On the Integrability of Birkhoff Billiards

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(joint work with Guan Huang, Vadim Kaloshin)

A *Birkhoff billiard* is a dynamical model describing the motion of a billiard ball inside a strictly convex domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$. The massless ball moves with unit velocity and without friction following a rectilinear path; when it hits the boundary it reflects elastically according to the standard *reflection law*: the angle of reflection is equal to the angle of incidence.

This conceptually simple model, yet dynamically very rich, has been proposed by G. D. Birkhoff as a mathematical playground where "[...]*the formal side, usually so formidable in dynamics, almost completely disappears, and only interesting qualitative questions need to be considered*" [3, pp. 155-156].

Since then, billiards have captured the attention of many researchers in various areas of mathematics. Whereas it is clear how the geometry (*i.e.*, the shape) of the domain determines the billiard dynamics, a more subtle and intriguing question is to which extent dynamical information can be used to reconstruct the shape of the billiard domain. This translates into compelling inverse problems and rigidity questions, that provide the ground for some of the foremost conjectures in dynamical systems.

In this talk I shall focus on the so-called *Birkhoff conjecture*, namely the possibility of classifying billiard domains which admit an integrable dynamics.

The easiest example of billiard is given by a billiard in a disc: in this case it is easy to check that the angle of reflection remains constant at each reflection, hence it is an *integral of motion*, which makes the circular billiard an *integrable* dynamical system.

Integrability is one of the most important issue in the study of dynamical systems. In the case of billiards, it translates into a very peculiar geometric property: the existence of so-called *caustics*. For circular billiards, for example, the fact that the angle of reflection remains constant implies that each trajectory is tangent to a concentric circle, which is an example of a caustic. The family of all these caustics foliates the whole circular billiard domain.

More precisely, we say that a curve Γ is a *caustic* for a billiard, if every time a trajectory is tangent to Γ, then it remains tangent after each reflection.

Whereas the mere existence of caustics does not provide significant information on the shape of the domain[1], the presence of a foliation of the billiard table by caustics seems to be a more peculiar property.

Billiards in an ellipse have a similar dynamical picture: trajectories not passing through a focal point are tangent to a confocal conic section, either a confocal ellipse or the two branches of a confocal hyperbola. Thus confocal ellipses are convex caustics, and they foliate the whole domain with the exception of the segment between the foci.

[1] A striking result by Lazutkin [9] shows that all Birkhoff billiards with sufficiently smooth boundary admit a positive measure set of caustics, accumulating to the boundary of the billiard.

Are there other billiards admitting an integrable dynamics? This appearantly naïve question has given rise to one of the most famous (and impenetrable) problems in dynamical systems:

Conjecture (Birkhoff[2]). *Integrable Birkhoff billiards correspond to ellipses.*

Despite its long history and the amount of attention that it has captured, this conjecture is still open. Some of the most relevant contributions are:

- Bialy [2] proved that the only Birkhoff billiard fully foliated by caustics is in the disc. This result was also proved by Wojtkowski [12], by an integral-geometric approach.
- Innami [7] proved, using Aubry-Mather theory, that the existence of caustics with rotation numbers accumulating to 1/2 implies that the billiard domain must be an ellipse.
- The analogue of this conjecture under the assumption that there exists an integral of motion polynomial in the velocity (*Algebraic Birkhoff conjecture*), has been recently proved by Glutsyuk [5].

Instead of considering all possible Birkhoff billiards, one could restrict the analysis to domains that are sufficiently close to ellipses and study the same question in this context (*Perturbative Birkhoff Conjecture*):

- Levallois & Tabanov [10] proved the non-integrability of algebraic perturbations of ellipses.
- Delshams & Ramírez-Ros [4] showed the non-integrability of entire symmetric perturbations of ellipses.

In this talk I shall describe a recent development obtained in collaboration with Vadim Kaloshin, proving that the Perturbartive Birkhoff Conjecture holds true. For nearly circular domains, this result was firstly proved in [1].

Theorem (Kaloshin, S. [8]). *Let $\mathcal{E}_0$ be an ellipse of eccentricity $0 \leq e_0 < 1$ and semi-focal distance c; let $k \geq 39$. For every $K > 0$, there exists $\varepsilon = \varepsilon(e_0, c, K)$ such that if Ω is C^k-smooth domain and*

i) *the billiard map in Ω admits invariant curves/caustics foliated by periodic points for all rotation numbers $\frac{1}{q}$, $q \geq 3$,*
ii) *$\partial\Omega$ is K-close to $\mathcal{E}_0$, with respect to the C^k-norm,*
iii) *$\partial\Omega$ is ε-close to $\mathcal{E}_0$, with respect to the C^1-norm,*

then Ω is an ellipse.

Notice that he notion of integrability i) that we require is very weak. A natural question is what happens if only a small neighbourhood of the boundary is foliated by caustics, or in another words there are invariant curves/caustics corresponding to rotation numbers in $(0, \delta)$, for some $0 < \delta < \frac{1}{3}$.

A partial answer to this question was recently provided in collaboration with Guan Huang and Vadim Kaloshin in [6] for domains that are a sufficiently smooth

[2]Although some vague indications of this question can be found in [3], its first appearance was in a paper by Poritsky [11], so sometimes it is referred to as *Birkhoff-Poritsky conjecture*.

perturbation of ellipses of small eccentricities, under the assumption that for a given $q_0 \geq 3$ there exist invariant curves foliated by periodic points, for all rotation numbers $\frac{j}{q}$, with $q \geq q_0$ and $j = 1, 2, 3$ such that $\gcd(j, q) = 1$. The upper bound on the eccentricity, the smallness condition on the perturbation and the smoothness requirements, depend all on the choice of q_0.

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References

[1] Artur Avila, Jacopo De Simoi and Vadim Kaloshin. *A*n integrable deformation of an ellipse of small eccentricity is an ellipse, Ann. of Math., **184** (2016): 527–558.
[2] Misha Bialy, *C*onvex billiards and a theorem by E. Hopf, Math. Z., **124** (1993): 147–154.
[3] George D. Birkhoff, *O*n the periodic motions of dynamical systems, Acta Math. **50** (1927), 359–379.
[4] Amadeu Delshams and Rafael Ramírez-Ros. *P*oincaré-Melnikov-Arnold method for analytic planar maps, Nonlinearity, **9** (1996): 1–26.
[5] Alexey Glutsyuk, *O*n polynomially integrable Birkhoff billiards on surfaces of constant curvature, J. Eur. Math. Soc. (JEMS), to appear.
[6] Guan Huang, Vadim Kaloshin and Alfonso Sorrentino *N*early circular domains which are integrable close to the boundary are ellipses, Geom. & Funct. Analysis (GAFA), **28** (2018): 334–392.
[7] Nobohuiro Innami, *G*eometry of geodesics for convex billiards and circular billiards, Nihonkai Math. J., **13** (2002): 73–120.
[8] Vadim Kaloshin and Alfonso Sorrentino, *O*n the local Birkhoff conjecture for convex billiards, Ann. of Math. (2), **188** (2018): 315–380.
[9] Vladimir F. Lazutkin, *E*xistence of caustics for the billiard problem in a convex domain. (Russian), Izv. Akad. Nauk SSSR Ser. Mat., **37** (1973): 186–216.
[10] Philippe Levallois and Mikhail Tabanov, *S*éparation des séparatrices du billard elliptique pour une perturbation algébrique et symétrique de l'ellipse, C. R. Acad. Sci. Paris Sér. I Math., **316** (1993): 589–592, 1993.
[11] Hillel Poritsky, *T*he billiard ball problem on a table with a convex boundary: an illustrative dynamical problem, Ann. of Math., **51** (1950): 446–470.
[12] Maciej P. Wojtkowski, *T*wo applications of Jacobi fields to the billiard ball problem, J. Differential Geom., **40** (1994): 155–164.

On the measure of maximal entropy of Sinai billiards

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(joint work with M. Demers)

Sinai billiards maps and flows are uniformly hyperbolic — however grazing orbits give rise to singularities. Most existing works on the ergodic properties of billiards are about the SRB measure (i.e. the Liouville measure in the case of flows), for which exponential mixing is known (both in discrete [6] and continuous time [2]). Another natural equilibrium state is the measure of maximal entropy. Since the discrete-time billiard map T is discontinuous, the mere existence of this measure is not granted a priori. The results of [1] presented in this talk are the following: