the projection map induces an isomorphism of $H^1(Y; \mathbb{R})$ and $H^1(T^*; \mathbb{R})$. Since $S^* = ST$, we see that $N \setminus S^* = AT$, where $A = N \setminus S$. Since T is contractible in G , one sees that the first real homology group of X^* is naturally identified with A ; or Z is visibly the universal abelian covering space of X^* . It follows that $H^1(X; \mathbb{R})$ is naturally identified with A^W , the fixed-point set of W acting on A .

Let $\phi : G \rightarrow X$ be defined by $\phi(t) = \Gamma g t^{-1} g'$ for some $g, g' \in G$. A few applications of Lemma 4.3 imply that one can suppose, without changing the rotation map, that $\phi(t) = \Gamma \kappa \alpha^{-1} \kappa^{-1} b$, where $a, b \in S, \kappa \in K$ and $t = \beta \alpha$ is the decomposition into

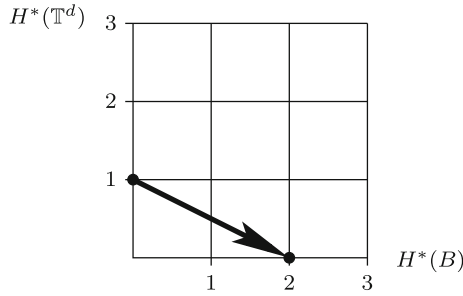


Fig. 1. E_2 page of the spectral sequence

$\beta \in K$ and $\alpha \in S$. Let $\hat{F} : Z \rightarrow A = N \backslash G / K$ be the map that induces the isomorphism of $[X^*, S^1] \otimes \mathbb{R}$ with A . Concretely, if $Nt \in Z$, let $t = \beta_t \alpha_t$ be the decomposition of t into $\beta_t \in K, \alpha_t \in S$; then $\hat{F}(Nt) = K\alpha_t N$. One computes that

$$\zeta_s(t) = -K(\kappa\beta_t^{-1}) \cdot \alpha_s \cdot (\kappa\beta_t^{-1})^{-1}N \quad s, t \in G. \tag{14}$$

It is clear that ζ_s is S -invariant since $t \mapsto \beta_t$ is the projection $G \rightarrow K$. Since the restriction of any mean on a compact Lie group to its continuous functions is the Haar probability measure [28], one sees that for any $v \in \mathfrak{m}(G)^{G^-}$, $\rho_s(v) = -\bar{\alpha}_s N$ is the projection of $\alpha_s N$ onto the subspace of K -invariant vectors.

Note that if one restricts ϕ to S , then the rotation vector of $s \in S$ with respect to the mean $\nu \in \mathfrak{m}(S)^{S^-}$ is the projection of $-\kappa s \kappa^{-1} N$ onto the subspace of W -invariant vectors.

When K is not a virtual subgroup of $\text{Aut}(S)$. Let us now examine the case where the kernel of representation $K \rightarrow \text{Aut}(S)$ is not finite. Let $K_1 \triangleleft K$ be the identity component of this kernel. Since K is compact and simply-connected, K is semi-simple and so $K = K_0 \oplus K_1$ is a sum of semi-simple factors, and the representation of $K_0 \rightarrow \text{Aut}(S)$ has finite kernel. By construction, K_1 is a normal subgroup of G and the lattice Γ intersects K_1 in a compact set, hence $\Gamma \cap K_1$ is a finite, normal subgroup of Γ . We obtain the fibration

$$\Gamma \cap K_1 \backslash K_1 \longrightarrow \Gamma \backslash G \xrightarrow{-\rho} \bar{\Gamma} \backslash \bar{G} = (\Gamma \cap K_1 \backslash \Gamma) \backslash (K_1 \backslash G). \tag{15}$$

The quotient $\bar{G} = SK_0$ has the property that K_0 is a virtual subgroup of $\text{Aut}(S)$. The fibre $\Gamma \cap K_1 \backslash K_1$ has a finite fundamental group. It follows that the map $\rho^* : H^1(\bar{\Gamma} \backslash \bar{G}; \mathbb{R}) \rightarrow H^1(\Gamma \backslash G; \mathbb{R})$ is an isomorphism. From this, one concludes that the preceding computations of the ζ -map (14) and the rotation vectors of a mean remain correct in this enlarged setting.

4.1.4. *Quotients of amenable Lie groups of type (E) – II.* Let us continue with the notations of the previous example. Let $H = G \times G'$ be a product of simply-connected amenable Lie groups (in applications, $G' = \mathbb{R}$, but what follows is perfectly general). Let $\varphi : G' \rightarrow X$ be a uniformly continuous map and let

$$\phi : H \rightarrow X \quad \phi(h) = \Gamma g^{-1} \varphi(g'), \quad \text{where } h = (g, g') \in H. \tag{16}$$

Similar to that above, one computes that with $s = (1, b)$ and $t = (g, a)$, one has

$$\zeta_s(t) = -K \delta_b(a)N \quad \delta_b(a) = \text{the projection of } \varphi(ba)^{-1} \cdot \varphi(a) \text{ onto } S, \quad (17)$$

using the factorisation of an element in G as in the previous example. In particular, this implies that ζ_s is independent of g when $s = (1, b)$. This implies that if $\nu \in \mathfrak{m}(H)^{H-}$ is a mean on H , then the rotation vector $\rho_s(\nu)$ ($s = (1, b)$) equals the rotation vector $\rho_b(\bar{\nu})$ for the map φ and the projected mean $\bar{\nu} \in \mathfrak{m}(G')^{G'-}$.

In the next section we show how this result can be interpreted in terms of the rotation vector of two measures with differently-sized support.

4.2. Relation to Schwartzman cycles. Let us suppose that $\Phi : G \times X \rightarrow G$ is a left-action of G on X . For each $x \in X$, one has the orbit map $\phi_x(t) := \Phi(t, x)$. The action will also be denoted by $\Phi(t, x) = t \cdot x$.

Lemma 4.5. *The orbit map $\phi_x : G \rightarrow X$ is uniformly continuous for all $x \in X$.*

Proof. Let us define $\epsilon(\delta) = \max \{d(\Phi(1, x), \Phi(t, x)) : x \in X, d(1, t) \leq \delta\}$. By local compactness of G and compactness of X , the maximum is attained. Moreover, ϵ is a continuous increasing function of δ that vanishes at $\delta = 0$. This implies uniform continuity of the orbit map ϕ_x . \square

Let $\nu \in \mathfrak{m}(G)^{G-}$ be a left-invariant mean on G . For each $x \in X$, the pull-back of $C^0(X)$ by the orbit map ϕ_x lies inside $L^\infty(G)$. Thus, $\phi_{x,*}\nu$ determines a positive, continuous linear functional on $C^0(X)$ and so by the Riesz representation theorem, $\phi_{x,*}\nu$ induces a Borel probability measure μ_x on X . It is clear that μ_x is G -invariant. The support of μ_x is clearly contained in the ω -limit set of x ,

$$\omega_G(x) = \bigcap_{T>0} \overline{\{t \cdot x : d(1, t) > T\}}. \quad (18)$$

In [16, App. A], one finds a definition of the rotation vector of an invariant measure of a flow (an \mathbb{R} -action). Let μ be an invariant Borel probability measure of the flow $\varphi : \mathbb{R} \times X \rightarrow X$ and $[f] \in [X, S^1]$ a cohomology class. The rotation vector of μ is defined as

$$\langle [f], \rho_\varphi(\mu) \rangle = \int_{x \in X} \zeta_\varphi(x) d\mu(x), \quad (19)$$

where $\zeta_\varphi(x) = f(\varphi_1(x)) - f(x)$ similar to (6). We have:

Theorem 4.2. *Let $\Phi : G \times X \rightarrow X$ be a G -action, φ be an action of a 1-dimensional subgroup with $\varphi_1 = s$, and let $\nu \in \mathfrak{m}(G)^{G-}$, $\mu_x = \phi_{x,*}\nu$ for some $x \in X$. Then*

$$\rho_s^x(\nu) = \rho_\varphi(\mu_x), \quad (20)$$

where ρ^x is the rotation map for the orbit map ϕ_x .

The proof is an application of change of variables.

4.3. Averaged rotation vectors. In this subsection, let us suppose that G fits in the exact sequence of (amenable) groups

$$H \hookrightarrow G \twoheadrightarrow F. \quad (21)$$

Let $\nu_H \in \mathfrak{m}(H)^{H-}$ (resp. $\nu_F \in \mathfrak{m}(F)^{F-}$) be left-invariant means. One can define an invariant mean ν_G as follows: let $f \in L^\infty(G)$ and define $f_H \in L^\infty(F)$ by averaging over H , $f_H(Ht) = \langle \nu_H, f_t \rangle$, where $f_t(x) = f(tx)$. The normality of H and left-invariance of ν_H implies that f_H is well-defined and $f_H \in L^\infty(F)$. Then, one defines the left-invariant mean ν_G by $\langle \nu_G, f \rangle := \langle \nu_F, f_H \rangle$.

Definition 4.2. The mean $\nu_G \in \mathfrak{m}(G)^{G-}$ is denoted by $\nu_G = \nu_F \times \nu_H$ and called a **product mean**.

Let us suppose that H acts on X by an action φ and that there is a uniformly continuous map $\phi : G \rightarrow X$ satisfying

$$\phi(s \cdot t) = \varphi(s) \cdot \phi(t) \quad \forall s \in H, t \in G. \quad (22)$$

Let $t_0 \in G$, $x = \phi(t_0)$ and $\mu_{H,x} = \varphi_{x,*} \nu_H$ is the pushed forward measure on X . The measure $\mu_G = \phi_* \nu_G$ (where $\nu_G = \nu_F \times \nu_H$) is H -invariant due to the cocycle condition (22) and $\text{supp } \mu_{H,x} \subset \text{supp } \mu_G$.

The following lemma shows that under a suitable condition on the map ϕ , one can average over the group G to obtain a measure μ_G with a larger support and the same rotation set.

Lemma 4.6. Suppose that the lift $\hat{\phi}$ (see (5)) has the property that for each $t \in G$, there is a $K > 0$ such that $d(\hat{\phi}(s) \cdot \hat{\phi}(t_0), \hat{\phi}(s) \cdot \hat{\phi}(t)) < K$ for all $s \in H$. Then for all $s \in H$, $\rho_s(\mu_{H,x})$ is independent of the point $x \in \text{Im } \phi$. In particular,

$$\rho_s(\mu_{H,x}) = \rho_s(\mu_G). \quad (23)$$

To be clear, ρ_s refers to the rotation map of the flow generated by the 1-parameter group through s , as in (19). The proof of this lemma follows from Lemma 4.3 and Theorem 4.2 along with an unraveling of the product mean.

Note that the example in Sect. 4.1.3 does not contradict this lemma. In that example, the map ϕ does not satisfy the uniform boundedness condition.

5. Homogeneous Structures

This section proves Theorems 1.3 and 1.4. We begin by establishing some terminology and notation.

Let G be a connected Lie group. Define the left (resp. right) translation map by

$$L_h(g) := hg, \quad R_h(g) := gh \quad (24)$$

for all $g, h \in G$. These two maps define a left action of $G_- = G$ (resp. $G_+ = G^{op}$) on G and therefore on T^*G by Hamiltonian symplectomorphisms. The momentum maps of these actions are

$$\begin{aligned} \Psi_- : T^*G &\longrightarrow \mathfrak{g}_-^*, & \Psi_+ : T^*G &\longrightarrow \mathfrak{g}_+^*, \\ \Psi_-(g, \mu_g) &:= (T_1 R_g)^* \mu_g, & \Psi_+(g, \mu_g) &:= (T_1 L_g)^* \mu_g, \end{aligned} \quad (25)$$

for each $g \in G$, $\mu_g \in T_g^*G$.

A co-vector field $\mu : G \rightarrow T^*G$ is left- (resp. right-) invariant if $\mu(1) = (T_1 L_g)^* \mu(g)$ (resp. $\mu(1) = (T_1 R_g)^* \mu(g)$) for all $g \in G$. If one trivialises T^*G with respect to the left-invariant co-vectors, then the momentum maps are simply

$$\Psi_-(g, \mu) := \text{Ad}_g^* \mu, \quad \Psi_+(g, \mu) := \mu, \quad (26)$$

for all $g \in G$, $\mu \in \mathfrak{g}^* = T_1^*G$, where $\text{Ad}_g^* = (T_1 L_g R_{g^{-1}})^*$.

One says that a function $H : T^*G \rightarrow \mathbb{R}$ is *collective* for the left-action (resp. right-action) if $H = \Psi_-^* h$ (resp. $H = \Psi_+^* h$) for some $h : \mathfrak{g}^* \rightarrow \mathbb{R}$. If H is collective for the left-action (resp. right-action) then (25) shows it is right-invariant (resp. left-invariant). In particular, a Hamiltonian that is collective for the left-action [right-invariant] (resp. right-action [left-invariant]) Poisson-commutes with Ψ_+ (resp. Ψ_-).

Let $H : T^*G \rightarrow \mathbb{R}$ be a smooth, left-invariant (= right collective) Tonelli Hamiltonian. Therefore, there is a smooth convex Hamiltonian $h : \mathfrak{g}^* \rightarrow \mathbb{R}$ such that $H = \Psi_+^* h$. Moreover, since H is left-invariant, it Poisson-commutes with the momentum map of the left action Ψ_- .

Let $\Gamma \triangleleft G$ be a co-compact lattice subgroup and $M = \Gamma \backslash G$. It is assumed that G is simply-connected, so that the universal cover of M , \tilde{M} , is G . Let $[\Gamma, \Gamma] = \Gamma_1$ be the commutator subgroup of Γ , which is the fundamental group of the universal abelian cover \hat{M} . This leads to the commuting diagram of covering maps:

$$\begin{array}{ccc} T^*G = T^*\tilde{M} & \twoheadrightarrow & G = \tilde{M} \\ \downarrow \hat{\Pi} & & \downarrow \tilde{\pi} \\ \Pi \left(T^*(\Gamma_1 \backslash G) = T^*\hat{M} \right) & \twoheadrightarrow & \Gamma_1 \backslash G = \hat{M} \\ \downarrow \hat{\Pi} & & \downarrow \hat{\pi} \\ T^*(\Gamma \backslash G) = T^*M & \twoheadrightarrow & \Gamma \backslash G = M \end{array} \quad (27)$$

We adopt the notational convention that the pull-back of x to \hat{M} (resp. \tilde{M}) is denoted by \hat{x} (resp. \tilde{x}).

Let $c \in H^1(M; \mathbb{R})$ be a cohomology class, let $(x, p) \in \mathcal{M}_c^*(H)$ be a recurrent point in the Mather set and let $\delta : \mathbb{R} \rightarrow M$ be the minimizer with initial conditions $\delta(0) = x$ and $\mathcal{L}(x, \dot{\delta}(0)) = (x, p)$, where \mathcal{L} denotes the associated Legendre transform (see Sect. 2). By the arguments of [23], we can suppose that the rotation set of δ is a singleton $\{h\} \subset H_1(M; \mathbb{R})$ and any weak-* limit of uniform measures along the orbit is a minimizing measure. Fix a lift $\tilde{\delta}$ of δ to \tilde{M} . For each $g \in G$, let $\tilde{\delta}_g = L_{g^{-1}} \circ \tilde{\delta}$ be a left-translate of this lift. Left invariance of H implies that $\tilde{\delta}_g$ is the projection of an integral curve, which implies that the projection of $\tilde{\delta}_g$ to \hat{M} and M are also projections of orbits. All of this allows the definition of a map

$$\begin{array}{ccc} G \times \mathbb{R} & \xrightarrow{\phi} & T^*M \\ & \searrow \bar{\phi} & \downarrow \\ & & M \end{array} \quad \begin{array}{l} \phi(g, t) = \Pi \circ (TL_{g^{-1}})^* \tilde{\varphi}_t(x, p), \\ \bar{\phi}(g, t) = \Gamma g^{-1} \tilde{\delta}(t) = \Gamma \delta_g(t), \end{array} \quad (28)$$

where $\tilde{\varphi}$ is the flow of H on the universal cover T^*G . By the example in Sect. 4.1.4, the rotation vector of the map δ_g is independent of g for any mean on $G \times \mathbb{R}$. This implies the same is true for $\phi(g, t)$.

Let $\nu_{\mathbb{R}} \in \mathfrak{m}(\mathbb{R})^{\mathbb{R}}$ be an invariant mean such that the rotation vector of $\nu_{\mathbb{R}}$ at $s = 1$ under the map δ is h . By hypothesis, there is such a mean. The preceding discussion proves the following lemma.

Lemma 5.1. *Let $\nu_{\mathbb{R}} \in \mathfrak{m}(\mathbb{R})^{\mathbb{R}}$, $\nu_G \in \mathfrak{m}(G)^{G-}$ and $\mu = \phi_*(\nu_{\mathbb{R}} \times \nu_G)$. Then μ minimizes A_c - i.e. it is c -action minimizing - and the projection of $\text{supp } \mu$ covers M .*

Proof (Theorem 1.3). By [23, Thm. 2], we know that $\text{supp } \mu$ is a Lipschitz graph over M . Therefore, the lift to $T^*\tilde{M}$ contains the smooth manifold $\tilde{\phi}(G \times 0)$ which is a smooth graph over \tilde{M} . Therefore $\text{supp } \mu$ is a smooth Lagrangian graph over M , $\text{supp } \mu = \text{graph}(\eta)$, and lifting this picture to $T^*\tilde{M}$ shows that $\tilde{\eta}$ is closed and left-invariant. Therefore, $\tilde{\eta}$ is a bi-invariant 1-form. Since $\mathcal{M}_c^*(H) = \text{graph}(\eta)$, where c is the cohomology class of η , this proves item (i) of Theorem 1.3. Lemma 5.1 and the preceding discussion implies that the rotation set of $\mathcal{M}_c^*(H)$ is a singleton, which implies item (iii).

Let us now examine Hamilton's equations for H on the Mather set $\mathcal{M}^*(H) = \text{graph}(\eta)$. Since H is left-invariant, it follows that

$$H(q, \eta(q)) = h \circ \Psi_+(q, \eta(q)) = h((T_1 L_g)^* \eta(q)) = h(\eta(1)) = E, \tag{29}$$

$$\Psi_-(q, \eta(q)) = \eta(1), \tag{30}$$

for all $q \in M$. Equation (30) follows because η is bi-invariant, which implies that the co-adjoint orbit of $\eta(1)$ is a single point.

Hamilton's equations for the Hamiltonian H are

$$X_H(g, \mu) : \begin{cases} \dot{g} = (T_1 L_g) \cdot dh(\mu), \\ \dot{\mu} = -\text{ad}_{dh(\mu)}^* \mu. \end{cases} \quad \forall g \in G, \mu \in \mathfrak{g}^*. \tag{31}$$

In particular, if μ is a closed form, then $\text{ad}_{\xi}^* \mu$ vanishes for all $\xi \in \mathfrak{g}$. Therefore, the orbit of (g, μ) is $\{(T_1 R_{\exp(t\xi)})^*(g, \mu) = (g \exp(t\xi), \mu) : t \in \mathbb{R}\}$, where $\xi = dh(\mu)$, i.e. it is the orbit of a 1-parameter subgroup. This proves item (ii).

Finally, the discussion around (1) and (29) shows that the following diagram commutes:

$$\begin{array}{ccc} H^1(M; \mathbb{R}) & \xrightarrow{\alpha_H} & \mathbb{R} \\ \cong \uparrow & \nearrow h & \\ (T_1^*G)^G & & \end{array} \qquad \begin{array}{ccc} c & \xrightarrow{\alpha_H(c) = h(\eta(1))} & \\ \uparrow & \nearrow & \\ \eta & & \end{array} \tag{32}$$

where $(T_1^*G)^G$ is the set of bi-invariant 1-forms on G . By hypothesis, h is C^r . This completes the proof. \square

Proof (Theorem 1.4). The sole remaining thing to prove is that if H is weakly integrable and $\Lambda \subset T^*M$ lies inside an iso-energy surface and intersects $\text{Reg } F$, then M is a homogeneous space of a compact reductive Lie group. By Theorem 1.3, $\Lambda = \text{graph}(\eta)$, where η is a bi-invariant, closed 1-form on G . By Theorem 1.2, M is diffeomorphic to $\mathbb{T}^b \times B$, where B is a parallelisable manifold with finite coverings having zero first Betti

number. Therefore, the lattice $\Gamma = \pi_1(M)$ splits as $\Gamma = \mathbb{Z}^b \oplus P$, where $P = \pi_1(B)$. From the description in (13), one knows that B and hence N must be trivial. This implies that $\dim S = b$ (we do not claim that the \mathbb{Z}^b factor is a lattice in S). On the other hand, one also sees that $P = \pi_1(B)$ must be finite: since Γ is virtually polycyclic, so is $\mathbb{Z}^b \backslash \Gamma = P$, but a virtually polycyclic group is either finite or it contains a finite index subgroup that has non-zero first Betti number.² Additionally, since $P < G$ is a finite subgroup, it is compact and therefore a subgroup of a maximal compact subgroup; up to an inner automorphism, we can assume that $P < K$.

Therefore M is finitely covered by $\hat{M} = \mathbb{T}^b \times \hat{B}$ and Theorems 1.2 & 1.3 show that \mathbb{T}^b is the closure of the projection of a 1-parameter subgroup of S . This proves that S is abelian.

Finally, let $\Gamma_1 < \Gamma$ be a torsion-free subgroup such that $\hat{M} = \Gamma_1 \backslash G$. One knows that Γ_1 is generated by elements $\epsilon_i = e_i \delta_i$ for $i = 1, \dots, b$, where $e_i \in S$, $\delta_i \in K$. Since Γ_1 is abelian, the δ_i pairwise commute and e_i commutes with δ_j for all $i \neq j$. From the argument of Sect. 4.1.3 one knows that there are integers $n_i > 0$ such that $\delta_i^{n_i}$ generate a torus subgroup $T < K$. It follows that there are torsion elements $c_i \in K$ and $\xi_i \in \text{Lie } T$ such that $\delta_i = c_i \exp(\xi_i)$ and the c_i pairwise commute and commute with all δ_j . Let us define $\epsilon_{i,t} = e_i c_i \exp(t\xi_i)$ and Γ_t be the lattice subgroup of G generated by $\epsilon_{i,t}$. The identity map on G induces a diffeomorphism of $\Gamma_0 \backslash G$ with $\Gamma_1 \backslash G = \hat{M}$. The lattice Γ_0 is generated by $\epsilon_{i,0} = e_i c_i$. Since Γ_0 is abelian, the c_i must fix each c_j , $j \neq i$, and c_i must send e_i to $\pm e_i$. If $c_i e_i c_i^{-1} = -e_i$, then $\epsilon_{i,0}$ is a torsion element in the free abelian group Γ_0 , hence it is 1, absurd. Therefore, c_i fixes e_i , too. Since $\{e_i\}$ generates a lattice in S , each c_i commutes with S . Therefore, $c_i \in \ker(K \rightarrow \text{Aut}(S))$ for each i .

To sum up: let $\Gamma'_0 < \Gamma_0$ be the sublattice generated by the pure translations in Γ_0 . Then $\Gamma'_0 \backslash G$ is diffeomorphic to $\mathbb{T}^b \times K$, a reductive Lie group and it is a smooth covering space of M . \square

Acknowledgements. L. B. thanks K. F. Siburg for helpful discussions and the MFO for its hospitality. A. S. would like to acknowledge the support of the *Herchel-Smith foundation* and the French ANR project *Hamilton-Jacobi et théorie KAM faible*.

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² If D is solvable, then the derived series $D_k = [D_{k-1}, D_{k-1}]$, $D_0 = D$, terminates at 1 for some k . If each quotient D_{k-1}/D_k is finite, then D is finite; if D is not finite, then there is a least k such that D_k/D_{k+1} is infinite. This D_k is therefore of finite index with non-zero first Betti number.

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Communicated by G. Gallavotti