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Inverse problems and rigidity questions in billiard dynamics

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To the memory of Anatole Katok (1944–2018)

Abstract. A Birkhoff billiard is a system describing the inertial motion of a point mass inside a strictly convex planar domain, with elastic reflections at the boundary. The study of the associated dynamics is profoundly intertwined with the geometric properties of the domain: while it is evident how the shape determines the dynamics, a more subtle and difficult question is the extent to which the knowledge of the dynamics allows one to reconstruct the shape of the domain. This translates into many intriguing inverse problems and unanswered rigidity questions, which have been the focus of very active research in recent decades. In this paper we describe some of these questions, along with their connection to other problems in analysis and geometry, with particular emphasis on recent results obtained by the authors and their collaborators.

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1. *Introduction*

In this paper we would like to describe some recent results obtained by the authors and their collaborators in the study of spectral and dynamical properties of the so-called *mathematical billiards*. More specifically, a major thread running through them relies on the attempt to understand the nature and the significance of important *rigidity phenomena* appearing in the study of billiard dynamics, which translate into important problems at the crossroads of dynamical systems, analysisand geometry.

A *mathematical billiard* is a dynamical model describing the motion of a billiard ball inside a domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary. The massless ball moves with unit velocity and without friction following a rectilinear path. If the ball hits the boundary at a point of non-differentiability (a sort of 'hole'), then the motion stops; otherwise, when it hits the boundary, it reflects elastically according to the standard *reflection law*: the angle of reflection is equal to the angle of incidence.

Billiards have been capturing the attention of researchers in various areas of mathematics. Despite their simple dynamical law, their qualitative dynamical properties are extremely non-local and profoundly intertwined with the geometry (e.g., the *shape*) of the domain. While it is clear how the shape of the domain completely determines the billiard dynamics, a more subtle and intriguing question is the extent to which dynamical information can be used to reconstruct the shape of the billiard domain.

In the following we will be interested in a special class of billiards, consisting of strictly convex planar domains with smooth boundary; these billiards will henceforth be called *Birkhoff billiards*. More specifically, we would like to address this question in several different contexts, where it translates into unanswered questions and enthralling conjectures that have been the focus of very active research in recent decades.

(A) Can a billiard be recovered from its *length spectrum* (that is the set of lengths of its periodic orbits)?. In other words, *can one hear the shape of a billiard?* This question, in fact, turns out to be tightly related to the classical *spectral problem*, *can one hear the shape of a drum?*, as formulated in a very suggestive way by Kac [**57**]. See §3.

(B) Which billiards admit *integrable dynamics*? This is related to the celebrated *Birkhoff conjecture*. More generally, what information on the geometry of the billiard domain can be deduced from the existence of invariant curves for the corresponding billiard map (e.g., *caustics*) or from their properties? See §4.

(C) Do *action-minimizing properties* of the billiard map (related to the so-called *Aubry–Mather theory*) encode any information on the billiard domain and its dynamical properties? See §5.

2. *The billiard map*

Let us first recall the definition of the billiard map and its main properties. We refer to [**87**, **94**, **95**] for a more comprehensive introduction to the study of billiards.

Let Ω be a strictly convex compact domain in $\mathbb{R}^2$ with C^r boundary $\partial\Omega$, $r \geq 3$ (by strict convexity of Ω, we mean that the curvature of $\partial\Omega$ is strictly positive at every point). The phase space M of the billiard map consists of unit vectors (x, v) whose foot points x are on $\partial\Omega$ and which have inward directions. The billiard ball map $B_\Omega : M \longrightarrow M$ takes (x, v) to (x', v'), where x' represents the point where the trajectory starting at x with velocity v hits the boundary $\partial\Omega$ again, and v' is the *reflected velocity*, according to the standard reflection law: the angle of incidence is equal to the angle of reflection (Figure 1).

Remark 2.1.

(i) The dynamical properties of billiards are strongly related to the geometric properties of its shape. Besides the study of Birkhoff billiards, very active areas of research focus on the study of polygonal billiards, in particular *rational billiards*, whose dynamics can be related to geodesic flows on *translation surfaces* and *Teichmüller theory* (see, for example, [**34**]) or billiards with concave boundary (so-called

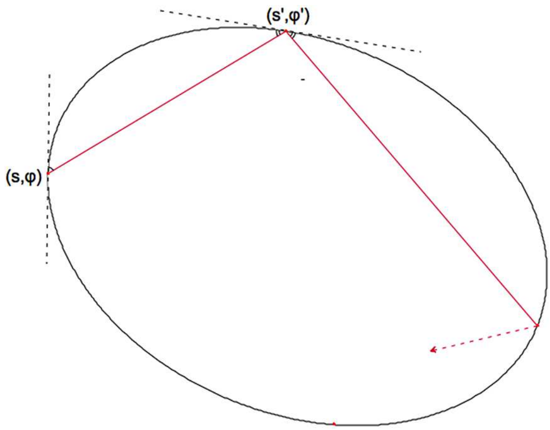


FIGURE 1. Example of billiard dynamics inside a planar domain.

dispersive billiards) of particular interest as models in statistical mechanics and mathematical physics (see [88]).

(ii) More generally, one could consider a Riemannian manifold with smooth boundary $(M, \partial M, g)$: the trajectory starting at $x \in \partial M$ with (inward) unit velocity v will follow the corresponding geodesic: either the geodesic remains in M for all positive times, or when it hits the boundary at $x' \in \partial M$, it will 'reflect' with a new inward velocity v' determined in the following way: the normal component of the hitting velocity instantaneously changes sign, while the tangential one stays unchanged. Observe that in the Euclidean planar case, this gives exactly the standard reflection law that we have described above.

(iii) Observe that if Ω is not convex, then the billiard map is not continuous.

(iv) As pointed out by Halpern [47], if the boundary is not at least C^3 (actually, it is enough for it to be C^2 with bounded third derivative), then strange phenomena can occur; for example, there might be orbits with infinitely many bounces, but of finite total length (since we are considering unit velocities, this could be interpreted as a sort of incompleteness of the billiard flow, namely the velocity becomes tangent to the boundary in finite time/length).

Let us introduce coordinates on M. We suppose that $\partial\Omega$ is parametrized by arc length s and let $\gamma : \mathbb{R}/\ell\mathbb{Z} \longrightarrow \mathbb{R}^2$ denote such a parametrization, where ℓ denotes the length of $\partial\Omega$. Without loss of generality, fix an orientation of γ. Let φ be the angle between v and the positive tangent to $\partial\Omega$ at x. Hence, M can be identified with the annulus $\mathbb{A}_\ell = \mathbb{R}/\ell\mathbb{Z} \times (0, \pi)$ and the billiard map B_Ω can be described as

$$\begin{aligned} B_\Omega : \mathbb{A}_\ell &\longrightarrow \mathbb{A}_\ell \\ (s, \varphi) &\longmapsto (s', \varphi'). \end{aligned}$$

Here are some properties of the billiard map.

- $B_\Omega \in C^{r-1}(\mathbb{A}_\ell)$ (see, for example, [59, Theorem 4.1]). Moreover, B_Ω can be continuously extended to $\overline{\mathbb{A}}_\ell = \mathbb{R}/\ell\mathbb{Z} \times [0, \pi]$ by fixing $B_\Omega(s, 0) = (s, 0)$ and $B_\Omega(s, \pi) = (s, \pi)$ for all $s \in \mathbb{R}/\ell\mathbb{Z}$.
- B_Ω is a symplectic map, that is, it preserves the exact symplectic form $\omega = \sin\varphi\, d\varphi \wedge ds = -d(\cos\varphi\, ds) =: -d\alpha$, namely $B_\Omega^*\omega = \omega$, where B_Ω^* denotes the pull-back (observe that the form ω becomes degenerate on $\partial\mathbb{A}_\ell$); see, for example, [95, Theorem 3.1].

  Moreover, B_Ω is an *exact symplectic map*, that is, $B_\Omega^*\alpha - \alpha = dh$ is an exact 1-form; the corresponding generating function is given by

$$h(s, s') := -\|\gamma(s) - \gamma(s')\|,$$

  which is minus the Euclidean distance between two points on $\partial\Omega$. It is easy to check that

$$\begin{cases} \partial_1 h(s, s') = \cos\varphi, \\ \partial_2 h(s, s') = -\cos\varphi', \end{cases} \tag{1}$$

  where ∂_i denotes the derivative with respect to the ith variable ($i = 1, 2$).
- If we lift B_Ω to the universal cover of $\mathbb{A}_\ell$ and introduce new coordinates $(x, y) = (s, -\cos\varphi) \in \mathbb{R} \times (-1, 1)$, then the billiard map becomes a monotone twist map with h as generating function and preserves the area form $dx \wedge dy$. See §5 and [87, 94, 95] for more details.

Remark 2.2. It follows from (1) that $\{(s_i, \varphi_i)\}_{i\in\mathbb{Z}}$ is an orbit of B_Ω if and only if $\{s_i\}_{i\in\mathbb{Z}}$ is a 'critical configuration' for the *action functional*

$$\{s_i\}_{i\in\mathbb{Z}} \longmapsto \sum_{i\in\mathbb{Z}} h(s_i, s_{i+1}),$$

in the usual sense of statistical mechanics and $\varphi_i = \arccos \partial_1 h(s_i, s_{i+1})$; in fact, while this latter sum is infinite, its derivatives are well defined:

$$\frac{\partial}{\partial s_n}\left(\sum_{i\in\mathbb{Z}} h(s_i, s_{i+1})\right) = \partial_1 h(s_n, s_{n+1}) + \partial_2 h(s_{n-1}, s_n).$$

These critical configurations are not necessarily (global) minima. One could be wondering whether (global) minima exist and if they have special dynamical features. More specifically, orbits that are *global minimizers* of the action functional, that is, every finite segment minimizes the action functional among all configurations with the same number of elements and the same end-points. This is the content of the so-called *Aubry–Mather theory* (see §5 for more details). Observe that in the billiard case, since the generating function (and hence the action) is given by minus the Euclidean length, *action minimization* can be rephrased in terms of *length maximization*.

3. *Periodic orbits and marked length spectrum: can you hear the shape of a billiard?*

Periodic orbits and their properties are among the first dynamical features of billiards to have been investigated. One of the first results in the theory of billiards, for example, can be considered to be Birkhoff's application of Poincaré's last geometric theorem to show the existence of infinitely many *distinct* periodic orbits [18]. Since then, new phenomena have been pointed out and many interesting questions have been raised.

How do we distinguish *distinct* periodic orbits? One could try to classify them in terms of their *period*, that is, the minimal number of times that the ball reflects before going back to the initial position in the initial direction. However, while in some cases this quantity allows one to distinguish different periodic orbits, in many cases it is not sufficient anymore: periodic orbits with the same periods may wind a different number of times before closing; this will clearly translate into a different topological shape.

A better invariant that one should consider is the so-called *rotation number*. The rotation number of a periodic billiard trajectory (respectively, a closed broken geodesic) is a rational number

$$\frac{p}{q} = \frac{\text{winding number}}{\text{number of reflections}} \in \left(0, \tfrac{1}{2}\right],$$

where the winding number $p \geq 1$ is defined as follows (see also [87, Definition 3.1.2]).

Fix the positive orientation of $\partial\Omega$ and pick any reflection point of the closed geodesic on $\partial\Omega$; then follow the trajectory and count how many times it goes around $\partial\Omega$ in the positive direction until it comes back to the starting point. Notice that inverting the direction of motion for every periodic billiard trajectory of rotation number $p/q \in (0, 1/2]$, we obtain a trajectory with rotation number $(q-p)/q \in [1/2, 1)$.

In [18], Birkhoff proved that for every $p/q \in (0, 1/2]$ in lowest terms, there are at least two closed orbits of rotation number p/q: one maximizing the total length and the other obtained by min-max methods (see also [87, Theorem 1.2.4]). This result is clearly optimal: in the case of a billiard in an ellipse (see §4.2), there are only two periodic orbits of period 2, also called *diameters*, which correspond to the two semiaxes of the ellipse. However, it is easy to find cases in which there are more than two periodic orbits for any given rotation number: think, for example, of a billiard in a disk where, due to the existence of a one-dimensional group of symmetries (rotations), each periodic orbit generates a one-dimensional family of similar ones; for example, all diameters are periodic orbits with period 2 (see §4.1).

Remark 3.1. A famous conjecture by Ivrii [56] states that in every billiard with infinitely smooth boundary in a Euclidean space of any dimension the set of periodic orbits has measure zero. As was shown by Ivrii [56], this implies the famous Weyl conjecture on the second-term asymptotics of the spectrum of the Laplacian. An interesting historical survey of both conjectures with references can be found in [38]. The conjecture is still open, although very interesting results towards its proof have been provided in [8, 82, 92, 99, 100] (for triangular orbits) and [37, 38] (for quadrilateral orbits); see also the results in [35].

3.1. *Laplace spectrum and length spectrum.* We define the *length spectrum* of Ω as the set

$$\mathcal{L}_\Omega := \mathbb{N}^+ \cdot \{\text{lengths of periodic orbits in } \Omega\} \cup \mathbb{N}^+ \cdot \ell(\partial\Omega),$$

where $\ell(\partial\Omega)$ denotes the length of the boundary, that is, the set of multiples of the lengths of all periodic orbits and multiples of the perimeter of Ω.

A remarkable relation exists between the length spectrum of a billiard in a convex domain Ω and the spectrum of the Laplace operator in Ω with Dirichlet boundary condition:

$$\begin{cases} \Delta f + \lambda^2 f = 0 & \text{in } \Omega, \\ f|_{\partial\Omega} = 0. \end{cases}$$

From the physical point of view, the eigenvalues are the eigenfrequencies of the membrane Ω with a fixed boundary. Denote by $\mathrm{Spec}_\Delta(\Omega) = \{0 < \lambda_1 \le \lambda_2 \le \cdots\}$ the Laplace spectrum of eigenvalues solving this problem.

The famous question of M. Kac in its original version asks *if one can recover the domain from the Laplace spectrum*. For general manifolds there are counterexamples (see [39]).

Anderson and Melrose [2] proved the following relation between the Laplace spectrum and the length spectrum (see also [44, 76, 83]).

THEOREM 3.2. (Anderson and Melrose) *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex compact domain with smooth boundary and let $\mathcal{L}_\Omega$ denote its length spectrum. Then, the wave trace*

$$w(t) := \mathrm{Re}\Big(\sum_{\lambda_n \in \mathrm{Spec}_\Delta(\Omega)} e^{i\lambda_n t} \Big)$$

is well defined as a distribution and smooth away from the length spectrum:

$$\text{sing.supp.}(w(t)) \subseteq \pm\mathcal{L}_\Omega \cup \{0\}. \tag{2}$$

That is, if $\xi > 0$ belongs to the singular support of this distribution, then there exists either a closed billiard trajectory of length ξ, or a closed geodesic of length ξ in the boundary of the billiard table. Generically, equality holds in (2).

Remark 3.3.

(i) We have stated the above theorem in the case of convex planar domains (that is the setting in which we are interested). However, the above inclusion also holds for more general classes of domains, for example for non-convex C^∞ domains in arbitrary dimension (see [76, Theorem 5.4.6]).

(ii) An easy example to convince the reader of this relation is the following. Consider $\Omega = (0, \pi) \times (0, \pi)$; then, it is easy to check that its Laplace spectrum is given by

$$\mathrm{Spec}(\Omega) = \{\sqrt{n^2 + m^2} : \ (n, m) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0)\}\}$$

which corresponds to the lengths of periodic orbits in $\overline{\Omega}$.

(iii) Observe that there are no known examples of domains in which the singular support of the wave trace is strictly included inside the length spectrum: the equivalence between these sets is strictly related to the problem whether Laplace spectral rigidity implies length spectral rigidity. A very interesting result in this direction has recently been provided in [49], where the authors prove that ellipses of sufficiently small eccentricities are Laplace spectrally unique (up to isometry) among all smooth domains (without any assumption on symmetry, convexity, or closeness to other ellipses). A key result in their proof involves showing that for nearly circular domains, the lengths of periodic orbits of rotation number $1/q$ are contained in the singular support of the wave trace [49, Theorem 1.4].

3.2. *Laplace spectral rigidity.* Given a class $\mathcal{M}$ of domains and a domain $\Omega \in \mathcal{M}$, we say that Ω is *spectrally determined in* $\mathcal{M}$ if it is the unique element (modulo isometries) of $\mathcal{M}$ with its Laplace spectrum: if $\Omega, \Omega' \in \mathcal{M}$ are *isospectral*, that is, $\mathrm{Spec}_\Delta(\Omega') = \mathrm{Spec}_\Delta(\Omega)$, then Ω' is the image of Ω by an isometry (that is a composition of translations and rotations).

Kac's question can be thus formulated as follows, assuming we have fixed a class of domains $\mathcal{M}$: *is every* $\Omega \in \mathcal{M}$ *spectrally determined?*

If $\mathcal{M}$ is the space of all planar domains, the answer is well known to be negative (see, for example, [39], which generalizes some results previously obtained for compact manifolds without boundary; see also [93, 98]). Remarkably, Sunada (see [93]) exhibits isospectral sets (that is sets of isospectral manifolds) of arbitrarily large cardinality. However, all known examples of domains that are not spectrally determined are not convex; moreover, they are bounded by curves that are only piecewise analytic (e.g. plane domains with corners). On the other hand, Zelditch proved in [101] that the inverse spectral problem has a positive answer when $\mathcal{M}$ is a generic class of analytic $\mathbb{Z}_2$-symmetric convex domains (that is symmetric with respect to reflection about a given axis). More recently, as we have already mentioned, Hezari and Zelditch [49] (see also [50]) proved that ellipses of sufficiently small eccentricities are Laplace spectrally unique (up to isometry) among all smooth domains (without any assumption on symmetry, convexity, or closeness to other ellipses).

The problem for non-analytic domains is substantially more challenging. In the C^∞ category, Osgood, Phillips and Sarnak [72–74] showed that isospectral sets of surfaces are necessarily compact in the C^∞ topology. Sarnak (see [85]) also conjectured that an isospectral set of surfaces consists of isolated domains. In other words, in dimension 2, C^∞-close to a C^∞ domain there should be no isospectral domains, except those that can be obtained by an isometry.

A weaker version of this conjecture can be stated as follows. A domain Ω is said to be *spectrally rigid in* $\mathcal{M}$ if any C^1-smooth one-parameter isospectral family $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{M}$ with $\Omega_0 = \Omega$ is necessarily an isometric family. We can then ask: *are all* C^∞ *domains spectrally rigid?*

The problem of spectral rigidity is in principle much simpler than the inverse spectral problem; yet it turns out to be extremely challenging. Hezari and Zelditch (see [48]) provided a result in the affirmative direction. Let Ω_0 be bounded by an ellipse $\mathcal{E}$. Then any one-parameter isospectral C^∞-deformation $(\Omega_\tau)_{|\tau|\le 1}$ which additionally preserves the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group of the ellipse is necessarily flat (that is all derivatives have to vanish for $\tau = 0$) (results of this kind are usually referred to as *infinitesimal spectral rigidity*). Popov and Topalov [78] recently extended these results (see also [79]).

Further historical remarks on the inverse spectral problem can also be found in [48] and in the surveys [102, 103].

In the case of Riemannian manifolds, we mention that Guillemin and Kazhdan [43] showed that any negatively curved surface is spectrally rigid among negatively curved surfaces. This result was later extended to compact manifolds of negative curvature in [26].

3.3. *Length spectral rigidity.* The relation between the Laplace spectrum and the length spectrum, immediately raises the following question: *does the knowledge of the lengths of periodic orbits determine the shape of the billiard domain?*

In [30], the following dynamical problem corresponding to spectral rigidity was investigated: we say that a domain $\Omega_0 \in \mathcal{M}$ is *dynamically spectrally rigid in* $\mathcal{M}$ if any C^1-smooth one-parameter dynamically isospectral family $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{M}$ is necessarily an isometric family. More specifically, the authors proved the following theorem.

THEOREM 3.4. (De Simoi, Kaloshin and Wei [30]) *Let $\mathcal{M}$ be the set of strictly convex domains with sufficiently (finitely) smooth boundary and axial symmetry and that are sufficiently close to a circle. Then $\Omega \in \mathcal{M}$ is dynamically spectrally rigid in $\mathcal{M}$.*

Remark 3.5. Let us point out that the above-mentioned results are concerned with spectral rigidity for smooth domains. Some results in the analytic category, but for non-Birkhoff billiards, are contained in:

- [29], where under suitable symmetry and genericity assumptions, it is proved that the marked length spectrum determines the geometry of billiard tables obtained by removing from the plane finitely many strictly convex analytic obstacles satisfying the so-called non-eclipse condition;
- [24], where the dynamical spectral rigidity for piecewise analytic Bunimovich stadia and squash-type stadia is established.

Remark 3.6. In the context of polygonal billiards, a positive answer to a related inverse problem was studied in [32], in which the authors provide a complete characterization of the relationship between the shape of a Euclidean polygon and the symbolic dynamics of its billiard flow. In particular, they consider the so-called *bounce spectrum*, namely, the set of all bounce sequences (sequences of letters from a given alphabet that identifies the sides of the polygon) that can occur along billiard trajectories; as a result, they prove that the only pairs of tables that can have the same bounce spectrum are right-angled tables that differ by an affine map.

3.4. *Some ideas on the proof of deformational spectral rigidity (Theorem 3.4).* Here we introduce the key elements of the proof of Theorem 3.4. Let $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{M}$ be a isospectral family of domains.

The first step is to establish the existence of a countable family of maximal periodic orbits given by q-gons for all $q \geq 2$.

LEMMA 3.7. (See [30, Lemma 4.3]) *Let $\Omega \in \mathcal{M}$. For any $q \geq 2$, there exists a periodic orbit of rotation number $1/q$ passing through the marked point of $\partial\Omega$ and having maximal length among other periodic orbits passing through the marked point. We call such an orbit a marked symmetric maximal periodic orbit and denote it by $S^q(\Omega)$.*

Let $S^q = (s_q^k, \varphi_q^k)_{k=0}^{q-1}$ be the maximal symmetric periodic orbit. Associate to S^q and a continuous function $\nu : \mathbb{T} \to \mathbb{R}$ a linear functional

$$\ell_{\Omega,q}(\nu) = \sum_{k=0}^{q-1} \nu(s_q^k) \sin \varphi_q^k.$$

Given a parametrization γ of a family $(\Omega_\tau)_{|\tau|\leq 1}$ in $\mathcal{M}$, we define the infinitesimal deformation function:

$$n_\gamma(\tau, \xi) = \langle \partial_\tau \gamma(\tau, \xi), N_\tau(\tau, \xi)\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^2$ and $N_\tau(\tau, \xi)$ is the outgoing unit normal vector to $\partial\Omega_\tau$ at the point $\gamma(\tau, \xi)$. Observe that n_γ is continuous in τ and $n_\gamma(\tau, \cdot)$ is smooth for any $|\tau| \leq 1$. By the normalization condition of $(\Omega_\tau)_{|\tau|\leq 1}$, we conclude that $n_\gamma(\tau, \cdot)$ is an even function, that is, $n_\gamma(\tau, \xi) = n_\gamma(\tau, -\xi)$, and $n_\gamma(\tau, 0) = 0$ for any $|\tau| \leq 1$. Naturally the space of perturbations can be identified with the space of smooth even functions on the circle denoted C_{sym}.

PROPOSITION 3.8. (See [30, Proposition 4.6]) *Let $(\Omega_\tau)_{|\tau|\leq 1}$ be an isospectral family. Then for any $|\tau| \leq 1$, $q \geq 2$, and having fixed arbitrarily $\bar{S}_\tau^q$ a maximal marked symmetric periodic orbit for Ω_τ, we have*

$$\ell_{\Omega_\tau,q}(n(\tau, \cdot)) = 0.$$

For any domain Ω (parametrized by the length s) with radius of curvature ρ, we define the linear functional

$$\ell_{\Omega,0}(\nu) := \int_0^1 \frac{\nu(s)}{\rho(s)}\, ds.$$

As shown in [30, (4.3)], if $(\Omega_\tau)_{|\tau|\leq 1}$ is an isospectral family, then for any $|\tau| \leq 1$ we have $\ell_{\Omega,0}(n(\tau, \cdot)) = 0$.

Define the following key notion. Call the linearized isospectral operator $\mathcal{T}_\Omega : C_{\mathrm{sym}} \to \mathbb{R}^{\mathbb{N}}$:

$$\mathcal{T}_\Omega \nu = (\ell_{\Omega,0}(n(\tau, \cdot)), \ell_{\Omega,1}(n(\tau, \cdot)), \ldots \ell_{\Omega,q}(n(\tau, \cdot)), \ldots).$$

In fact, $\mathcal{T}_\Omega$ has range in ℓ^∞, by definition of the functionals $\ell_{\Omega,q}$, since by [7, Lemma 8] there exists some $C > 0$ so that for any $q \geq 2$ we have $\sin \varphi_q^k \leq (C/q)$.

The linearized isospectral operator bears a strong analogy with the X-transform (see [41, §2.2]).

THEOREM 3.9. [30, Theorem 4.9] *In the space of sufficiently smooth axis symmetric domains there is a neighborhood of the circular domain such that the operator $\mathcal{T}_\Omega : C_{sym} \to \ell^\infty$ is injective.*

This theorem implies the rigidity theorem above. In the case of the domain Ω_0 being the circle, the linearized isospectral operator $\mathcal{T}_{\Omega_0}$ is easy to compute. For $j \geq 1$ and $q \geq 2$.

$$\ell_q(e_j) = \delta_{q|j},$$

where $\delta_{q|j} = 1$ is j is divisible by q and zero otherwise. For the circle $\mathcal{T}_{\Omega_0}$ is clearly injective. In [30, Lemma B.1] we compute a perturbative expression for $\ell_{\Omega,q}(e_j)$ when a domain Ω is close to the circle. In a proper sense perturbation of $\mathcal{T}_{\Omega_0}$ is also injective.

3.5. *Related prior results.* The problem of isospectral deformations of manifolds without boundary was considered in some early works on variations of the spectral functions and wave invariants.

Let (M, g) be a compact boundaryless Riemannian manifold. A family $(g_\tau)_{|\tau|\le 1}$ of Riemannian metrics on M depending smoothly on the parameter $|\tau| \le 1$ is called a *deformation of the metric* g if $g_0 = g$. A deformation is called *trivial* if there exists a one-parameter family of diffeomorphisms $\varphi_\tau : M \to M$ such that $\varphi_0 = \mathrm{id}$, and $g_\tau = (\varphi_\tau)^* g_0$. For each homotopy class of closed curves in M, consider the infimum of g-lengths of curves belonging to the given homotopy class. The *length spectrum* $\mathcal{L}(M, g)$ is defined as the union of these lengths over all homotopy classes. The *inverse spectral problem* in this setting is to show that two metrics with the same length spectrum are isometric.

Likewise, a deformation $(g_\tau)_{|\tau|\le 1}$ is said to be *isospectral* if $\mathcal{L}(M, g_\tau) = \mathcal{L}(M, g)$. We say that a Riemannian manifold (M, g) is *length spectrally rigid* if it does not admit non-trivial isospectral deformations.

It is worth mentioning that there is a partial solution of the inverse spectral problem due independently to Croke [25] and Otal [75] which can be stated as follows: any negatively curved manifold is uniquely determined by its *marked length spectrum* (see §3.6 for the corresponding billiard problem) (The *marked length spectrum* in the case of negatively curved surfaces without boundary consists of the set of pairs of free homotopy classes and length of the shortest geodesic in that homotopy class.)

Recently, Guillamou and Lefeuvre [41] proved that in all dimensions, the marked length spectrum of a Riemannian manifold (M, g) with Anosov geodesic flow and non-positive curvature locally determines the metric in the sense that two close enough metrics with the same marked length spectrum are isometric.

Remark 3.10. Observe that the inverse (length) spectral problem makes sense only under additional assumptions on (M, g), such as no conjugate points or non-positive (respectively, negative) curvature. Otherwise the union of the closed geodesics (minimizing in their homotopy class) may not be dense in M so that one can arbitrarily increase the metric outside this set without changing the length spectrum. More specifically, one can modify a Riemannian manifold by strongly increasing an arbitrary metric in a ball around a given point (see, for example, [17, Figure 1]); as a consequence, minimal (in the universal cover) geodesics are not dense on the manifold and therefore the union of maximal geodesics cannot determine the geometric of a manifold (they cannot determine this 'big bump'). Compare also with Remark 3.11.

Another example of deformational spectral rigidity appears in De la Llave, Marco and Moriyón [28]. Recall that one can associate to a symplectic map a generating function. Then, for each periodic orbit, one can define the corresponding *action* by summing the generating function along the orbit. This value of the action is invariant under symplectic coordinate changes. The union of the values of all these actions over all periodic orbits is called the action spectrum of the symplectic map. In [28, Theorem 1.3], it is proved that there are no non-trivial deformations of exact symplectic mappings $B_\tau,\ \tau \in [-1, 1]$, leaving the action spectrum fixed, when B_τ are Anosov's mappings on a symplectic manifold. One of the reasons for symplectic rigidity in [28] is that all periodic points of B_τ are hyperbolic and form a dense set.

3.6. *Marked length spectral rigidity.* One of the difficulties in working with the length spectrum is that all of this information on the periodic orbits comes in a non-formatted form. For example, we lose track of the rotation numbers corresponding to each length. A way to overcome this difficulty is to 'organize' this set of information more systematically, for instance, by labeling each length with corresponding rotation number. This new set is called the *marked length spectrum* of Ω and is denoted by $\mathcal{ML}_\Omega$:

$$\mathcal{ML}_\Omega := \{(\text{length}(\gamma), \text{rot}(\gamma)) :\ \gamma \text{ periodic orbit of the billiard in } \Omega\},$$

where $\text{rot}(\gamma)$ denotes the rotation number of γ.

One could also reduce this set of information by not considering the lengths of all orbits, but selecting some of them. More precisely, for each rotation number p/q in lowest terms, one could consider the maximal length among those having rotation number p/q. We call this map the *maximal marked length spectrum* of Ω, namely $\mathcal{ML}_\Omega^{\max} : \mathbb{Q} \cap [0, 1/2] \to \mathbb{R}$, given by

$$\mathcal{ML}_\Omega^{\max}(p/q) = \max\{\text{lengths of periodic orbits with rotation number } p/q\}. \qquad (3)$$

Marked spectral Rigidity Question. Let Ω_1 and Ω_2 be two strictly convex planar domains with smooth boundaries and assume that they are isospectral, that is, $\mathcal{ML}_{\Omega_1} \equiv \mathcal{ML}_{\Omega_2}$. Is it true that Ω_1 and Ω_2 are isometric? Similarly, one could ask whether this same question has an affirmative answer by asking only that $\mathcal{ML}_{\Omega_1}^{\max} \equiv \mathcal{ML}_{\Omega_2}^{\max}$.

Remark 3.11.

(i) The above question could be reformulated—and remains meaningful and interesting—by asking that the two domains are only isospectral near the boundary, that is, $\mathcal{ML}_{\Omega_1}^{\max}(p/q) = \mathcal{ML}_{\Omega_2}^{\max}(p/q)$ for all $p/q \in \mathbb{Q} \cap [0, \varepsilon)$, for some $0 < \varepsilon \le 1/2$.

(ii) In this regard, it is interesting to notice that, according to Lazutkin's result [61], there does not exist a billiard table Ω (strictly convex and with a sufficiently smooth boundary $\partial\Omega$) with a non-empty open subset of the boundary $U \subset \partial\Omega$ such that no maximal orbit hits U (in contrast to the Riemannian situation; see Remark 3.10). On the other hand, for every $\varepsilon > 0$ there exist a table Ω and an open subset of the boundary $U \subset \partial\Omega$, such that no maximal orbit of rotation number larger than ε hits U. In fact, given $\varepsilon > 0$, one can suitably deform a convex curve containing a point x_0 with curvature 0 to a (strictly convex) billiard. Since there are no maximal orbits through x_0 (see [68, Formula (4)]), one obtains examples analogous to the Riemannian case (Remark 3.10), but only for rotation numbers greater than $\varepsilon > 0$.

See §5 for a reformulation of this question in terms of what is known as Mather's minimal average action (or β*-function*) and for some partial answers to the marked length rigidity question related to the proof of the perturbative Birkhoff conjecture (see Corollary 5.9).

4. *Caustics: existence and integrability of the billiard map*

In this section we would like to recall the concept of a *caustic* of a billiard and discuss its relations with invariant curves for the billiard map. Let us introduce the concepts of caustic and integrability by means of two examples; the definition of caustic will be given in §4.3.

4.1. *Example 1: circular billiards.*

The easiest example of a billiard is given by a billiard in a disc $\mathcal{D}$ (e.g. of radius R). It is easy to check in this case that the angle of reflection remains constant at each reflection (see also [**95**, Ch. 2]). If we denote by s the arc-length parameter (that is $s \in \mathbb{R}/2\pi R\mathbb{Z}$) and by $\varphi \in (0, \pi/2]$ the angle of reflection, then the billiard map has a very simple form (see Figure 2):

$$f(s, \varphi) = (s + 2R\,\varphi,\ \varphi).$$

In particular, φ stays constant along the orbit and represents an *integral of motion* for the map; hence, the property of the orbit is determined by the corresponding angle $\varphi = \pi\omega$, with $\omega \in (0, 1)$.

- If $\omega = (p/q) \in (0, 1/2] \cap \mathbb{Q}$, in lowest terms, then the orbit is periodic with minimal period q. In particular, it closes after q rebounces and it winds p times around the disc before closing.
- If $\omega \in (0, 1/2] \setminus \mathbb{Q}$, then the orbit is not periodic and it hits the boundary $\partial\mathcal{D}$ on a dense set of points (by Kronecker's theorem).

Moreover, this billiard enjoys the peculiar property that all orbits with $\varphi = \pi\omega$ are tangent to the same concentric circle of radius $R \cos \pi\omega$ (see Figure 2); this concentric circle is an example of *caustics* (see Definition 4.3) and it is related to the existence of a homotopically non-trivial invariant curve for the corresponding billiard map, namely, $\mathcal{C}_\omega = \mathbb{R}/2\pi R\mathbb{Z} \times \{\pi\omega\}$ (this relation between caustics and invariant curves is more subtle; see Remark 4.5). Observe that the whole phase space of the circular billiard map (which is topologically a cylinder) is completely foliated by these $\mathcal{C}_\omega$ and, looking at the billiard table, this is completely foliated by caustics (this foliation is a singular foliation, due to the special role of the center of the disc): in this regard, circular billiards are example of *integrable billiards* (see Figure 3).

4.2. *Example 2: elliptic billiards.*

As a second example, let us consider the billiard inside an ellipse

$$\mathcal{E} = \left\{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$$

with $0 < b \le a$.

Optical properties of conics (an alternative way to consider the billiard ball motion inside a conic) were already well known to ancient Greeks. We refer to [**95**] for a more detailed discussion (see also [**87**]). In particular, each trajectory which does not pass

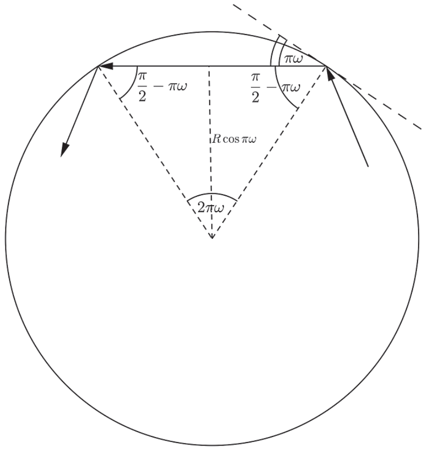
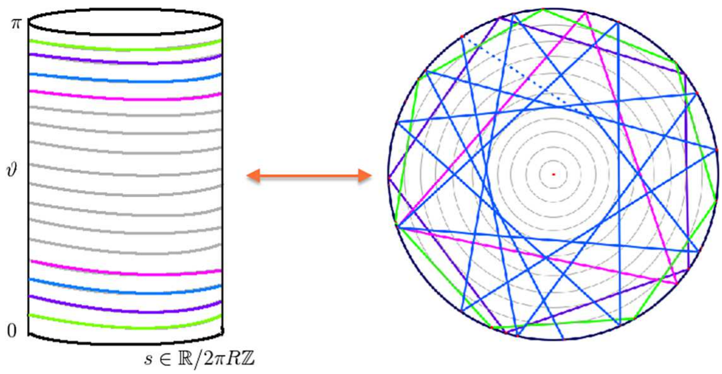


FIGURE 2. Billiard in a disc.



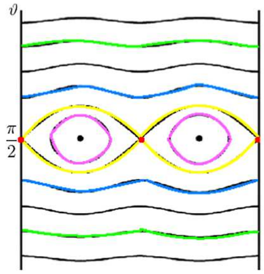
(a) (b)

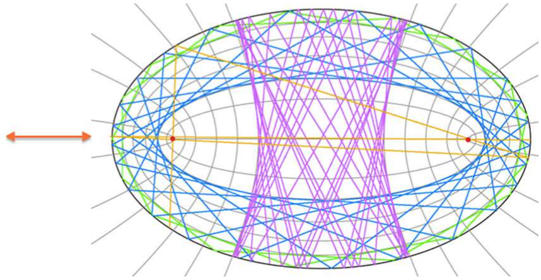
FIGURE 3. Circular billiard. (a) Phase space of the circular billiard map. (b) Foliation of the circular billiard table by caustics.

through a focal point is always tangent to precisely one confocal conic section. More specifically, billiard trajectories can be classified in the following way:

(a) trajectories that always intersect the open segment between the two foci,
(b) trajectories that never intersect the closed segment between the two foci, and
(c) trajectories that alternatively pass through one of the two foci.

In particular, each trajectory in (a) is tangent to a confocal hyperbola, each trajectory in (b) is tangent to a confocal ellipse, while trajectories in (c) tend asymptotically to the major semiaxis (which corresponds to a hyperbolic orbit of period 2). Confocal ellipses are therefore examples of caustics and they foliate everything but the closed segment between the two foci (see Figure 4). Hence, this could also be considered as an example of an integrable billiard. Observe that hyperbolae can also be considered examples of caustics, although, differently from concentric circles or confocal ellipses, they are not connected, closed or convex; see §4.3 for a more precise discussion.





(a) (b)

FIGURE 4. Elliptic billiard. (a) Phase space of an elliptic billiard map. (b) Dynamics inside an elliptic billiard and caustics.

Analytic descriptions of the dynamics and the integral of motion are not as easy as in the circular case, yet they can be done by means of elliptic functions and elliptic integrals; we refer the reader to [**23**, **96**] for more details.

4.3. *Caustics.* Let us introduce the concept of *convex caustic* and its relation to invariant curves for the billiard map. (*Caustic* comes from the Greek word καυστικός (kaustikós), meaning 'burning'; this terminology is related to optics and refers to the envelope of reflected or refracted rays of light, that is, the concentration of lights that can potentially lead to burns.) We refer to [**46**] for a more detailed (and extended) presentation of these topics. We also discuss some results and questions about their existence.

Let us start by recalling the definition of *invariant circle* for a billiard map.

Definition 4.1. We say that a curve $\gamma \subset M$ is an *invariant circle* for the billiard map B_Ω (we recall that M denotes the phase space of the billiard map), if γ is isotopic to a boundary component of M and $B_\Omega(\gamma) = \gamma$.

Remark 4.2.

(i) Observe that both boundary components of M are trivial invariant circles. It follows from Birkhoff's theorem that invariant circles must be Lipschitz graphs (see [**19**] and also [**87**, Theorem 1.3.3]).

(ii) Clearly, a billiard map may possess invariant curves that are not invariant circles: see, for example, the billiard map in an ellipse (see §4.2) and its homotopically trivial (disconnected) invariant curves, corresponding to orbits intersecting the segment between foci).

In the spirit of what we have seen in the examples of circular and elliptic billiards (see §§4.1 and 4.2), let us give the following definition.

Definition 4.3. A C^1 simple closed curve Γ in the interior of Ω is called a *convex caustic* for the billiard map B_Ω, if γ bounds a convex set D_Γ and any supporting line to D_Γ remains a supporting line to D_Γ after the billiard reflection in Ω. In other words, every time a trajectory is tangent to Γ, it remains tangent after every each reflection (Figure 5).

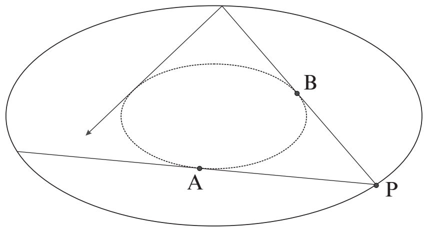


FIGURE 5. Caustic and Lazutkin invariant (from [87, Figure 3.6], used with kind permission).

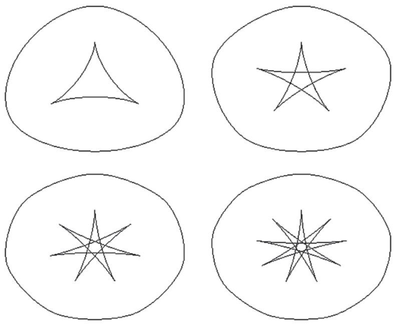


FIGURE 6. Examples of non-convex caustics in billiards of constant width (from [60, Figure 6], used with kind permission).

In our discussion, we will focus on convex caustics; however, one could consider a more general notion of caustic that does not require the properties of bounding a convex region, of being closed (see, for example, confocal hyperbolas for elliptic billiards in §4.2), or of being necessarily C^1. Since this will not be the object of our investigation, we refer to the discussion in [4, 46, 60]. See Figure 6 for some examples.

Remark 4.4. An interesting example of billiard maps with invariant circles are billiards whose boundary is a curve of *constant width*, that is, a curve that bounds a convex planar region whose width (defined as the perpendicular distance between two distinct parallel lines each having at least one point in common with the region's boundary but none with its interior) is the same regardless of the orientation of the curve (to construct such curves, see, for example, [60, §4] and [95, Exercise 3.13]). The corresponding billiard map has an invariant circle consisting of 2-periodic orbits. These curves correspond to caustics that, in general, may have cusps; see [60, §4 and Figure 6]. Billiard tables (other than ellipses) with a one-parameter family of 3-periodic trajectories have been constructed by Innami in [54].

Remark 4.5. One could wonder about the relation between caustics for the billiard in Ω and invariant circles for the corresponding billiard map B_Ω. One can show that to a convex caustic in Ω (not necessarily C^1) there corresponds an invariant circle for the billiard map. However, caustics corresponding to an arbitrary invariant circle may have a complicated structure and may not in general be convex or differentiable; we refer to [**4**, **46**, **60**] for more details.

Remark 4.6.

(i) Observe that every convex caustic has a well-defined rotation number. In fact, the dynamics tangent to it (that is the map that associates to each point on the boundary the first point (with respect to a chosen orientation) such that the segment joining these two points is tangent to the caustic) induces a circle homeomorphism from the boundary to itself; the rotation number of the caustic corresponds to the Poincaré rotation number of this circle homeomorphism.

(ii) The notion of caustics is often connected to the so-called *whispering gallery*, a phenomenon that can be detected under some particular domes, in which whispers can be clearly transmitted and received from distant parts of the gallery, as long as the speaker and listener are close the wall.

(iii) If Γ_ω is a convex caustic with rotation number $\omega \in (0, 1/2]$, then one can associate to it an invariant, the so-called *Lazutkin invariant* $Q(\Gamma_\omega)$. More precisely,

$$Q(\Gamma_\omega) = |A - P| + |B - P| - |AB| \tag{4}$$

where $|\cdot|$ denotes the Euclidean length and $|AB|$ the length of the arc on the caustic joining A to B (see Figure 5). This quantity is connected to the value of Mather's α-function (see §5).

4.4. *Existence of (convex) caustics.* A natural question that one could ask is whether the existence of (convex) caustics is a common or a rare phenomenon. As we have seen before, circular and elliptic billiards possess many convex caustics.

Furthermore, are there other Birkhoff billiards with (convex) caustics? And given an affirmative answer, *how many of them is it reasonable to expect?*

In the following we will often write 'caustic' in place of 'convex caustic' (unless differently specified). However, most of these questions can be addressed for more general notions of caustics.

Constructing a Birkhoff billiard with *at least one* caustic is easy: it is enough to perform the so-called *string construction*, similarly to the well-known one of drawing a circle as the set of points equidistant from a fixed center, or of constructing an ellipse as the locus of points whose distances from two fixed points have a constant sum. More specifically (see, for example, [**95**, Ch. 5] for a more precise construction), given a curve γ, one could wrap a closed non-stretchable string around it (of length longer than that of γ), pull it tight at a point and move this point around γ: the curve that one obtains corresponds to a billiard domain that has γ as a caustic.

Are there other billiards with *infinitely many* caustics? Quite surprisingly, the answer is affirmative: all (sufficiently smooth) Birkhoff billiards have infinitely many smooth convex caustics that accumulate to the boundary of the billiard domain. In [**61**], in fact, Lazutkin introduced a very special change of coordinates that reduces the billiard map B_Ω to a very simple form. Let $L_\Omega : \mathbb{R}/\ell\mathbb{Z} \times [0,\pi] \to \mathbb{R}/\mathbb{Z} \times [0,\delta]$ with small $\delta > 0$ be given by

$$L_\Omega(s,\varphi) := \left(x \;=\; C_\Omega^{-1} \int_0^s \rho^{-2/3}(s)\,ds,\;\; y \;=\; 4C_\Omega^{-1}\rho^{1/3}(s)\sin\varphi/2\right),$$

where ρ denotes the radius of curvature of $\partial\Omega$, and $C_\Omega := \int_0^\ell \rho^{-2/3}(s)\,ds$ (sometimes called the *Lazutkin perimeter*). In these new coordinates the billiard map has a simpler expression:

$$B_\Omega^L(x,y) = (x + y + O(y^3),\, y + O(y^4)).$$

In particular, near the boundary $\{y = 0\}$, this map can be seen as a small perturbation of the integrable map $(x,y) \longmapsto (x+y,y)$, and hence, under suitable regularity assumptions, the Kolmogorov–Arnold–Moser theorem can be applied (it is sufficient, for example, that $\partial\Omega$ is C^6, so that the map is at least C^5). Hence, there exists a positive-measure Cantor set of smooth invariant circles for the map which accumulates on $\{y = 0\}$ and on which the motion is smoothly conjugate to a rigid rotation with Diophantine rotation number (see [**61**, **77**] for a refined version); this translates into the existence of a positive measure set of caustics, accumulating to the boundary of the billiard table.

Remark 4.7. Observe that in this context it is extremely important that Ω is strictly convex. In [**68**], in fact, Mather proved the non-existence of caustics if the curvature of the boundary vanishes at one point. An alternative proof of this result is provided by Gutkin and Katok in [**46**], where the authors also investigate how the shape of the domain determines the location of caustics, establishing the existence of open regions which are free of caustics and estimating (from below) the size of these regions. More specifically, given a caustic Γ with Lazutkin invariant $L = L(\Gamma)$, if we denote by $\delta_{\max}(\Gamma,\partial\Omega)$ the maximum distance of Γ from the boundary $\partial\Omega$, they prove the following estimates (see [**46**, Propositions 1.2 and 1.3]):

$$\frac{\delta_{\max}^2(\Gamma,\partial\Omega)}{d} \le L \le \min\{2d^3\kappa^2, 2/K\},$$

where $d = d(\Omega)$ denotes the diameter of Ω, while κ and K are respectively the minimum and the maximum of the curvature of $\partial\Omega$. It follows from this that if $\kappa = 0$ at some point, then caustics cannot exist.

The next step then is to ask in which cases these caustics *foliate* the whole billiard table or an open dense subset of it, as happens in the circular and elliptic cases. In other words, *are there other examples of integrable billiards?* This apparently naïve question turns out to be much more difficult to tackle, and has given rise to one of the most famous (and somehow impenetrable) open problem in dynamical systems: the so-called *Birkhoff conjecture*.

4.5. *Integrable billiards and Birkhoff conjecture.* As we have seen in the previous subsection, billiards in a disc or in an ellipse are examples of *integrable billiards*. A natural question is the following: which Birkhoff billiards are integrable?

Birkhoff Conjecture. Circular and elliptic billiards are the only examples of integrable Birkhoff billiards.

Remark 4.8. Although some vague indications of this question can be found in [**18**], to the best of our knowledge, its first appearance as a conjecture was in a paper by Poritsky [**80**], where the author attributes it to Birkhoff himself. (Poritsky was a National Research Fellow in Mathematics at Harvard University, presumably under the supervision of Birkhoff. and [**80**] was published several years after Birkhoff's death.) Thereafter, references to this conjecture (as either the *Birkhoff conjecture* or the *Birkhoff–Poritsky conjecture*) repeatedly appeared in the literature: see, for example, Gutkin [**45**, §1], Moser [**71**, Appendix A] and Tabachnikov [**94**, §2.4].

Remark 4.9. We remark that the existence of caustics in dimension at least 3 is a much rarer phenomenon. In fact, it was proved in [**9**, **40**] that *among all convex billiard tables in* $\mathbb{R}^d$*,* $d \geq 3$*, only the solid ellipsoids have convex caustics, which are precisely the confocal solid ellipsoids contained in their interiors and, moreover, the intersection of these.* In particular, the analog of the Birkhoff conjecture in dimension at least 3 trivially holds.

The above conjecture assumes very different connotations and levels of complexity, according to the notion of integrability that one takes into account. Despite its long history and the amount of attention that it has captured in recent decades, many interesting formulations of this conjecture still remain unanswered.

We shall see in §5 how this conjecture/question can also be rephrased as a regularity question for Mather's minimal average action (or β*-function*).

4.5.1. *Global integrability.* In [**12**], Bialy proved the following result under the assumption of full global integrability.

THEOREM 4.10. (Bialy) *If the phase space of the billiard ball map is fully foliated by continuous invariant circles, then it is a circular billiard.*

Remark 4.11. An integral-geometric approach to proving Bialy's result was proposed by Wojtkowski in [**100**], by means of the so-called mirror formula. This approach was later exploited by Bialy [**13**] for billiards on the sphere and the hyperbolic plane, as well as for magnetic billiards.

Observe that Bialy and Wojtkowski's result is not in contrast with what we have discussed in the case of elliptic billiards. In fact, in that case the family of convex caustics represented by confocal ellipses do not foliate the whole domain (the segment between the two foci is left out) nor does the set of homotopically non-trivial invariant curves (invariant circles) have full ω-measure in the phase space: the homotopically trivial invariant curves corresponding to orbits tangent to confocal hyperbolae foliate a positive ω-measure set (in the phase portrait—see Figure 4—this set corresponds to the area below the separatrix, that is, the stable/unstable manifold of the hyperbolic 2-periodic orbit corresponding to the major semiaxis of the ellipse).

What about other notions of integrability? In the study of integrable systems, in fact, in most cases integrals of motion are non-degenerate not everywhere, but either on an open-dense subset of the phase space (we shall refer to this as *global integrability*) or on just a proper (non-trivial) open subset (we shall refer to this as *local integrability*).

Remark 4.12.

(i) An interesting result by Innami [55] shows that the existence of convex caustics with rotation numbers accumulating to $1/2$ implies that the billiard must be an ellipse. This regime of integrability is somehow opposite to the one we are interested in, which is concerned with caustics near the boundary of the billiard table, that is, with small rotation numbers. Innami's proof is based on Aubry–Mather theory; a simpler and more geometric proof of Innami's result was recently given in [4]. Observe that in this result it is decisive that the caustics are convex.

(ii) In this regard, Treschev [97] gave a numerical indication that there might exist analytic billiards, different from ellipses, for which the dynamics in a neighborhood of the elliptic period-2 orbit is conjugate to a rigid rotation. These billiards could be seen as an instance of *local integrability*; however, as we have already remarked, this regime is somehow complementary to the one usually considered for the Birkhoff conjecture, since it is concerned with integrability in a neighborhood of an elliptic periodic orbit of period 2. Very interestingly, this fact—if verified—would provide an intriguing indication that these regimes of integrability are significantly different.

4.5.2. *Perturbative Birkhoff conjecture.* Instead of considering all possible Birkhoff billiards, one could restrict the analysis to what happens for domains that are sufficiently close to ellipses and try to study the Birkhoff conjecture in this class of domains, which can be considered as *perturbations* of ellipses. More specifically, we can state the following perturbative version of Birkhoff conjecture.

Birkhoff Conjecture. (Perturbative version) A smooth strictly convex domain that is sufficiently close (with respect to some topology) to an ellipse and whose corresponding billiard map is integrable is necessarily an ellipse.

First results in this direction have been obtained:

- non-integrability of certain algebraic perturbations of elliptic billiards (Levallois [63] and Levallois and Tabanov [64]);
- non-integrability of entire symmetric perturbations of ellipses (these perturbations break integrability near the homoclinic solutions (Delshams and Ramírez-Ros [31]).

More recently, Avila, De Simoi and Kaloshin proved in [7] that the claim of the perturbative version of Birkhoff conjecture is true for domains that are sufficiently close to a circular billiard. The complete proof for domains sufficiently close to an ellipse of any eccentricity has been provided in [58].

Let us describe this result more precisely, starting with the following definition.

Definition 4.13. Let Ω be a strictly convex domain.

(i) We say Γ is an integrable rational caustic for the billiard map in Ω, if the corresponding invariant circle Γ consists of periodic points; in particular, the corresponding rotation number is rational.

(ii) Let $q_0 \geq 2$ be a positive integer. If the billiard map inside Ω admits integrable rational caustics for all rotation numbers $0 < (p/q) < (1/q_0)$, we say that Ω is q_0-rationally integrable.

The main result proved in [58] is the following theorem.

THEOREM 4.14. (Kaloshin and Sorrentino [58]) *Let $\mathcal{E}_{e_0,c}$ be an ellipse of eccentricity $0 \leq e_0 < 1$ and semifocal distance c; let $k \geq 39$. For every $K > 0$, there exists $\varepsilon = \varepsilon(e_0, c, K)$ such that if Ω is a 2-rationally integrable C^k-smooth domain, whose boundary $\partial\Omega$ is*

- *K-close to $\mathcal{E}_{e_0,c}$, with respect to the C^k-norm,*
- *ε-close to $\mathcal{E}_{e_0,c}$, with respect to the C^1-norm,*

then Ω is an ellipse.

Remark 4.15. Actually, it is sufficient to require only the existence of rational integral caustics of rotation number $1/q$, for all $q \geq 3$.

4.5.3. *Local integrability and Birkhoff conjecture.* What can be said for *locally integrable* Birkhoff billiards? As we have noticed in Remark 4.12, the correct regime that one should consider seems to be integrability in a neighborhood of the boundary of the billiard table, that is, for small rotation numbers.

Let us denote by $\mathcal{E}_{e,c} \subset \mathbb{R}^2$ an ellipse of eccentricity e and semifocal distance c. We state the following local version of Birkhoff conjecture.

Local Birkhoff Conjecture. For any integer $q_0 \geq 3$, there exist $e_0 = e_0(q_0) \in (0, 1)$, $m_0 = m_0(q_0)$, $n_0 = n_0(q_0) \in \mathbb{N}$ such that the following assertion holds. For each $0 < e \leq e_0$ and $c \geq 0$, there exists $\varepsilon = \varepsilon(e, c, q_0) > 0$ such that if $\mathcal{E}_{e,c}$ is an ellipse of eccentricity e and semifocal distance c, and Ω is a q_0-rationally integrable C^{m_0}-smooth domain, whose boundary $\partial\Omega$ is ε-close to $\mathcal{E}_{e,c}$, with respect to the C^{n_0}-norm, then Ω must be an ellipse.

This conjecture has been studied in [52]. More precisely, the following results have been proved.

THEOREM 4.16. (Huang, Kaloshin and Sorrentino [52])

(i) *The local Birkhoff conjecture holds true for $q_0 = 2, 3, 4, 5$, with $m_0 = 40q_0$ and $n_0 = 3q_0$.*
(ii) *The local Birkhoff conjecture holds true for $q_0 > 5$ with $m_0 = 40q_0$ and $n_0 = 3q_0$, subject to checking that $q_0 - 2$ matrices (which are explicitly described) are invertible.*

Remark 4.17.

(i) Case $q_0 = 2$ was proven in [7] (see also [55, 58]).
(ii) Smoothness exponents are probably not optimal.
(iii) Notice that in the proof we actually need only the existence of rationally integrable caustics of rotation numbers, less than $1/q_0$, of the form j/q for $j = 1, 2, 3$.
(iv) The invertibility condition on finitely many matrices, to which the claim of part (ii) of Theorem 4.16 is subject, is explicit and computable. In [53] it is

described how to implement an algorithm to verify it by means of symbolic computations. The coefficients of these matrices are completely determined by the e-expansion of the action-angle parametrization of the ellipse, which, in turn, is explicitly given by elliptic integrals; it turns out that the entries of these matrices are either 0, 1 or of the form $\xi \cos^{-2j}(w\pi)e^{2j}$, where $\xi \in \mathbb{Q}$, $j \in \mathbb{N}$, $w \in \{1/(2k+1), 2/(2k+1), 1/2k, 3/2k : k > j\}$.

4.5.4. *Non-perturbative results?* A possible strategy to extend our results to a non-perturbative version of this conjecture involves the use of some *geometric flow* to transform the domain into a small perturbation of an ellipse. Roughly speaking, the most important features of this flow should be:

- (i) preservation of strictly convexity and smoothness of the boundary;
- (ii) convergence (possibly up to some renormalization of the length or the area) to the set of elliptic domains, which must be an invariant set for the flow;
- (iii) preservation of integrability.

It is clear that if such a flow exists, then (i)–(iii) imply that any integrable Birkhoff billiard Ω can be mapped into an integrable Birkhoff billiard Ω' close to an ellipse; using our perturbative result, we can deduce that Ω' must be an ellipse; since the set of ellipses is invariant under the (backward) flow, it follows that Ω must also be an ellipse.

In [58, Appendix G] we suggested as a possible candidate the so-called *affine length shortening (ALS) flow* a flow describing the evolution of plane curves in the direction of the *affine* normal, with speed proportional to the *affine* curvature (see, for instance, [84] for more details). This flow satisfies properties (i) and (ii) (see [84]); the main obstacle lies in proving that property (iii) holds (if one believes in the Birkhoff conjecture, then it should hold, since ellipses are an invariant set for the flow). In [58, Appendix G] we proposed to prove this by introducing a family of functions measuring the *non-integrability* of the domains, and conjecturing that they behave as Lyapunov functions for the ALS flow.

We remark that property (iii) for the classical *Euclidean curve shortening flow* (that is the evolution is in the direction of the Euclidean normal with speed proportional to the Euclidean curvature) does not hold in general, as proved in [27].

4.6. *Some ideas on the proofs of the perturbative Birkhoff conjecture and its local version (Theorems 4.14 and 4.16).*

4.6.1. *Perturbative Birkhoff conjecture (Theorem 4.14).* Let us provide a description of the strategy that we adopted in [58] to prove Theorem 4.14.

For small eccentricities, Theorem 4.14 was proven in [7]. Let us start by describing the simplified setting of integrable infinitesimal deformations of a circle. This provides an insight into the strategy of the proof in the general case.

Let Ω_0 be a circle centered at the origin and with radius $\rho_0 > 0$. Let Ω_ε be a one-parameter family of smooth deformations given in the polar coordinates (ρ, φ) by

$$\partial\Omega_\varepsilon = \{(\rho, \varphi) = (\rho_0 + \varepsilon\rho(\varphi) + O(\varepsilon^2), \varphi)\}.$$

Consider the Fourier expansion of ρ:

$$\rho(\varphi) = \rho_0' + \sum_{k>0} \rho_k \sin(k\varphi) + \rho_{-k} \cos(k\varphi).$$

THEOREM 4.18. (Ramírez-Ros [81]) *If Ω_ε has an integrable rational caustic $\Gamma_{1/q}$ of rotation number $1/q$, for any ε sufficiently small, then $\rho_{kq} = \rho_{-kq} = 0$ for any integer k.*

Let us now assume that the domains Ω_ε are 2-rationally integrable for all sufficiently small ε and ignore for a moment the dependence on the parametrization. Then the above theorem implies that $\rho_k' = \rho_k'' = 0$ for $k > 2$, that is,

$$\begin{aligned}\rho(\varphi) &= \rho_0' + \rho_1' \cos\varphi + \rho_1'' \sin\varphi + \rho_2' \cos 2\varphi + \rho_2'' \sin 2\varphi \\ &= \rho_0' + \rho_1^* \cos(\varphi - \varphi_1) + \rho_2^* \cos 2(\varphi - \varphi_2)\end{aligned}$$

where φ_1 and φ_2 are appropriately chosen phases.

Remark 4.19. Observe that:

- ρ_0 corresponds to a homothety;
- ρ_1^* corresponds to a translation in the direction forming an angle φ_1 with the polar axis $\{\varphi = 0\}$;
- ρ_2^* corresponds to a deformation of the circle into an ellipse of small eccentricity, whose major axis forms an angle φ_2 with the polar axis.

This implies that, infinitesimally (as $\varepsilon \to 0$), rationally integrable deformations of a circle are tangent to the five-parameter family of ellipses.

In order to extend these ideas to the case of an integrable perturbation (not necessarily a deformation) of an ellipse, a more elaborate strategy is needed, involving more quantitative estimates and approximation procedure (we refer to [7, 58] for more technical details). In particular, Fourier modes are replaced by new functions determined by the dynamics inside the approximating ellipse, which we call *dynamical modes* $\{c_q, s_q\}_{q\geq 3}$, given by

$$\begin{aligned}c_q(\varphi) &:= \frac{\cos(2\pi q/4K(k_q)F(\varphi; k_q))}{\sqrt{1 - k_q^2 \sin^2\varphi}}, \\ s_q(\varphi) &:= \frac{\sin(2\pi q/4K(k_q)F(\varphi; k_q))}{\sqrt{1 - k_q^2 \sin^2\varphi,}}\end{aligned}$$

where k_q denotes the eccentricity of the confocal ellipse corresponding to the caustic of rotation number $1/q$, while

$$F(\varphi; k) := \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\tau}} \quad \text{and} \quad K(k) := F\left(\frac{\pi}{2}; k\right)$$

are the elliptic integrals of first kind (see, for example, [1] for more details on these functions and their properties).

The core of the proof consists in showing that these dynamical modes together with the infinitesimal generators of homotheties, translations, rotations and hyperbolic rotations (that is those transformations preserving the set of ellipses), form a basis of $L^2(\mathbb{R}/2\pi\mathbb{Z})$.

This is one of the main difficulties (perhaps the hardest) involved in the extension of the perturbative result in [7] to the case of perturbations of any ellipse, as studied in [58]. While in the former case one can take advantage of the fact that these functions can be considered small perturbations of the Fourier modes, in the latter new strategies need to be exploited. In [58] we consider analytic extensions of the action-angle coordinates of the elliptic billiard, more specifically, of the boundary parametrizations induced by each integrable caustic (these functions can be explicitly expressed in terms of elliptic integrals and Jacobi elliptic functions). A detailed study of their complex singularities and the size of their maximal strips of analyticity allowed us to deduce their linear independence (both for finite and infinite combinations) and, by a suitable codimension argument, to show that they form a complete set of generators, thus completing the proof that they are a basis of $L^2(\mathbb{R}/2\pi\mathbb{Z})$.

4.6.2. *Local Birkhoff conjecture for nearly circular domains (Theorem 4.16).* The main difficulty in this case (in comparison with the one discussed in Theorem 4.14 and §4.6.1) is that we cannot use the preservation of integrable rational caustics for all rotation numbers $1/q$, with $q \geq 3$; hence, we need to recover the missing conditions on the corresponding Fourier coefficients of the perturbation.

Our key idea is the following: for ellipses of small eccentricity $e > 0$, we study the Taylor expansion, with respect to e, of the corresponding action-angle coordinates. Using this expansion, we derive the necessary condition for the preservation of integrable rational caustics, in terms of the Fourier coefficients of the perturbation, up to precision of order e^{2N}, for some positive integer $N = N(q_0)$.

Let us outline our strategy, starting from some special cases.

- *Case* $q_0 = 3$. We lose a pair of conditions corresponding to Fourier coefficients of order 3. We exploit the conditions obtained from the existence of integrable rational caustics of rotation numbers $1/5$, $1/7$, $2/7$: we use the corresponding expansions, with respect to e, up to precision $O(e^6)$, to derive a system of linear equations for the third, fifth and seventh Fourier coefficients. Solving this linear system will provide us with the estimates needed for Fourier coefficients of order 3.
- *Case* $q_0 = 4$. In this case we lose two pairs of conditions corresponding to Fourier coefficients of order $q = 3, 4$. These will be recovered in two steps.
  - To recover the one corresponding to Fourier coefficients of order 3, we study the necessary conditions for the existence of integrable rational caustics of rotation numbers $1/5$, $1/7$, $1/9$, $2/9$, written in terms of the Fourier coefficients of the perturbation, and consider their expansions, with respect to e, up to order $O(e^8)$. We then derive a linear system for the third, fifth, seventh and ninth Fourier coefficients, whose solution will provide us with the estimates needed for the Fourier coefficients of order 3.
  - To recover the one corresponding to Fourier coefficients of order 4, we study the necessary conditions for the existence of integrable rational caustics of rotation numbers $1/6$, $1/8$, $1/10$, $1/12$, $1/14$, $3/14$, which give rise to a system of linear equation for the fourth, sixth, eighth, tenth, twelfth and fourteenth Fourier coefficients; as before, the solution of this linear system will give us the estimates needed for the Fourier coefficients of order 4.

- *The general case*. Along the same lines described in the previous two items, we outlined in [53] a general (conditional) procedure to deal with this problem for any $q_0 \geq 3$; the implementation of this scheme is based on the assumption that certain explicit non-degeneracy conditions for the corresponding linear systems hold. We remark, however, that all of these conditions are very explicit and the algorithm is explicitly described for implementation on a computer.

4.7. *Integrable Riemannian geodesic flows on the torus.* We conclude this section by drawing some connections between the Birkhoff conjecture and a problem in Riemannian geometry. The Birkhoff conjecture can be also thought of as an analog, in the case of billiards, to the task of classifying integrable (Riemannian) geodesic flows on $\mathbb{T}^2$. The complexity of this question, of course, depends on the notion of integrability that one considers. If one assumes that the whole phase space is foliated by invariant Lagrangian graphs (that is the system is C^0-*integrable* (see [3, Definition 4.19]), in particular, the integral of motion is only assumed to be continuous), then it follows from Hopf's result [51] (see also [22] for the proof in dimension greater than 2) that the associated metric must be flat. Bialy and Wojtkowski's results in the billiard setting can be considered as the analogs of this result.

However, the question becomes more challenging (and is still open) if one considers integrability only on an open and dense set (global integrability), or assumes the existence of an open set foliated by invariant Lagrangian graphs (local integrability). Examples of globally integrable (non-flat) geodesic flows on $\mathbb{T}^2$ are those associated to *Liouville-type metrics*, namely metrics of the form

$$ds^2 = (f_1(x_1) + f_2(x_2))\,(dx_1^2 + dx_2^2).$$

A folklore conjecture states that these metrics are the only globally (respectively, locally) integrable metrics on $\mathbb{T}^2$, which, in some sense, can be interpreted as the analog of the Birkhoff conjecture, in the realm of integrable geodesic flows on $\mathbb{T}^2$.

A partial answer to this conjecture (global case) is provided in [21], where the authors prove it under the assumption that the system admits an integral of motion which is quadratic in the momenta. Observe that while the case of quadratic integral of motion reduces to a system of linear partial differential equations, the case of higher degree integrals of motions is very challenging and it turns out to be equivalent to delicate questions on non-linear partial differential equations of hydrodynamic type (see, for example, [14, 15]). This notion of integrability is related to the so-called *algebraic integrability*, namely the existence of integrals of motion that are polynomial in the velocity. The relation between this notion of integrability and the Birkhoff conjecture (*algebraic Birkhoff conjecture*) has been studied and has lead to interesting results [16, 20]. Recently, using previous results of [16], Glutsyuk [36] proved the algebraic Birkhoff conjecture.

Finally, we point out that the topological structure of the torus plays a fundamental role in the above-mentioned conjectures and results. For example, on the two-dimensional sphere there are plenty of non-trivial integrable metrics: the so-called *Zoll surfaces*. A Zoll surface is a surface homeomorphic to the 2-sphere, equipped with a Riemannian metric all of whose geodesics are closed and of equal length (the first non-trivial example was discovered by Zoll in [104]). While the usual unit-sphere metric on $\mathbb{S}^2$ obviously has this property, there also exists an infinite-dimensional family of geometrically distinct deformations that are still Zoll surfaces (see, for example, [42]). In particular, most Zoll surfaces do not have constant curvature. See [11, 62] for more details.

5. *Aubry–Mather theory and billiard dynamics*

In this section we discuss how the study of *action-minimizing properties* of billiards can be used to shed some light on their dynamical properties. In particular, we shall see how many of the questions discussed in the previous sections can be rephrased in these terms. Let us start by briefly recalling the main ideas at the heart of this approach.

5.1. *Aubry–Mather theory for twist maps of the annulus.* At the beginning of the 1980s Serge Aubry and John Mather developed, independently, what nowadays is commonly called *Aubry–Mather theory*. This novel approach to the study of the dynamics of twist diffeomorphisms of the annulus pointed out the existence of many *action-minimizing orbits* for any given rotation number. For a more detailed introduction, see, for example, [33, 87, 90]).

Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$, with $a < b$, and let

$$f : \mathbb{R}/\mathbb{Z} \times (a, b) \longrightarrow \mathbb{R}/\mathbb{Z} \times (a, b)$$

be a monotone twist map, that is, a C^1 diffeomorphism such that its lift to the universal cover $\tilde{f}$ satisfies the following properties (we denote $(x_1, y_1) = \tilde{f}(x_0, y_0)$).

(i) $\tilde{f}(x_0 + 1, y_0) = \tilde{f}(x_0, y_0) + (1, 0)$ and $x_0 \le x_1 < x_0 + 1$.

(ii) $\tilde{f}$ is orientation preserving and preserves the boundaries of $\mathbb{R} \times (a, b)$:

$$y_1(x_0, y_0) \to a \quad \text{as} \quad y_0 \to a \quad \text{and} \quad y_1(x_0, y_0) \to b \quad \text{as} \quad y_0 \to b.$$

(iii) If $a > -\infty$, then $\tilde{f}$ extends continuously to $\mathbb{R} \times \{a\}$ by a rotation:

$$\tilde{f}(x, a) = (x + \omega_-, a).$$

Similarly, if $b < +\infty$, then $\tilde{f}$ extends continuously to $\mathbb{R} \times \{b\}$ by a rotation:

$$\tilde{f}(x, b) = (x + \omega_+, b).$$

(iv) $(\partial x_1/\partial y_0) \ge c > 0$ (monotone twist condition).

(v) $\tilde{f}$ admits a (periodic) generating function h (that is it is an exact symplectic map):

$$y_1\, dx_1 - y_0\, dx_0 = dh(x_0, x_1).$$

We call the interval $(\omega_-, \omega_+) \subset \mathbb{R}$ the twist interval of f (we remark that if $a = -\infty$ then $\omega_- = -\infty$, and if $b = +\infty$ then $\omega_+ = +\infty$.

In particular, it follows from (v) that

$$\begin{cases} y_1 = \dfrac{\partial h}{\partial x_1}(x_0, x_1), \\ y_0 = -\dfrac{\partial h}{\partial x_0}(x_0, x_1). \end{cases} \tag{5}$$

Remark 5.1. The billiard map is an example of monotone twist map (to fit with the above definition, one can normalize the boundary length to be equal to 1). In particular, as we have already pointed out, its generating function (see (1)) is given by $h(x_0, x_1) = -\ell(x_0, x_1)$, where $\ell(x_0, x_1)$ denotes the Euclidean distance between the two points on the boundary of the billiard domain corresponding to $\gamma(x_0)$ and $\gamma(x_1)$.

As follows from (5), orbits $(x_i)_{i\in\mathbb{Z}}$ of the monotone twist diffeomorphism f correspond to 'critical points' of the *action functional*

$$\{x_i\}_{i\in\mathbb{Z}} \longmapsto \sum_{i\in\mathbb{Z}} h(x_i, x_{i+1}).$$

Aubry–Mather theory is concerned with the study of orbits that minimize this action-functional among all configurations with a prescribed rotation number; recall that the rotation number of an orbit $\{x_i\}_{i\in\mathbb{Z}}$ is given by $\omega = \lim_{i\to\pm\infty}(x_i/i)$, if this limit exists (in the billiard case, this definition leads to the same notion of rotation number introduced in §3). In this context, *minimizing* is meant in the statistical mechanical sense, that is, every finite segment of the orbit minimizes the action functional with fixed end-points.

THEOREM 5.2. (Aubry [**5**, **6**], Mather [**33**, **67**]) *A monotone twist map possesses minimal orbits for every rotation number in its twist interval* (ω_-, ω_+). *Moreover, every minimal orbit lies on a Lipschitz graph over the x-axis.*

We can now introduce the *minimal average action* (or *Mather's β-function*).

Definition 5.3. Let $x^\omega = \{x_i\}_{i\in\mathbb{Z}}$ be any minimal orbit with rotation number ω. Then, the value of the *minimal average action* at ω is given by

$$\beta(\omega) := \lim_{N\to+\infty} \frac{1}{2N} \sum_{i=-N}^{N-1} h(x_i, x_{i+1}). \tag{6}$$

This value is well defined, since it does not depend on the chosen orbit.

The function $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ enjoys many properties and encodes interesting information on the dynamics. In particular:

(i) β is strictly convex and, hence, continuous (see [**33**]);
(ii) β is differentiable at all irrationals (see [**69**]);
(iii) β is differentiable at a rational p/q if and only if there exists an invariant circle consisting of periodic minimal orbits of rotation number p/q (see [**69**]).

In particular, β being a convex function, one can consider its convex conjugate,

$$\alpha(c) = \sup_{\omega\in\mathbb{R}}[\omega\, c - \beta(\omega)].$$

This function (which is generally called *Mather's α-function*) also plays an important role in the study of minimal orbits and in Mather's theory (particularly in higher dimension; see, for example, [**66**, **91**]). We refer interested readers to the surveys [**33**, **87**, **90**].

Observe that for each ω and c one has

$$\alpha(c) + \beta(\omega) \geq \omega c,$$

where equality is achieved if and only if $c \in \partial\beta(\omega)$ or, equivalently, if and only if $\omega \in \partial\alpha(c)$ (the symbol ∂ denotes in this case the set of 'subderivatives' of the function, which is always non-empty and is a singleton if and only if the function is differentiable).

5.2. *Action-minimizing properties of billiards.* In the billiard case, since the generating function of the billiard map is the Euclidean distance $-\ell$, the action of the orbit coincides (up to sign) with the length of the trajectory that the ball traces on the table Ω. In particular, these two functions encode many dynamical properties of the billiard (see [86, 87] for more details).

- For each $0 < p/q \leq 1/2$, one has $\beta(p/q) = -(1/q)\mathcal{ML}_{\Omega}^{\max}(p/q)$; see (3) for the definition of $\mathcal{ML}_{\Omega}^{\max}(p/q)$.
- β is differentiable at p/q if and only if there exists an invariant circle of rotation number p/q foliated by periodic orbits.
- If Γ_ω is a convex caustic with rotation number $\omega \in (0, 1/2]$, then β is differentiable at ω and $\beta'(\omega) = -\text{length}(\Gamma_\omega) =: -|\Gamma_\omega|$ (see [87, Theorem 3.2.10]). In particular, β is always differentiable at 0 and $\beta'(0) = -|\partial\Omega|$, where $|\partial\Omega|$ denotes the length of the boundary of Ω.
- If Γ_ω is a convex caustic with rotation number $\omega \in (0, 1/2]$, then its Lazutkin invariant $Q(\Gamma_\omega)$ (see §4.3) can be related to the value of the α-function. In fact, one can show that (see [87, Theorem 3.2.10])

$$Q(\Gamma_\omega) = \alpha(\beta'(\omega)) = \alpha(-|\Gamma_\omega|).$$

In [10, 70, 86, 87, 89] properties of Mather's β- and α-functions have been studied more in depth. In particular, explicit expressions for their (formal) Taylor expansions at, respectively, $\omega = 0$ and $c = -|\partial\Omega|$ have been obtained. The coefficients in these expressions will be obtained in terms of the curvature of the boundary and its derivatives.

THEOREM 5.4. *Let Ω be a strictly convex planar domain with smooth boundary. Denote by $k(s) > 0$ the curvature of $\partial\Omega$ with arc-length parametrization s. Let $\ell_0 := |\partial\Omega|$ be the length of the boundary and denote*

$$\mathcal{I}_1 := \int_0^{\ell_0} ds = \ell_0,$$

$$\mathcal{I}_3 := \int_0^{\ell_0} k^{2/3}\, ds,$$

$$\mathcal{I}_5 := \int_0^{\ell_0} \left(9\, k^{4/3} + \frac{8\, \dot{k}^2}{k^{8/3}}\right) ds,$$

$$\mathcal{I}_7 := \int_0^{\ell_0} \left(9\, k^2 + \frac{24\, \dot{k}^2}{k^2} + \frac{24\, \ddot{k}^2}{k^4} - \frac{144\, \dot{k}^2 \ddot{k}}{k^5} + \frac{176\, \dot{k}^4}{k^6}\right) ds,$$

$$\mathcal{I}_9 := \int_0^{\ell_0} \left[\frac{281}{44800} k^{8/3} + \frac{281\, \dot{k}^2}{8400\, k^{4/3}} + \frac{167\, \ddot{k}^2}{4200\, k^{10/3}} - \frac{167\, \dot{k}^2\, \ddot{k}}{700\, k^{13/3}} + \frac{\dddot{k}^2}{42\, k^{16/3}} \right.$$
$$+ \frac{559\, \dot{k}^4}{2100\, k^{16/3}} - \frac{473\, \ddot{k}^3}{4725\, k^{19/3}} - \frac{10\, \dddot{k}\, \dot{k}\, \ddot{k}}{21\, k^{19/3}} + \frac{5\, \dddot{k}\, \dot{k}^3}{7\, k^{22/3}} + \frac{13142\, \dot{k}^2\, \ddot{k}^2}{4725\, k^{22/3}}$$
$$\left. - \frac{10777\, \dot{k}^4\, \ddot{k}}{1575\, k^{25/3}} + \frac{521897\, \dot{k}^6}{127575\, k^{28/3}} \right] ds.$$

Then:

- *the formal Taylor expansion of β at $\omega = 0$, $\beta(\omega) \sim \sum_{k=0}^{\infty} \beta_k (\omega^k / k!)$, has coefficients*

$$\begin{aligned}
\beta_{2k} &= 0 \quad \textit{for all } k,\\
\beta_1 &= -\mathcal{I}_1,\\
\beta_3 &= \tfrac{1}{4}\, \mathcal{I}_3^3,\\
\beta_5 &= -\frac{1}{144}\, \mathcal{I}_3^4\, \mathcal{I}_5,\\
\beta_7 &= \frac{1}{320}\, \mathcal{I}_3^5 \left(\frac{14}{81} \mathcal{I}_5^2 - \mathcal{I}_3 \mathcal{I}_7 \right), \; = \frac{\mathcal{I}_3^5 (14\, \mathcal{I}_5^2 - 81\, \mathcal{I}_3 \mathcal{I}_7)}{25920}\\
\beta_9 &= -7\, \mathcal{I}_3^6 \left(\mathcal{I}_3^2\, \mathcal{I}_9 - \frac{1}{5600} \mathcal{I}_3\, \mathcal{I}_5\, \mathcal{I}_7 + \frac{7}{583200} \mathcal{I}_5^3 \right);
\end{aligned}$$

- *the (formal) Taylor expansion of $(c+\ell_0)^{-3/2}\alpha(c)$ at $c = -\ell_0$ (note that α has in fact a square-root type singularity at the boundary), $(c+\ell_0)^{-3/2}\alpha(c) \sim \sum_{k=0}^{\infty} \alpha_k ((c+\ell_0)^k)/k!$, has coefficients*

$$\begin{aligned}
\alpha_0 &= \frac{4\sqrt{2}}{3} \mathcal{I}_3^{-3/2},\\
\alpha_1 &= \frac{\sqrt{2}}{135} \mathcal{I}_3^{-7/2} \mathcal{I}_5,\\
\alpha_2 &= \frac{1}{56700\sqrt{2}} \left(\frac{72\, \mathcal{I}_3 \mathcal{I}_7 + 7\, {\mathcal{I}_5}^2}{{\mathcal{I}_3}^{11/2}} \right),\\
\alpha_3 &= \frac{1}{826686000\sqrt{2}} \left(\frac{261273600\, {\mathcal{I}_3}^2 \mathcal{I}_9 + 21384\, \mathcal{I}_3 \mathcal{I}_5 \mathcal{I}_7 + 1001\, {\mathcal{I}_5}^3}{{\mathcal{I}_3}^{15/2}} \right).
\end{aligned}$$

Remark 5.5.

(i) The techniques used in the proof of the Theorem 5.4, allow one to obtain explicit expressions up to any arbitrary high order (we restrict to order 11 just for the sake of this presentation).

(ii) The coefficients β_k are algebraically related to the set of spectral invariants introduced by Marvizi and Melrose [65] for strictly convex planar regions in order to investigate and give some partial answers to Kac's question on the isospectrality of planar domains. These computations provide explicit expressions for those invariants as well (see the expressions for the $\mathcal{I}_k$).

An easy consequence of these formulae is the following corollary, which is a direct consequence of the isoperimetric inequality (see [89, Corollary 1] and [86, 87]).

COROLLARY 5.6. *Let Ω be a strictly convex planar domain with smooth boundary. Then*

$$\beta_3 + \pi^2 \beta_1 \leq 0,$$

and equality holds if and only if Ω is a disc.

Proof. The proof easily follows from the expressions for β_1 and β_3, found in Theorem 5.4. In fact, observe that

$$\beta_3 + \pi^2 \beta_1 \leq 0 \quad \Longleftrightarrow \quad \mathcal{I}_3^3 - 4\pi^2 \mathcal{I}_1 \leq 0.$$

Now, using Hölder's inequality (with $p = \frac{3}{2}$ and $q = 3$) and the Gauss–Bonnet theorem,

$$\begin{aligned} \mathcal{I}_3 = \int_0^{\ell_0} k^{2/3}\, ds &\leq \left(\int_0^{\ell_0} (k^{2/3})^{3/2}\, ds \right)^{2/3} \left(\int_0^{\ell_0} 1^3\, ds \right)^{1/3} \\ &= (2\pi)^{2/3} {\ell_0}^{1/3} = (4\pi^2 \mathcal{I}_1)^{1/3}. \end{aligned}$$

Moreover, equality holds if and only if it holds in the Hölder inequality. This means that k must be constant (and strictly positive), and therefore the curve must be a circle. □

Remark 5.7. In particular, the above corollary says that if the first two coefficients β_1 and β_3 coincide to those of the β-function of a disc, then the domain must be a disc. Therefore, the β-function univocally determines discs amongst all possible Birkhoff billiards. It would be interesting to find a similar characterization for elliptic billiards. We can prove the following result: the β-function determines univocally a given ellipse in the family of all ellipses.

PROPOSITION 5.8. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two ellipses such that $\beta_{\mathcal{E}_1} \equiv \beta_{\mathcal{E}_2}$, then $\mathcal{E}_1$ and $\mathcal{E}_2$ are the same ellipse. More generally, if the Taylor coefficients $\beta_{\mathcal{E}_1,1} = \beta_{\mathcal{E}_2,1}$ and $\beta_{\mathcal{E}_1,3} = \beta_{\mathcal{E}_2,3}$, then the same conclusion remains true.*

The proof easily follows from expressing these coefficients by means of elliptic integrals (see [89, Proposition 1]).

5.3. *Birkhoff conjecture and spectral rigidity questions (revisited).* We can now rephrase the spectral rigidity question for the maximal length spectrum (see §3.1) and Birkhoff conjecture (see §4.5) in terms of these new objects.

Spectral Rigidity Question. (Revisited) Let Ω_1 and Ω_2 be two strictly convex planar domains with smooth boundaries and assume that $\beta_{\Omega_1} \equiv \beta_{\Omega_2}$. Is it true that Ω_1 and Ω_2 are isometric? More generally, if $\beta_{\Omega_1}(\omega) = \beta_{\Omega_2}(\omega)$ for all $\omega \in (0, \varepsilon)$ for some small $\varepsilon > 0$, is it true that Ω_1 and Ω_2 are isometric?

Similarly, keeping in mind the relation between the differentiability properties of Mather's β-function at rational rotation numbers and the existence of invariant circles foliated by periodic points (see §5.2), we can also rephrase Birkhoff conjecture in this context.

Birkhoff Conjecture. (Revisited) Let Ω be a strictly convex planar domain with smooth boundary and assume that β_Ω is differentiable in $[0, 1/2)$. Is it true that Ω is an ellipse? More generally, if β_Ω is differentiable in $[0, \varepsilon)$ for some small $0 < \varepsilon < 1/2$, is it true that Ω is an ellipse?

In fact, if β_Ω is differentiable in an open interval, then the billiard map is locally integrable in an open set. In fact, β_Ω will be differentiable at all rationals in that interval and therefore there will be caustics corresponding to these rotation numbers. By semicontinuity arguments, one obtains caustics corresponding to irrational rotation numbers and hence a family of caustics that foliate an open set. Observe that if β is differentiable in the whole domain of definition $(0, 1/2]$, then it must be a circle by the aforementioned result of Bialy.

The relation between the integrability of the billiard map and the differentiability of the corresponding Mather β-function implies that a solution to the Birkhoff conjecture would lead to a solution to the question whether ellipses are uniquely spectrally determined in the class of convex domains. The final result follows from the proof of the perturbative Birkhoff conjecture (see [**58**, Corollaries 13 and 14]).

COROLLARY 5.9. (Kaloshin and Sorrentino) *Let Ω be a smooth strictly convex domain Ω sufficiently close to an ellipse.*

(i) *If Ω has the same maximal marked length spectrum (or Mather's β-function) as an ellipse, then it is an ellipse.*
(ii) *If its Mather's β-function is differentiable at all rationals $1/q$ with $q \geq 3$, then Ω is an ellipse.*
(iii) *Ellipses are* (maximal) marked-length-spectrally rigid*, meaning that if Ω_t is a smooth deformation of an ellipse which keeps the (maximal) marked length spectrum fixed, then it consists of a rigid motion.*
(iv) *Ellipses are* length-spectrally rigid*, meaning that if Ω_t is a smooth deformation of an ellipse which keeps the length spectrum fixed, then it consists of a rigid motion.*

Remark 5.10.

(i) Compare this result with the previously mentioned result of Hezari and Zelditch [**49**] (see §3.2), where it is proved that ellipses of sufficiently small eccentricities are Laplace spectrally unique (up to isometry) among all smooth domains (without any assumption on symmetry, convexity, or closeness to other ellipses).
(ii) In a recent paper [**50**], Hezari and Zelditch prove that, for any invariant curve for the billiard map on the boundary phase space of an ellipse, there exists a sequence of eigenfunctions whose Cauchy data (that is eigenfunctions of the semiclassical eigenvalue problem; see [**50**, Formula (1)]) concentrate on the invariant curve; in particular, they use this result to give a new proof that ellipses are infinitesimally spectrally rigid among C^∞ domains with the symmetries of the ellipse.

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